Getting from One Colouring to Another

Jan van den Heuvel

Department of Mathematics

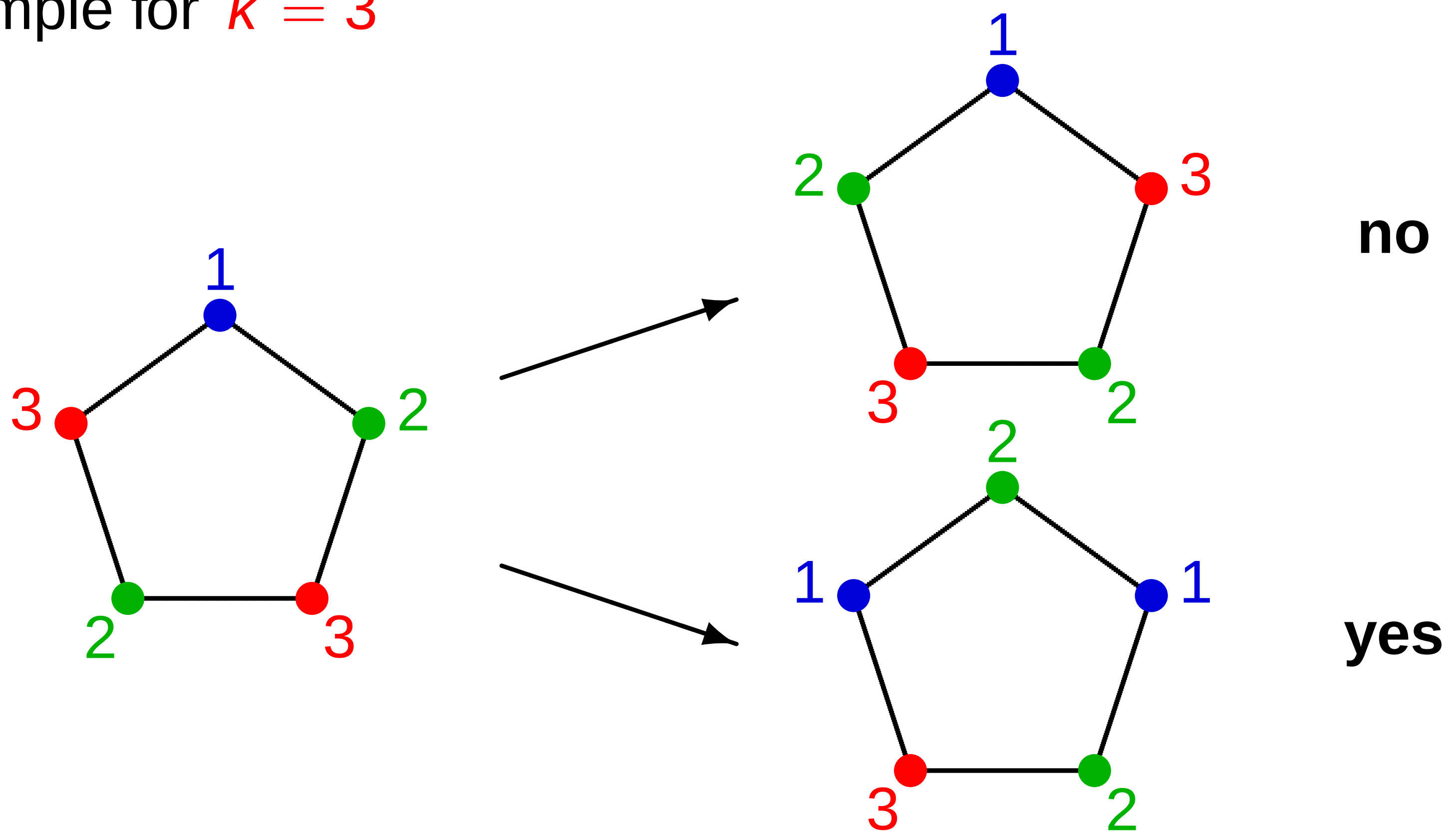
London School of Economics and Political Science

First definitions

- *k-colouring* of G : proper vertex-colouring using colours from $\{1, 2, \ldots, k\}$
- *list k-colouring* of G : proper vertex-colouring, using colours from each vertex' own list $L(v)$, with $|L(v)| = k$
- *recolouring* :
  - changing the colour of one vertex v
  - still using colours from $\{1, 2, \ldots, k\}$ or $L(v)$
  - and still maintaining a proper colouring

The basic question

- given a graph and two colourings α and β
  - can we recolour α to β ?
- example for $k = 3$



Some further definitions

- colourings α and β are *connected* : α can be recoloured to β
- graph G is *k-mixing* : any two k-colourings are connected
- G is *k-list mixing* : similar, with given lists of size k

First properties

Easy fact

- $k \geq \boldsymbol{\Delta}(G) + 2 \quad \Longrightarrow \quad G$ is k-(list) mixing
- requires at most $\boldsymbol{\Delta} \cdot |V|$ steps

- *degeneracy* $\deg(G) = \max\{\,\delta(H) \mid H \subseteq G\,\}$

Property (Dyer, et al., 2004)

- $k \geq \deg(G) + 2 \quad \Longrightarrow \quad G$ is k-(list) mixing

First properties

Property (Dyer, et al., 2004)

- $k \geq \deg(G) + 2 \quad \Longrightarrow \quad G$ is k-(list) mixing

Question

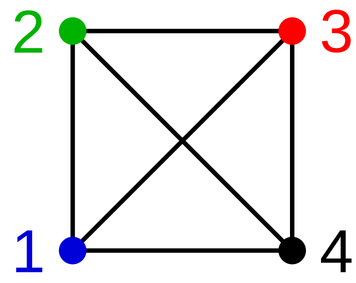
- how many steps are required ?
  - proof gives exponential upper bound (in $|V|$)
  - is a polynomial upper bound possible ?
  - maybe even a quadratic one ?

Theorem (Bonsma & Cereceda)

- $k \geq 2\deg(G) + 1 \quad \Longrightarrow$ requires at most $O(|V|^2)$ steps

Extremal graphs for k-mixing

- "boring" extremal graph : complete graph K_m
  - $deg(K_m) + 1 = m$
  - all m -colourings look the same :
  - no vertex can change colour

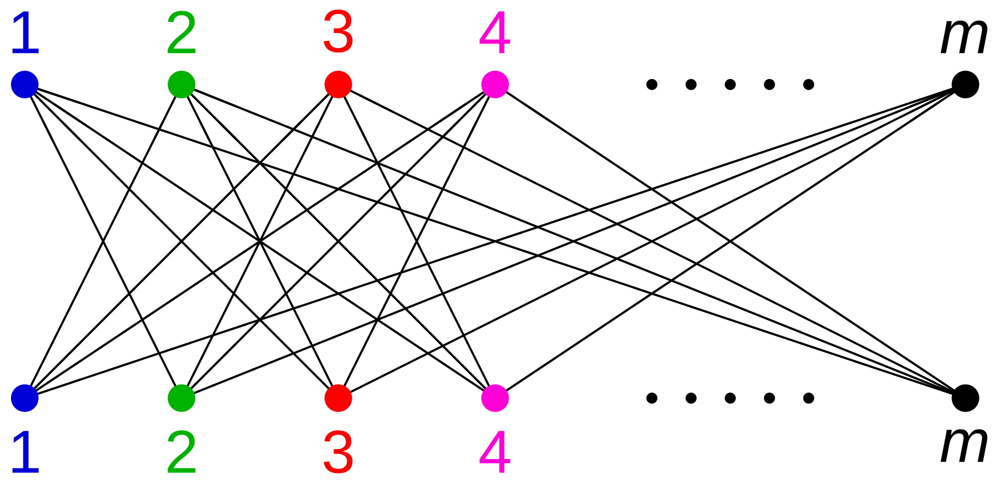


Terminology

- *frozen k-colouring* : colouring in which no vertex can change colour
  - frozen colourings immediately mean G is not k-mixing

More interesting extremal graphs

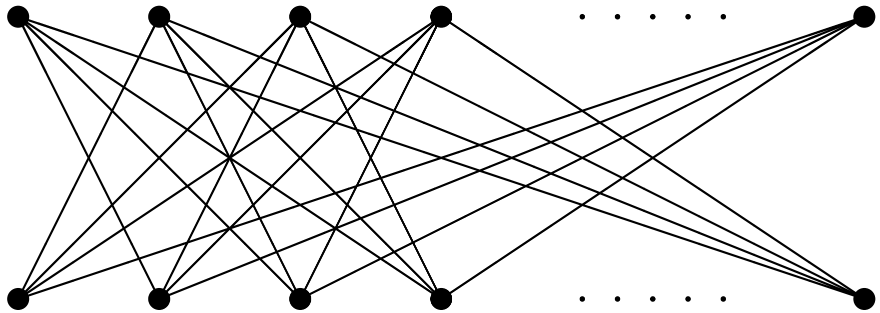
- graph L_m: $K_{m,m}$ minus a perfect matching ($m \geq 3$)
  - $deg(L_m) + 1 = m$



  - has frozen m-colourings – hence L_m is not m-mixing
- **so**:
  bipartite graphs can be non-k-mixing for arbitrarily large k

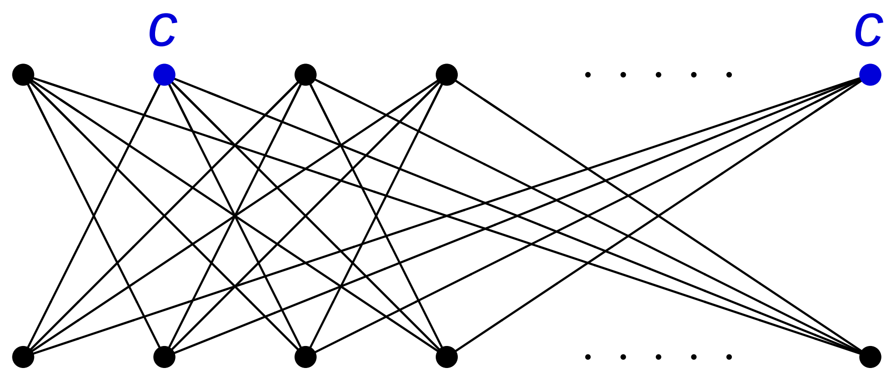
More interesting properties of L_m

- non-k-mixing for $k = m$ colours
- but k-mixing for $3 \leq k \leq m - 1$
  - suppose L_m coloured with $k \leq m - 1$ colours



More interesting properties of L_m

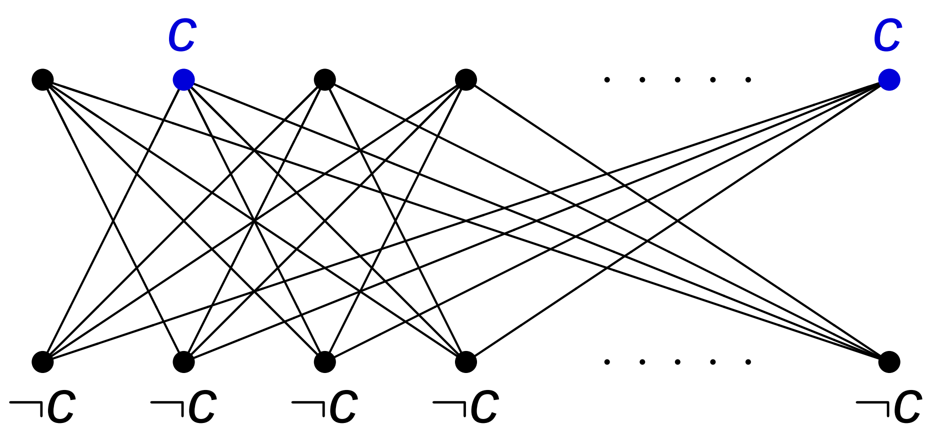
- non-k-mixing for $k = m$ colours
- but k-mixing for $3 \le k \le m-1$
  - suppose L_m coloured with $k \le m-1$ colours



  - some colour c must appear more than once on the top

More interesting properties of L_m

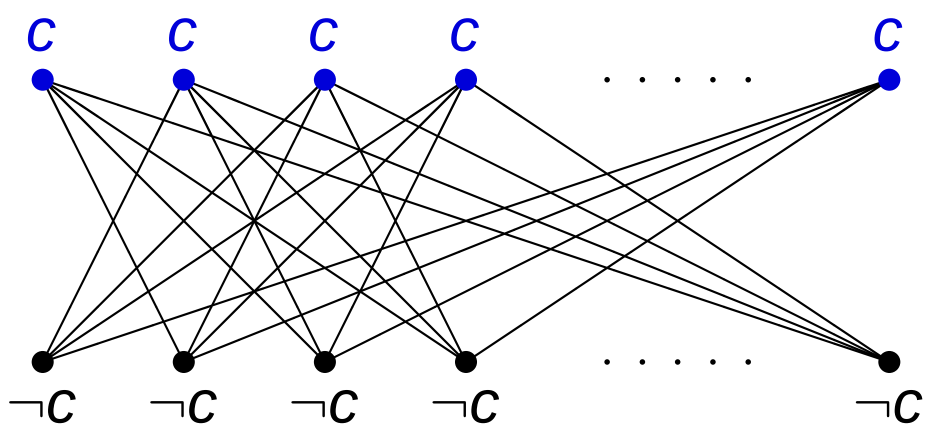
- non-k-mixing for $k = m$ colours
- but k-mixing for $3 \leq k \leq m-1$
  - suppose L_m coloured with $k \leq m-1$ colours



  - that colour c can't appear among the bottom vertices

More interesting properties of L_m

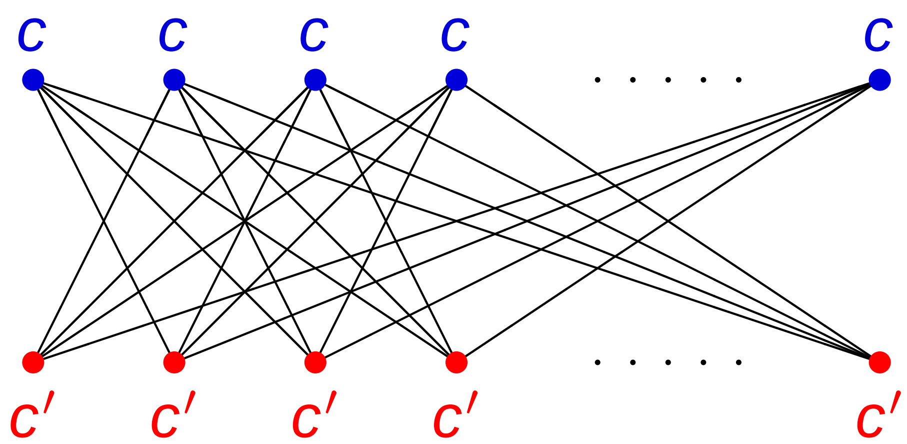
- non-k-mixing for $k = m$ colours
- but k-mixing for $3 \leq k \leq m - 1$
  - suppose L_m coloured with $k \leq m - 1$ colours



  - so all vertices on the top can be recoloured to c

More interesting properties of L_m

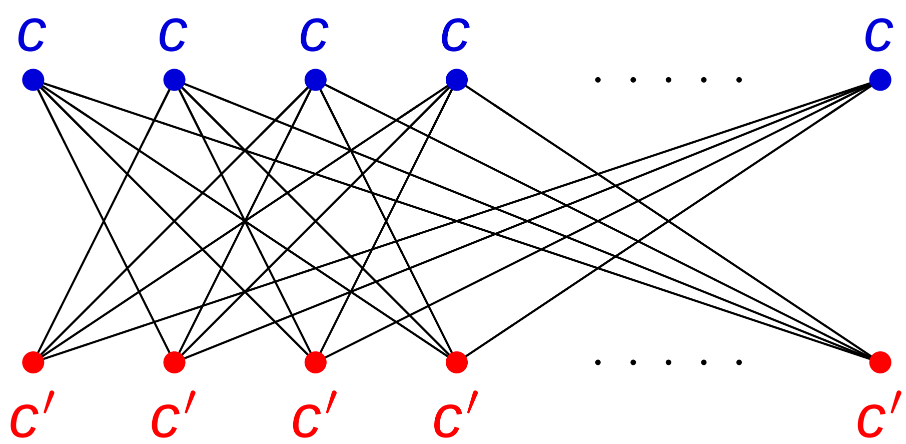
- non-k-mixing for $k = m$ colours
- but k-mixing for $3 \leq k \leq m - 1$
  - suppose L_m coloured with $k \leq m - 1$ colours



  - then the bottom can be recoloured to some $c' \neq c$

More interesting properties of L_m

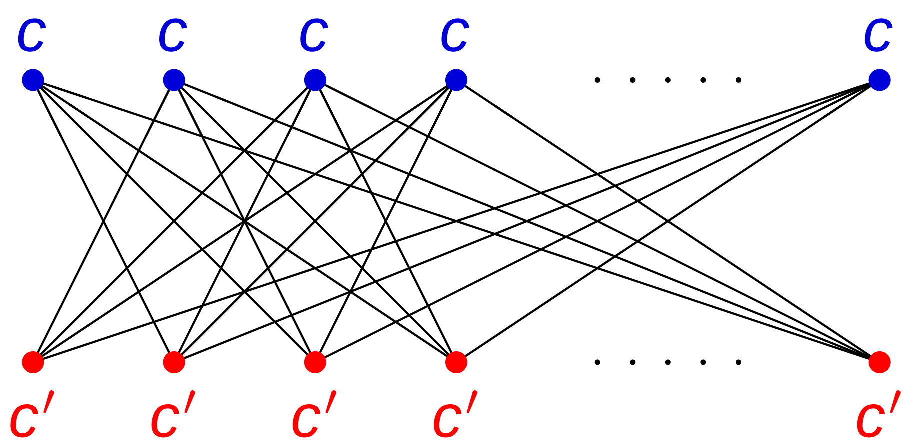
- non-k-mixing for $k = m$ colours
- but k-mixing for $3 \le k \le m-1$
  - suppose L_m coloured with $k \le m-1$ colours



  - hence any colouring is connected to a 2-colouring
  - easy to see that all these 2-colourings are connected

More interesting properties of L_m

- non-k-mixing for $k = m$ colours
- but k-mixing for $3 \leq k \leq m-1$
  - suppose L_m coloured with $k \leq m-1$ colours



  - hence any colouring is connected to a 2-colouring
  - easy to see that all these 2-colourings are connected
- **so**: mixing is not a monotone property

Decision problems

k-COLOUR-PATH

Input : graph G and two k-colourings α and β

Question : are α and β connected ?

k-LIST-COLOUR-PATH

Input : graph G , lists $L(v)$ of size k for each vertex v , and two colourings α and β

Question : are α and β connected ?

" k-(LIST)-MIXING "

Input : graph G (and lists $L(v)$ of size k for each vertex)

Question : is G k-(list) mixing ?

Decision problems for mixing

k-MIXING

Input: graph G and a k-colouring α

Question: is G k-mixing?

BIPARTITE-k-MIXING

Input: bipartite graph G

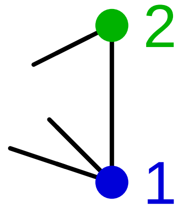
Question: is G k-mixing?

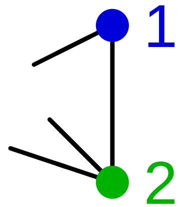
k-LIST-MIXING

Input: graph G, lists $L(v)$ of size k for each vertex v, proper colouring α using those lists

Question: is G k-list mixing?

2-MIXING

- 

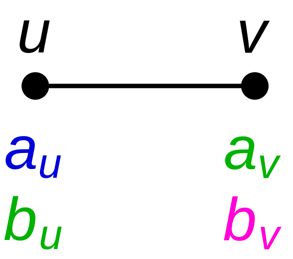
can't become


so

- two 2-colourings of a bipartite graph are connected $\iff$ they are identical on non-trivial components
- a bipartite graph is 2-mixing $\iff$ there are no edges

2-LIST-MIXING

- **Input** : graph G
  lists $L(v) = \{ a_v, b_v \}$ for each vertex v

- translate to 2-SAT problem :
  - for each vertex v introduce Boolean variable x_v
    - $x_v = \mathrm{T}$ $\leftrightarrow$ v is coloured a_v
    - $x_v = \mathrm{F}$ $\leftrightarrow$ v is coloured b_v
  - suppose uv is an edge with $b_u = a_v$
    - 

      $\longleftrightarrow$ $x_u \vee \overline{x_v}$

Recolouring and satisfiability

- 2-list colouring problem

  $\longrightarrow$ 2-sat problem $(x_u \vee \overline{x_v}) \wedge (x_s \vee x_t) \wedge \cdots$

- recolouring a vertex v $\longleftrightarrow$ "flipping" the value of x_v

SAT-ST-CONNECTED

Input : Boolean expression φ, satisfying assignments $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$

Question : can we get from $\boldsymbol{x}$ to $\boldsymbol{y}$ using single variable flips ?

SAT-CONNECTED

Input : Boolean expression φ

Question : can we go between **any** two satisfying assignments ?

Connectivity of satisfying assignments

Theorem (Gopalan, et al., 2007)

- SAT-ST-CONNECTED restricted to 2-sat expressions is in P
- SAT-CONNECTED restricted to 2-sat expressions is in P
- requires at most n flips

Corollary

- 2-LIST-COLOUR-PATH is in P
- 2-LIST-MIXING is in P
- requires at most $|V|$ steps

The case $k \geq 3$

Theorem (Bonsma & Cereceda)

- for all $k \geq 3$: k-LIST-COLOUR-PATH is PSPACE-complete
- for all $k \geq 4$: k-COLOUR-PATH is PSPACE-complete
- problems remain PSPACE-complete for bipartite graphs
- for all the cases above, there exist examples of graphs G and two colourings α and β, so that going from α to β takes $\Omega(2^{\sqrt{|V|}})$ steps
- similar, but weaker, results earlier obtained by Jacob (1997)

Recolouring with 3 colours

Theorem (Cereceda, vdH & Johnson)

- 3-COLOUR-PATH is in P

Theorem (Cereceda, vdH & Johnson)

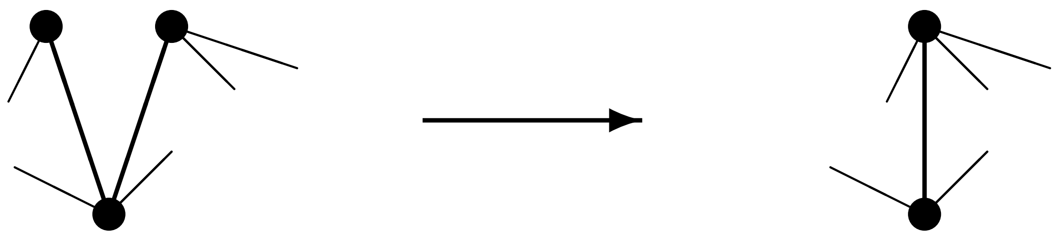
- BIPARTITE-3-MIXING is coNP-complete

- a non-bipartite 3-colourable graph is never 3-mixing

- requires at most $O(|V|^2)$ steps

Folding of graphs

- fold of two vertices at distance 2 :



- G foldable to H : sequence of folds changes G to H

Theorem (Cook & Evans, 1979)

- G connected

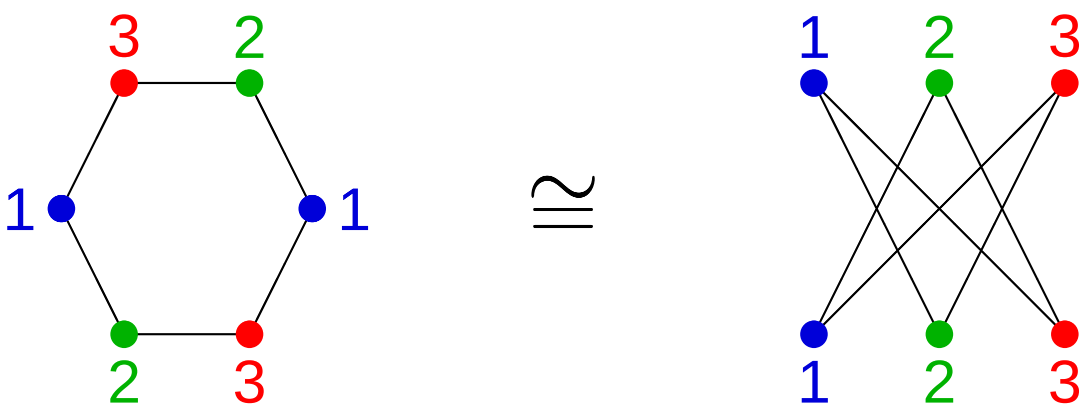
$$\Longrightarrow \quad \chi(G) = \min\{\, k \mid G \text{ is foldable to } K_k \,\}$$

A structural certificate for bipartite non-3-mixing

Theorem

- connected bipartite G is not 3-mixing
  $\iff$ G is foldable to a chordless 6-cycle

- $C_6 \cong L_3$ – so C_6 is not 3-mixing



- **note** : C_4 is 3-mixing

Why are 3 colours so much easier ?

- because everything just works !
  (and fails horribly for more colours)

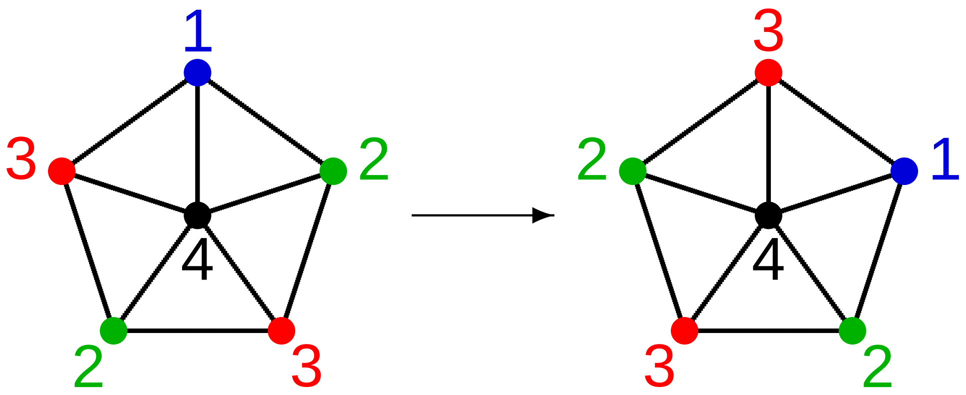
Why are 3 colours so much easier ?

example

- given graph G and k-colouring α ,
  a vertex v is fixed (for α) if v can never be recoloured
- if a vertex v is adjacent to fixed vertices of all colours (except from its own), then v itself is fixed

Lemma

- for 3 colours, this determines exactly the fixed vertices
- but not for 4 colours :



Open problems I

we know

- BIPARTITE-3-MIXING is coNP-complete
- BIPARTITE-4-COLOUR-PATH is PSPACE-complete

- what is the complexity of BIPARTITE-4-MIXING ?
- maybe easier if the graph is cubic ?

- what can we say if $k = \Delta + 1$ or $k = \Delta$?

Open problems II

- what can we say about the structure of the set of all k-colourings of a graph ?

- G is k-mixing $\iff$ the set of k-colourings is connected

Theorem (Achlioptas & Coja-Oghlan, 2008)

- G a random graph
  k close to the chromatic number of G
  $\Longrightarrow$ the set of all k-colourings is shattered

Open problems III

- what happens if we use a different recolouring rule ?
- *Kempe recolouring* :
  - changing the colour of one vertex v from c_1 to c_2
  - by swapping colours on the component induced by vertices coloured c_1 or c_2 containing v

Folklore

- G bipartite $\Longrightarrow$ G is Kempe-k-mixing for all k
- what is the complexity of KEMPE-k-PATH or KEMPE-k-MIXING for non-bipartite graphs ?