Further Mathematical Methods (Linear Algebra) 2002

Problem Sheet 6

(To be discussed in week 7 classes. Please submit answers to the asterisked questions only.)

This week, we shall investigate the use of the Gram-Schmidt procedure to find orthonormal bases, spectral decomposition and some of the properties of certain types of complex matrix. We shall denote the complex conjugate transpose of a matrix A by $\mathsf{A}^\dagger$ and you should feel free to use the fact that

$$\text{For any two matrices } \mathsf{A} \text{ and } \mathsf{B},\ (\mathsf{AB})^\dagger = \mathsf{B}^\dagger \mathsf{A}^\dagger$$

(but, see Question 5).

1. Use the Gram-Schmidt procedure to find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

Verify that your new basis is indeed orthonormal. Hence, or otherwise, find the Cartesian equation of the hyperplane that contains these vectors.

2. * Prove the following theorems:

- If A is a *symmetric* matrix with real entries, then all eigenvalues of A are real. (Hint: Use the fact that symmetric matrices are Hermitian.)
- If A is a *normal* matrix and all of the eigenvalues of A are real, then A is Hermitian. (Hint: Use the fact that normal matrices are unitarily diagonalisable.)
- If P is a *unitary* matrix, then all eigenvalues of P have modulus one. (Hint: Use the definition of a unitary matrix and the fact that $\mathsf{A}\mathbf{x} = \lambda\mathbf{x}$ if $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue λ.)

Note that these can all be proved quite quickly.

3. * Consider the matrix

$$\mathsf{A} = \begin{bmatrix} 7 & 0 & 9 \\ 0 & 2 & 0 \\ 9 & 0 & 7 \end{bmatrix}$$

Find an orthogonal matrix P such that the matrix $\mathsf{P}^t\mathsf{AP}$ is diagonal. Express A in the form

$$\mathsf{A} = \lambda_1\mathsf{E}_1 + \lambda_2\mathsf{E}_2 + \lambda_3\mathsf{E}_3,$$

where (for $i = 1, 2, 3$) the matrix $\mathsf{E}_i = \mathbf{x}_i\mathbf{x}_i^t$ is formed from the eigenvector $\mathbf{x}_i$ corresponding to the eigenvalue λ_i of the matrix A. Further, verify that these matrices have the property that

$$\mathsf{E}_i\mathsf{E}_j = \begin{cases} \mathsf{E}_i & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

where $i, j = 1, 2, 3$. Consequently, prove that for any three matrices E_1, E_2 and E_3 with this property,

$$(\alpha_1\mathsf{E}_1 + \alpha_2\mathsf{E}_2 + \alpha_3\mathsf{E}_3)^3 = \alpha_1^3\mathsf{E}_1 + \alpha_2^3\mathsf{E}_2 + \alpha^3\mathsf{E}_3,$$

where α_1, α_2 and α_3 are arbitrary real numbers. Hence, find a matrix B such that $\mathsf{B}^3 = \mathsf{A}$.

Other Problems. (These are *not* compulsory, they are *not* to be handed in, and will *not* be covered in classes.)

Here are some more problems on these topics. As this is all new stuff, it might be useful if you try some of them to further your understanding of the myriad of terms that have been introduced recently. Maybe try them after the class when you will have a clearer understanding of what is going on.

4. Given that the Taylor expansion of the exponential function, e^x, is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

for all $x \in \mathbb{R}$ and that A is an $n \times n$ Hermitian matrix, use the spectral decomposition of such a matrix to deduce that the exponential function, e^{A}, is given by

$$e^{\mathsf{A}} = \sum_{i=1}^{n} e^{\lambda_i} \mathsf{E}_i$$

where $\mathbf{x}_i$ is an eigenvector of A corresponding to the eigenvalue λ_i and $\mathsf{E}_i = \mathbf{x}_i \mathbf{x}_i^t$. Verify that this function has the property that $e^{2\mathsf{A}} = e^{\mathsf{A}} e^{\mathsf{A}}$ which is analogous to the property that $e^{2x} = e^x e^x$ when $x \in \mathbb{R}$.

5. Show that for two matrices A and B, $(\mathsf{AB})^\dagger = \mathsf{B}^\dagger \mathsf{A}^\dagger$. In particular, prove that if A and B are real matrices, $(\mathsf{AB})^t = \mathsf{B}^t \mathsf{A}^t$.

6. If A is an $n \times n$ matrix with complex entries, then $\det(\mathsf{A}^*) = \det(\mathsf{A})^*$. Assuming the truth of this result, prove that for such a matrix, $\det(\mathsf{A}^\dagger) = \det(\mathsf{A})^*$. Also, establish that:

- If A is Hermitian, then $\det(\mathsf{A})^*$ is real.
- If A is unitary, then $|\det(\mathsf{A})| = 1$.

where, again, A is an $n \times n$ matrix with complex entries. Using these, establish the three corresponding results for a matrix A with real entries.

7. Prove the following theorems:

- If A is invertible, then so is $\mathsf{A}^\dagger$. In particular, $(\mathsf{A}^\dagger)^{-1} = (\mathsf{A}^{-1})^\dagger$.
- If A is a unitary matrix, then $\mathsf{A}^\dagger$ is unitary too.

Harder Problems. (These are *not* compulsory, they are *not* to be handed in, and will *not* be covered in classes.)

For those of you who find this *just* fascinating, there is a link between this sort of stuff and the stuff we did on inner products and vector spaces. If you are interested, you might care to try these questions. Note that these are *even more* optional than the previous four!

8. Prove that an $n \times n$ matrix with complex entries is unitary iff its column vectors form an orthonormal set in $\mathbb{C}^n$ with the [complex] Euclidean inner product (i.e. the inner product where for two vectors $\mathbf{x} = [x_1, x_2, \ldots, x_n]^t$ and $\mathbf{y} = [y_1, y_2, \ldots, y_n]^t$ we have $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1^* + x_2 y_2^* + \cdots + x_n y_n^*$).

9. Prove that if $\mathsf{A} = \mathsf{A}^\dagger$, then for every vector in $\mathbb{C}^n$, the entry in the 1×1 matrix $\mathbf{x}^\dagger \mathsf{A} \mathbf{x}$ is real.

10. Let λ and μ be distinct eigenvalues of a Hermitian matrix A. Prove that if $\mathbf{x}$ is an eigenvector corresponding to λ and $\mathbf{y}$ is an eigenvector corresponding to μ, then $\mathbf{x}^\dagger \mathsf{A} \mathbf{y} = \lambda \mathbf{x}^\dagger \mathbf{y}$ and $\mathbf{x}^\dagger \mathsf{A} \mathbf{y} = \mu \mathbf{x}^\dagger \mathbf{y}$.

Using this, prove that if A is a normal matrix, then the eigenvectors from different eigenspaces are orthogonal. (The *eigenspace* of A corresponding to the eigenvalue λ is the vector space spanned by the eigenvectors corresponding to λ. You may care to show that eigenspaces are subspaces of $\mathbb{C}^n$ (when A is an $n \times n$ matrix). You can do this by either noting that the eigenspace corresponding to the eigenvalue λ is the null-space of the matrix $\mathsf{A} - \lambda I$, or directly using Theorem 1.4.)