Radio Channel Assignment on 2-Dimensional Lattices *

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Abstract

Given a set V of points in the plane, a sequence $d_0, d_1, \ldots, d_k$ of non-negative numbers and an integer n, we are interested in the problem to assign integers from $\{0, \ldots, n-1\}$ to the points in V such that if $\underline{\mathbf{x}}, \underline{\mathbf{y}} \in V$ are two points with euclidean distance less than d_i, then the difference between the labels of $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$ is not equal to i. This question is inspired by problems occurring in the design of radio networks, where radio channels need to be assigned to transmitters in such a way that interference is minimised. In this paper we consider the case that the set of points are the points of a 2-dimensional lattice. Recent results by McDiarmid and Reed show that if only one constraint d_0 is given, good labellings can be obtained by using so-called strict tilings. We extend these results to the case that higher level constraints $d_0, d_1, \ldots, d_k$ occur. In particular we study conditions that guarantee that a strict tiling, satisfying only the one constraint d_0, can be transformed to a strict tiling satisfying the higher order constraints as well. Special attention is devoted to the case that the points are the points of a triangular lattice.

Keywords: radio channel assignment, optimal labelling, euclidean distance, minimum span, 2-dimensional lattice, strict tiling.

1 Introduction

Large-scale radio systems such as those used in mobile phone networks often exhibit a "cellular structure" [10, 14]: The service area is divided into cells and each cell is serviced by a transmitter. These transmitters communicate with users within the cell using a particular radio channel (or set of channels). In any particular application the available channels are uniformly spaced in the spectrum justifying integer labellings of these channels.

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Suppose that a radio receiver is tuned to a signal on channel c_0, broadcast by its local transmitter (i.e., the one at closest distance). Reception will be degraded if there is excessive interference from other transmitters in the vicinity. First there is 'co-channel' interference due to re-use of channel c_0 at nearby sites; but there are also contributions from sites using channels near c_0, since in practice neither transmitters nor receivers operate exclusively within the frequencies of their assigned channels. To ensure acceptable signal quality constraints are imposed on the allowed channel separations between pairs of potentially interfering transmitters.

As a first simplification, we assume that the allowed channel separation between a pair of transmitters is uniquely determined by the distance of the transmitters. That is, we assume that a sequence of numbers $d_0, \ldots, d_k$ is given such that if two transmitters have distance less than d_i, then the difference of their assigned channels is not allowed to be equal to i. The physical assumptions involved when this type of constraints is that signal propagation is isotropic and independent of frequency. Our main question is to determine the minimum span of channels needed to assign frequencies to the transmitters such that none of the constraints is violated.

A further simplification we assume in this paper is that the transmitters are placed at the points of an infinite 2-dimensional lattice. Regular lattices form the initial framework for many exercises in radio spectrum planning. The assumptions involved when using lattices are that all transmitters are identical. Restrictions on transmitter placement generally mean that the regular structure is distorted when the plan is actually implemented. Nevertheless, lattice planning remains an important tool for radio engineers and we will accept its inherent assumptions.

If one needs to cover the infinite plane with equal cells, then it is well-known that the most economic covering will use hexagonal cells. And the centres of these cells ("the transmitter sites") will form the points of a triangular lattice. It is because of this that much work has been done on channel assignment on the regular triangular lattice. In this paper we obtain several results that can be used for 2-dimensional lattices in general as well as some stronger results for the triangular lattice.

This paper is organised as follows. In the next section we describe the mathematical background used and give the main definitions. In Section 3 we look at the problem to estimate the span for general lattices. Our starting position is a recent result in [15] proving that so-called "strict tilings" give a good approximation for an optimal labelling if only a constraint on co-channel re-use is assumed (in other words, if only a distance d_0 is given). We extend this result to the case that higher level constrains are involved and give sufficient conditions that assure that the higher order constraints do not mean that more channels are needed than if just the co-channel constraint needs to be satisfied. In Section 4, again inspired by work in [15], we sharpen the results in the case that the lattice is a triangular lattice. In that case we also can give an indication how good the results are.

The last sections, Sections 5 to 7, contain the proofs of the results in Sections 2 to 4, respectively.

2 Mathematical background

We identify the 2-dimensional euclidean plane with the vector space $\mathbb{R}^2$. If $\underline{\mathbf{x}}, \underline{\mathbf{y}}$ are two vectors, then their inner product is denoted by $\underline{\mathbf{x}} \cdot \underline{\mathbf{y}}$; the norm of a vector $\underline{\mathbf{x}}$ is the standard euclidean norm $\|\underline{\mathbf{x}}\| = \sqrt{\underline{\mathbf{x}} \cdot \underline{\mathbf{x}}}$; and the distance between two vectors $\underline{\mathbf{x}}, \underline{\mathbf{y}}$ is $d(\underline{\mathbf{x}}, \underline{\mathbf{y}}) = \|\underline{\mathbf{x}} - \underline{\mathbf{y}}\|$. If $\underline{\mathbf{x}} \in \mathbb{R}^2$ and $V \subseteq \mathbb{R}^2$, $V \neq \emptyset$, then $d(\underline{\mathbf{x}}, V) = \inf\{\, d(\underline{\mathbf{x}}, \underline{\mathbf{v}}) \mid \underline{\mathbf{v}} \in V \,\}$.

2.1 Definition

Given a set $V \subseteq \mathbb{R}^2$, an *n-labelling of V*, where n is a positive integer, is a function $\varphi : V \to \{1, \ldots, n\}$. Given non-negative real numbers $d_0, d_1, \ldots, d_k$, we say that an n-labelling φ *satisfies* $(d_0, \ldots, d_k)$ if for all $\underline{\mathbf{x}}, \underline{\mathbf{y}} \in V$ and for all $i = 1, \ldots, k$,

$$d(\underline{\mathbf{x}}, \underline{\mathbf{y}}) \;<\; d_i \qquad \Longrightarrow \qquad |\varphi(\underline{\mathbf{x}}) - \varphi(\underline{\mathbf{y}})| \;\neq\; i.$$

The idea of the numbers d_i is that they determine the minimum distance that two transmitters need to have in order to be allowed to use channels that are distance i apart. In particular, the number d_0 gives the minimum distance that two transmitters using the same channel need to have.

For most radio systems it can be assumed that $d_0 \geq d_1 \geq \cdots \geq d_k$. But there exist radio systems that can be modelled with a description where this is not the case (see, e.g., [10, Section III E]) and we will not use the assumption in this paper.

2.2 Definition

The *span* $sp(V;\, d_0, \ldots, d_k)$ of a set $V \subseteq \mathbb{R}^2$ is the minimum n such that there exists an n-labelling of V satisfying $(d_0, \ldots, d_k)$.

In this paper we are interested in the case that the set V is a 2-dimensional lattice.

2.3 Definition

A *lattice* $\Lambda = L(\underline{\mathbf{m}}, \underline{\mathbf{n}})$ is a subset of $\mathbb{R}^2$ of the form $\{\, p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}} \mid p, q \in \mathbb{Z} \,\}$. Here $\underline{\mathbf{m}}, \underline{\mathbf{n}}$ are two linearly independent vectors in $\mathbb{R}^2$. The pair of vectors $\underline{\mathbf{m}}, \underline{\mathbf{n}}$ is called a *basis* of the lattice.

The *triangular lattice* Δ is the lattice generated by the vectors $\binom{1}{0}$ and $\binom{\frac{1}{2}}{\frac{1}{2}\sqrt{3}}$.

2.4 Definition

If $\underline{\mathbf{m}} = \binom{m_1}{m_2}$ and $\underline{\mathbf{n}} = \binom{n_1}{n_2}$ are the cartesian coordinates of $\underline{\mathbf{m}}$ and $\underline{\mathbf{n}}$, then the *cell-area* of the lattice $\Lambda = L(\underline{\mathbf{m}}, \underline{\mathbf{n}})$, denoted $\rho(\Lambda)$, is the number $\rho(\Lambda) = |m_1\, n_2 - m_2\, n_1|$.

The cell-area $\rho(L(\underline{\mathbf{m}}, \underline{\mathbf{n}}))$ is the area of the parallelogram spanned by the pair $\underline{\mathbf{m}}, \underline{\mathbf{n}}$. But this is also equal to the area of the set of all points in $\mathbb{R}^2$ closest to a given point in the lattice $L(\underline{\mathbf{m}}, \underline{\mathbf{n}})$. Interpreting the points in the lattice as the transmitters of a radio system this means that $\rho(L(\underline{\mathbf{m}}, \underline{\mathbf{n}}))$ is the area of the cell serviced by any one transmitter. See Figure 2.1.

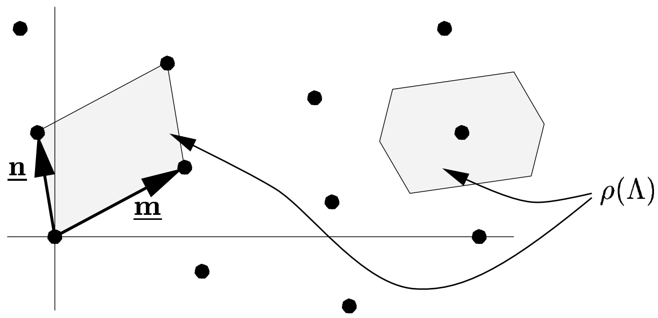

Figure 2.1. The lattice $\Lambda = L(\underline{\mathbf{m}}, \underline{\mathbf{n}})$ with the cell area $\rho(\Lambda)$.

2.5 Definition
If $\underline{\mathbf{m}}, \underline{\mathbf{n}} \in \mathbb{R}^2$ are two linearly independent vectors, then $M_{\underline{\mathbf{m}},\underline{\mathbf{n}}}$ denotes the 2×2 matrix which has the vectors $\underline{\mathbf{m}}, \underline{\mathbf{n}}$ as its columns. This means we can define the lattice $L(\underline{\mathbf{m}}, \underline{\mathbf{n}})$ as $\{ M_{\underline{\mathbf{m}},\underline{\mathbf{n}}} \underline{\mathbf{z}} \mid \underline{\mathbf{z}} \in \mathbb{Z}^2 \}$.

Also, if $\underline{\mathbf{x}} \in \mathbb{R}^2$, then $M^{-1}_{\underline{\mathbf{m}},\underline{\mathbf{n}}} \underline{\mathbf{x}}$ gives the coordinates of $\underline{\mathbf{x}}$ with respect to $\underline{\mathbf{m}}, \underline{\mathbf{n}}$. It follows that $\underline{\mathbf{x}} \in L(\underline{\mathbf{m}}, \underline{\mathbf{n}})$ if and only if $M^{-1}_{\underline{\mathbf{m}},\underline{\mathbf{n}}} \underline{\mathbf{x}} \in \mathbb{Z}^2$.

For a given lattice Λ there exists a wide choice of bases for this lattice. In fact, we have that $L(\underline{\mathbf{m}}_1, \underline{\mathbf{n}}_1) = L(\underline{\mathbf{m}}_2, \underline{\mathbf{n}}_2)$ if and only if $\underline{\mathbf{m}}_2 = A\,\underline{\mathbf{m}}_1$ and $\underline{\mathbf{n}}_2 = A\,\underline{\mathbf{n}}_1$ for a *unimodular* 2×2 matrix A, i.e., a matrix A with integral entries and determinant ± 1. In order to avoid too much ambiguity we define the following.

2.6 Definition
A *minimal basis* of a lattice Λ is a pair $\underline{\mathbf{m}}, \underline{\mathbf{n}} \in \Lambda$ chosen such that
(1) $\|\underline{\mathbf{m}}\|$ is minimum, subject to $\underline{\mathbf{m}} \in \Lambda \setminus \{\underline{\mathbf{0}}\}$;
(2) $\|\underline{\mathbf{n}}\|$ is minimum, subject to (1) and $\underline{\mathbf{n}} \in \Lambda \setminus \{ p\,\underline{\mathbf{m}} \mid p \in \mathbb{Z} \}$;
(3) $\underline{\mathbf{m}} \cdot \underline{\mathbf{n}} \geq 0$, subject to (1) and (2).

Notice that a minimal basis always exists. It is obvious that we can find a pair $\underline{\mathbf{m}}', \underline{\mathbf{n}}'$ satisfying (1) and (2). If we have $\underline{\mathbf{m}}' \cdot \underline{\mathbf{n}}' < 0$, then the pair $\underline{\mathbf{m}}', -\underline{\mathbf{n}}'$ will form a minimal basis.

Next note that a minimal basis is not unique. For instance, if $\underline{\mathbf{m}}, \underline{\mathbf{n}}$ is a minimal basis of Λ, then so is $-\underline{\mathbf{m}}, -\underline{\mathbf{n}}$. And lattices such as the triangular lattice have even more choices for a minimal basis. The problem to find a basis of a lattice satisfying certain minimality conditions such as those in Definition 2.6 is a hard problem in higher dimensions, see [9, Section 5.3]. Fortunately, for the 2-dimensional case the so-called "Gaussian Algorithm" provides a fast tool to find such a minimal basis (see, e.g., [4] and references therein).

One of the main advantages to assume that the basis of a lattice is a minimal basis is that it gives a straightforward way to determine the distance between a given vector and the points of the lattice, as expressed in the following lemma. The floor $\lfloor x \rfloor$ of a real number x is the largest integer $p \in \mathbb{Z}$ such that $p \leq x$.

2.7 Lemma
Let $\underline{\mathbf{x}} \in \mathbb{R}^2$ *and let* Λ *be a lattice with minimal basis* $\underline{\mathbf{m}}, \underline{\mathbf{n}}$*. Set* $\binom{y_1}{y_2} = M^{-1}_{\underline{\mathbf{m}},\underline{\mathbf{n}}}\,\underline{\mathbf{x}}$ *and define* $\underline{\mathbf{x}}_\Lambda = \lfloor y_1 \rfloor\,\underline{\mathbf{m}} + \lfloor y_2 \rfloor\,\underline{\mathbf{n}} \in \Lambda$*. Then*

$$d(\underline{\mathbf{x}}, \Lambda) \;=\; \min\{\, d(\underline{\mathbf{x}}, \underline{\mathbf{x}}_\Lambda),\, d(\underline{\mathbf{x}}, \underline{\mathbf{x}}_\Lambda + \underline{\mathbf{m}}),\, d(\underline{\mathbf{x}}, \underline{\mathbf{x}}_\Lambda + \underline{\mathbf{n}}),\, d(\underline{\mathbf{x}}, \underline{\mathbf{x}}_\Lambda + \underline{\mathbf{m}} + \underline{\mathbf{n}})\,\}.$$

An illustration of the essential points in Lemma 2.7 is given in Figure 2.2. The elementary but technical proof of the lemma can be found in Section 5.

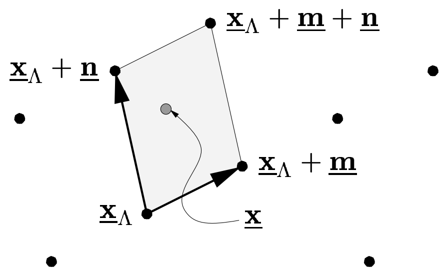

Figure 2.2. If $\underline{\mathbf{m}}, \underline{\mathbf{n}}$ are a minimal bases of $L(\underline{\mathbf{m}}, \underline{\mathbf{n}})$, then one of the four indicated lattice points is the lattice point closest to $\underline{\mathbf{x}}$.

3 Results for general lattices

Throughout this section we assume that Λ is a lattice and $d_0, d_1, \ldots, d_k$ are non-negative real numbers.

3.1 Definition
A labelling φ of a Λ is called an $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$*-labelling*, where $\underline{\mathbf{s}}, \underline{\mathbf{t}} \in \Lambda$ are linearly independent, if $\varphi(\underline{\mathbf{x}} + \underline{\mathbf{s}}) = \varphi(\underline{\mathbf{x}} + \underline{\mathbf{t}}) = \varphi(\underline{\mathbf{x}})$ for all $\underline{\mathbf{x}} \in \Lambda$. In the literature $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$-labellings are sometimes known as *regular tilings* or just *tilings*. The lattice $\Lambda^* = L(\underline{\mathbf{s}}, \underline{\mathbf{t}})$ forms a sub-lattice of Λ. This sub-lattice is often called the *co-channel lattice*. So an $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$-labelling φ has the property

$$\underline{\mathbf{x}} - \underline{\mathbf{y}} \in \Lambda^* \qquad \Longrightarrow \qquad \varphi(\underline{\mathbf{x}}) \;=\; \varphi(\underline{\mathbf{y}}).$$

A *strict* $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$*-labelling* or *strict tiling* is an $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$-labelling satisfying

$$\underline{\mathbf{x}} - \underline{\mathbf{y}} \in \Lambda^* \qquad \Longleftrightarrow \qquad \varphi(\underline{\mathbf{x}}) \;=\; \varphi(\underline{\mathbf{y}}).$$

Let $sp_T(\Lambda;\, d_0, \ldots, d_k)$ denote the minimum n such that there exists an n-labelling of Λ satisfying $(d_0, \ldots, d_k)$ which is a strict tiling.

Regular tilings and strict tilings have been an important tool in the construction of channel assignments for the triangular lattice [1, 5]. Because of this it is surprisingly that it was shown only recently that regular tilings provide good assignments, i.e., assignments with a span that is close to the optimal span.

3.2 Theorem [15]
The spans $sp(\Lambda;\, d_0)$ *and* $sp_T(\Lambda;\, d_0)$ *satisfy*

$$\tfrac{1}{2}\sqrt{3}\,\frac{d_0^2}{\rho(\Lambda)} \;\le\; sp(\Lambda;\, d_0) \;\le\; sp_T(\Lambda;\, d_0) \;\le\; \tfrac{1}{2}\sqrt{3}\,\frac{d_0^2}{\rho(\Lambda)} + \mathcal{O}(d_0) \qquad (\, d_0 \to \infty\,).$$

Theorem 3.2 shows that strict tilings give good channel assignments for lattices if we are only interested in the distance that channels can be re-used, and for large values of d_0. For more general constraints $d_0, \ldots, d_k$ very little is known. In light of this and of Theorem 3.2 the following seems an interesting problem.

3.3 Question
Given a lattice Λ, for which constraints $d_0, d_1, \ldots, d_k$ can we guarantee

$$sp_T(\Lambda;\, d_0, d_1, \ldots, d_k) \;=\; sp_T(\Lambda;\, d_0)\,?$$

A further idea behind Question 3.3 is that one might expect that for small values of $d_1, d_2, \ldots, d_k$, depending on d_0, the span is determined by the minimum co-channel re-use distance d_0 only. In this section we will try to give a partial answer to Question 3.3. First we take a closer look at the mathematics of strict tilings.

Given a lattice $\Lambda = L(\underline{\mathbf{m}}, \underline{\mathbf{n}})$ and a co-channel lattice $\Lambda^* = L(\underline{\mathbf{s}}, \underline{\mathbf{t}})$ generated by linearly independent vectors $\underline{\mathbf{s}}, \underline{\mathbf{t}} \in \Lambda$, we can consider Λ as an infinite abelian group (with standard vector addition as the group operation) and Λ^* as a subgroup of Λ. Then Λ^* is in fact a normal subgroup of Λ and, since $\underline{\mathbf{s}}, \underline{\mathbf{t}} \in \Lambda$ are linearly independent, the quotient group Λ/Λ^* is a finite abelian group. If $\underline{\mathbf{s}} = s_1\,\underline{\mathbf{m}} + s_2\,\underline{\mathbf{n}}$ and $\underline{\mathbf{t}} = t_1\,\underline{\mathbf{m}} + t_2\,\underline{\mathbf{n}}$, then it is an easy exercise to show that the order of the quotient group is $|\Lambda/\Lambda^*| = |s_1\,t_2 - s_2\,t_1|$.

Combining everything, we can view a strict $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$-labelling as a labelling of the group Λ/Λ^* in which every element (every coset of Λ^*) receives a different label. In particular this means that every strict $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$-labelling of Λ has at least $|\Lambda/\Lambda^*|$ channels. The following theorem gives a sufficient condition for the existence of a strict tiling labelling with exactly $|\Lambda/\Lambda^*|$ channels satisfying the constraints.

If X is a subset of $\mathbb{R}^2$ and $\lambda \in \mathbb{R}$, then $\lambda\,X$ denotes the set $\{\,\lambda\,\underline{\mathbf{x}} \mid \underline{\mathbf{x}} \in X\,\}$.

3.4 Theorem
Let $\underline{\mathbf{s}}, \underline{\mathbf{t}} \in \Lambda$ *be linearly independent vectors which generate the sub-lattice* $\Lambda^* = L(\underline{\mathbf{s}}, \underline{\mathbf{t}})$ *and such that there is a strict* $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$*-labelling of* Λ *satisfying* (d_0). *Suppose there exists a set* $B \subseteq \Lambda$ *such that the following two conditions are satisfied:*

(1) The set $\{\,\underline{\mathbf{b}} + \Lambda^* \mid \underline{\mathbf{b}} \in B\,\}$ *generates the quotient group* Λ/Λ^*.

(2) For each vector $\underline{\mathbf{a}}$ *in the convex hull of* B *(here we regard all points as vectors in* $\mathbb{R}^2$ *),*

$$d(\underline{\mathbf{a}}, \frac{1}{i}\,\Lambda^*) \;\ge\; \frac{1}{i}\,d_i \qquad \text{for all } i = 1, \ldots, k.$$

Then there is a strict $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$*-labelling of* Λ *with exactly* $|\Lambda/\Lambda^*|$ *channels satisfying* $(d_0, \ldots, d_k)$.

The proof of Theorem 3.4, as well as the proofs of the remaining results in this section, can be found in Section 6.

In [11] Theorem 3.4 is used to design an algorithm to find radio channel assignments on 2-dimensional lattices. In this section we look at some corollaries that give sufficient conditions for the existence of good labellings and such that the conditions are somewhat easier to check than the complicated condition in Theorem 3.4. The next result shows a corollary in which it is no longer needed to find a generating set for the quotient group Λ/Λ^*.

If $\underline{\mathbf{m}}, \underline{\mathbf{n}}$ is a minimal basis of the lattice Λ, then we define $\gamma_\Lambda = \|\underline{\mathbf{n}} - \underline{\mathbf{m}}\|$. Note that γ_Λ is independent of the choice of the minimal basis. For the triangular lattice Δ we get $\gamma_\Delta = 1$.

3.5 Theorem
Let $\underline{\mathbf{s}}, \underline{\mathbf{t}} \in \Lambda$ be linearly independent vectors which generate the sub-lattice $\Lambda^* = L(\underline{\mathbf{s}}, \underline{\mathbf{t}})$ and such that there is a strict $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$-labelling of Λ satisfying (d_0). Suppose there exists a vector $\underline{\mathbf{z}} \in \mathbb{R}^2$ such that

$$d(\underline{\mathbf{z}}, \frac{1}{i}\Lambda^*) \;\geq\; \frac{1}{i}d_i + \gamma_\Lambda \qquad \text{for all } i = 1, \ldots, k.$$

Then there is a strict $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$-labelling of Λ with exactly $|\Lambda/\Lambda^*|$ channels satisfying $(d_0, \ldots, d_k)$.

Theorem 3.5 gives a partial answer to Question 3.3, provided we know the co-channel lattice Λ^* which gives a labelling of Λ with $sp_T(\Lambda;\, d_0)$ channels. But since

$$sp_T(\Lambda;\, d_0) \;=\; \min\{\, |\Lambda/\Lambda^*| \;\big|\; \Lambda^* \text{ a sub-lattice of } \Lambda \text{ and } \min\{\, \|\underline{\mathbf{x}}\| \mid \underline{\mathbf{x}} \in \Lambda^* \setminus \{\underline{\mathbf{0}}\}\,\} \geq d_0 \,\}$$

and since there exist only a finite number of sub-lattices of Λ with $|\Lambda/\Lambda^*| \leq N$ for every N, we can fairly easily find the optimal strict tilings of Λ satisfying (d_0).

The last results of this section give some further corollaries of Theorem 3.4. These corollaries provide an answer to Question 3.3 independent of a co-channel lattice and only dependent on the value of the constant γ_Λ.

3.6 Theorem
(a) If $d_1 \leq \frac{1}{3}\sqrt{3}\, d_0 - \gamma_\Lambda$, then $sp_T(\Lambda;\, d_0, d_1) = sp_T(\Lambda;\, d_0)$.
(b) For all $\varepsilon > 0$ and d_0 large enough, if $d_1 > (\frac{1}{3}\sqrt{3} + \varepsilon)\, d_0$, then $sp_T(\Lambda;\, d_0, d_1) > sp_T(\Lambda;\, d_0)$.

Theorem 3.6 can be compared to similar bounds for the triangular lattices as they appear in [6, Section II.B] and [13, Section III].

3.7 Theorem
Let Λ_0 be the lattice generated by the vectors $\binom{d_0}{0}$ and $\binom{\frac{1}{2}d_0}{\frac{1}{2}d_0\sqrt{3}}$. If there is a vector $\underline{\mathbf{z}} \in \mathbb{R}^2$ such that

$$d(\underline{\mathbf{z}}, \frac{1}{i}\Lambda_0) \;\geq\; \tfrac{1}{2}\sqrt{6}\,\big(\frac{1}{i}d_i + \gamma_\Lambda\big) \qquad \text{for all } i = 1, \ldots, k,$$

then $sp_T(\Lambda;\, d_0, d_1, \ldots, d_k) = sp_T(\Lambda;\, d_0)$.

3.8 Corollary
If $d_1 \leq \frac{1}{3}\sqrt{2}\, d_0 - \gamma_\Lambda$ and $d_2 \leq \frac{1}{3}\sqrt{2}\, d_0 - 2\,\gamma_\Lambda$, then $sp_T(\Lambda;\, d_0, d_1, d_2) = sp_T(\Lambda;\, d_0)$.

4 Labellings and strict tilings for the triangular lattice

As mentioned before, the triangular lattice has been the object of significant research in the area of radio channel assignment. In [15] a stronger version of Theorem 3.2 is proved in the case that the lattice Λ is the triangular lattice. (See also [2, Theorem 3].)

For a non-negative real number d let d^+ denote the minimum value of $\sqrt{p^2 + p\,q + q^2}$ such that p, q are integers and $p^2 + p\,q + q^2 \geq d^2$. Note that $d \leq d^+ \leq \lceil d \rceil$ and that in fact d^+ is the minimum euclidean distance between two points in Δ subject to that distance being at least d.

4.1 Theorem [2, 15]
For any $d_0 > 0$, $sp(\Delta;\, d_0) = sp_T(\Delta;\, d_0) = (d_0^+)^2$.

Using Theorem 3.5 and the result above we can prove stronger versions of some of the results in Section 3 for the triangular lattice.

The proofs of the results in this section will be given in Section 7.

4.2 Theorem
(a) If $d_1 \leq \frac{1}{3}\sqrt{3}\, d_0 - 1$, then $sp_T(\Delta;\, d_0, d_1) = sp_T(\Delta;\, d_0)$.
(b) If $d_0 > \sqrt{3}$ and $d_1 \geq \frac{1}{3}\sqrt{3}\, d_0^+$, then $sp_T(\Delta;\, d_0, d_1) > sp_T(\Delta;\, d_0)$.

4.3 Theorem
Let Λ_0 be the lattice generated by the vectors $\binom{d_0}{0}$ and $\binom{\frac{1}{2}d_0}{\frac{1}{2}d_0\sqrt{3}}$. If there exists a vector $\underline{\mathbf{z}} \in \mathbb{R}^2$ such that

$$d(\underline{\mathbf{z}}, \frac{1}{i}\Lambda_0) \;\geq\; \frac{1}{i} d_i + 1 \qquad \text{for all } i = 1, \ldots, k,$$

then $sp_T(\Delta;\, d_0, d_1, \ldots, d_k) = sp_T(\Delta;\, d_0)$.

4.4 Corollary
If $d_1 \leq \frac{1}{3}\sqrt{3}\, d_0 - 1$ and $d_2 \leq \frac{1}{3}\sqrt{3}\, d_0 - 2$, then $sp_T(\Lambda;\, d_0, d_1, d_2) = sp_T(\Lambda;\, d_0)$.

A question that arises when studying sufficient conditions such as those in the theorems above is the sharpness of the conditions. It seems very unlikely that the conditions are sufficient and necessary in general. But for the special case of $k = 1$, we see in Theorems 3.6 and 4.2 that our results are close to best possible.

In general it is very hard to get an idea about the sharpness, mainly because we only know $sp_T(\Lambda;\, d_0, \ldots, d_k)$ for a very limited number of choices of $d_0, \ldots, d_k$. Nevertheless, in the remainder of this section we show that for some special cases the result in Theorem 4.3 is surprisingly close to a best possible result.

The following proposition can be proved using the ideas in the proofs of [7, Lemma 2.2], [8, Lemma 2.1] or [12, Proposition 2.5]. We use the notation $m * d$ for a sequence of m times d.

4.5 Proposition
Let Λ be a lattice. Then for all d_0 and k, $sp_T(\Lambda;\, (k+1) * d_0) = (k+1)\, sp_T(\Lambda;\, d_0) - k$.

From this proposition and Theorem 4.1 we easily deduce that for all k and $d_0 \geq d$ it holds that $sp_T(\Delta;\, d_0, k * d) \geq sp_T(\Delta;\, (k+1) * d) = (k+1)\,(d^+)^2 - k$. In particular we get the following.

4.6 Proposition
For all $\varepsilon > 0$, k and d_0, where d_0 is large enough and such that $d_0 \geq d > \big(\frac{1}{\sqrt{k+1}} + \varepsilon\big)\, d_0$, we have $sp_T(\Delta;\, d_0, k * d) > sp_T(\Delta;\, d_0)$.

The following result, which is a corollary of Theorem 4.3, shows that we can guarantee $sp_T(\Delta;\, d_0, k * d) = sp_T(\Delta;\, d_0)$ for d a little bit smaller than $\frac{d_0}{\sqrt{k+1}}$, provided d_0 and k are large enough.

4.7 Theorem
For all A with $0 \leq A < \sqrt{\frac{1}{2}\sqrt{3}}$ ($\approx$ 0.931), there exist k and d_0 large enough such that if $d \leq A\,\frac{d_0}{\sqrt{k+1}}$, then $sp_T(\Delta;\, d_0, k * d) = sp_T(\Delta;\, d_0)$.

5 Proof of Lemma 2.7

The following observation will be used several times in the proofs of this paper.

5.1 Lemma
Let $\underline{\mathbf{m}}, \underline{\mathbf{n}}$ be a minimal basis of a lattice Λ. Then

$$\|\underline{\mathbf{n}}\|^2 \;\geq\; \|\underline{\mathbf{m}}\|^2 \;\geq\; 2\,\underline{\mathbf{m}} \cdot \underline{\mathbf{n}}.$$

Proof Since $\underline{\mathbf{n}} - \underline{\mathbf{m}} \in \Lambda \setminus \{\, p\,\underline{\mathbf{m}} \mid p \in \mathbb{Z}\,\}$ we get by the choice of $\underline{\mathbf{n}}$ that $\|\underline{\mathbf{n}} - \underline{\mathbf{m}}\|^2 \geq \|\underline{\mathbf{n}}\|^2$, hence $\|\underline{\mathbf{m}}\|^2 + \|\underline{\mathbf{n}}\|^2 - 2\,\underline{\mathbf{m}} \cdot \underline{\mathbf{n}} \geq \|\underline{\mathbf{n}}\|^2$. This gives $\|\underline{\mathbf{m}}\|^2 \geq 2\,\underline{\mathbf{m}} \cdot \underline{\mathbf{n}}$. The fact that $\|\underline{\mathbf{n}}\|^2 \geq \|\underline{\mathbf{m}}\|^2$ follows from the definition of a minimal basis. ■

Proof of Lemma 2.7 Since $d(\underline{\mathbf{x}}, \Lambda) = d(\underline{\mathbf{x}} - \underline{\mathbf{x}}_\Lambda, \Lambda)$, we can assume that $\underline{\mathbf{x}}_\Lambda = \underline{\mathbf{0}}$ and hence $\underline{\mathbf{x}} = y_1\,\underline{\mathbf{m}} + y_2\,\underline{\mathbf{n}}$ for some $0 \leq y_1, y_2 < 1$. We first prove that

$$d(\underline{\mathbf{x}}, \Lambda) \;=\; \min\{\, d(\underline{\mathbf{x}}, \underline{\mathbf{0}}), d(\underline{\mathbf{x}}, \underline{\mathbf{m}}), d(\underline{\mathbf{x}}, \underline{\mathbf{n}}), d(\underline{\mathbf{x}}, \underline{\mathbf{n}} - \underline{\mathbf{m}}), d(\underline{\mathbf{x}}, \underline{\mathbf{m}} - \underline{\mathbf{n}}), d(\underline{\mathbf{x}}, \underline{\mathbf{m}} + \underline{\mathbf{n}})\,\}. \tag{5.1}$$

Let $p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}} \in \Lambda$. First consider the case $p, q \leq 0$. Then

$$\begin{aligned} d(\underline{\mathbf{x}}, p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}})^2 \;&=\; \|\underline{\mathbf{x}} - (p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}})\|^2 \\ &=\; \|\underline{\mathbf{x}}\|^2 + \|p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}}\|^2 - 2\,\underline{\mathbf{x}} \cdot (p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}}) \\ &=\; d(\underline{\mathbf{x}}, \underline{\mathbf{0}})^2 + \|p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}}\|^2 - 2\,(y_1\,p\,\|\underline{\mathbf{m}}\|^2 + y_2\,q\,\|\underline{\mathbf{n}}\|^2 + (y_1\,q + y_2\,p)\,\underline{\mathbf{m}} \cdot \underline{\mathbf{n}}). \end{aligned}$$

Now using that $p, q \leq 0$, $y_1, y_2 \geq 0$ and $\underline{\mathbf{m}} \cdot \underline{\mathbf{n}} \geq 0$, we get that $d(\underline{\mathbf{x}}, p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}})^2 \geq d(\underline{\mathbf{x}}, \underline{\mathbf{0}})^2$.

Next consider the case $p+q \le 0$, $q \ge 1$. Set $\underline{\mathbf{a}} = \underline{\mathbf{n}} - \underline{\mathbf{m}}$. Then $\underline{\mathbf{x}} = y_1\,\underline{\mathbf{m}} + y_2\,\underline{\mathbf{n}} = (y_1+y_2)\,\underline{\mathbf{m}} + y_2\,\underline{\mathbf{a}}$ and $p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}} = (p+q)\,\underline{\mathbf{m}} + q\,\underline{\mathbf{a}}$. This means that

$$\begin{aligned} d(\underline{\mathbf{x}}, p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}})^2 &= \|\underline{\mathbf{x}} - \underline{\mathbf{a}} + \underline{\mathbf{a}} - ((p+q)\,\underline{\mathbf{m}} + q\,\underline{\mathbf{a}})\|^2 \\ &= \|\underline{\mathbf{x}} - \underline{\mathbf{a}}\|^2 + \|\underline{\mathbf{a}} - ((p+q)\,\underline{\mathbf{m}} + q\,\underline{\mathbf{a}})\|^2 \\ &\quad + 2\,(\underline{\mathbf{x}} - \underline{\mathbf{a}}) \cdot (\underline{\mathbf{a}} - ((p+q)\,\underline{\mathbf{m}} + q\,\underline{\mathbf{a}})). \end{aligned} \tag{5.2}$$

Now use Lemma 5.1 to obtain $\underline{\mathbf{m}} \cdot \underline{\mathbf{a}} = \underline{\mathbf{m}} \cdot (\underline{\mathbf{n}} - \underline{\mathbf{m}}) = \underline{\mathbf{m}} \cdot \underline{\mathbf{n}} - \underline{\mathbf{m}} \cdot \underline{\mathbf{m}} \le -\underline{\mathbf{m}} \cdot \underline{\mathbf{n}} \le 0$. Using that $\underline{\mathbf{x}} - \underline{\mathbf{a}} = (y_1+y_2)\,\underline{\mathbf{m}} + (y_2-1)\,\underline{\mathbf{a}}$ and together with $y_1+y_2 \ge 0$, $p+q \le 0$, $y_2 - 1 \le 0$ and $q-1 \ge 0$, we obtain

$$\begin{aligned} (\underline{\mathbf{x}} - \underline{\mathbf{a}}) \cdot (\underline{\mathbf{a}} - ((p+q)\,\underline{\mathbf{m}} + q\,\underline{\mathbf{a}})) &= ((y_1+y_2)\,\underline{\mathbf{m}} + (y_2-1)\,\underline{\mathbf{n}}) \cdot (-(p+q)\,\underline{\mathbf{m}} - (q-1)\,\underline{\mathbf{n}}) \\ &= -(y_1+y_2)\,(p+q)\,\|\underline{\mathbf{m}}\|^2 - (y_2-1)\,(q-1)\,\|\underline{\mathbf{a}}\|^2 \\ &\quad - ((y_1+y_2)\,(q-1) + (y_2-1)\,(p+q))\,\underline{\mathbf{m}} \cdot \underline{\mathbf{a}} \\ &\ge 0. \end{aligned}$$

This means that from (5.2) we obtain $d(\underline{\mathbf{x}}, p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}})^2 \ge \|\underline{\mathbf{x}} - \underline{\mathbf{a}}\|^2$, hence $d(\underline{\mathbf{x}}, p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}}) \ge d(\underline{\mathbf{x}}, \underline{\mathbf{a}}) = d(\underline{\mathbf{x}}, \underline{\mathbf{n}} - \underline{\mathbf{m}})$.

Continuing in a similar way, we get the results

$$\begin{aligned} d(\underline{\mathbf{x}}, p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}}) &\ge d(\underline{\mathbf{x}}, \underline{\mathbf{0}}), && \text{if } p \le 0,\ q \le 0; \\ d(\underline{\mathbf{x}}, p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}}) &\ge d(\underline{\mathbf{x}}, \underline{\mathbf{n}} - \underline{\mathbf{m}}), && \text{if } p+q \le 0,\ q \ge 1; \\ d(\underline{\mathbf{x}}, p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}}) &\ge d(\underline{\mathbf{x}}, \underline{\mathbf{n}}), && \text{if } p \le 0,\ p+q \ge 1; \\ d(\underline{\mathbf{x}}, p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}}) &\ge d(\underline{\mathbf{x}}, \underline{\mathbf{m}} + \underline{\mathbf{n}}), && \text{if } p \ge 1,\ q \ge 1; \\ d(\underline{\mathbf{x}}, p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}}) &\ge d(\underline{\mathbf{x}}, \underline{\mathbf{m}}), && \text{if } p+q \ge 1,\ q \le 0; \\ d(\underline{\mathbf{x}}, p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}}) &\ge d(\underline{\mathbf{x}}, \underline{\mathbf{m}} - \underline{\mathbf{n}}), && \text{if } p \ge 1,\ p+q \le 0, \end{aligned}$$

which proves (5.1).

Some straightforward manipulations, using Lemma 5.1, give

$$\begin{aligned} \|\underline{\mathbf{x}} - (\underline{\mathbf{n}} - \underline{\mathbf{m}})\|^2 - \|\underline{\mathbf{x}} - \underline{\mathbf{n}}\|^2 &= (2\,y_1 + 1)\,\|\underline{\mathbf{m}}\|^2 + (2\,y_2 - 2)\,\underline{\mathbf{m}} \cdot \underline{\mathbf{n}} \\ &\ge (4\,y_1 + 2\,y_2)\,\underline{\mathbf{m}} \cdot \underline{\mathbf{n}} \ \ge\ 0, \end{aligned}$$

which proves $d(\underline{\mathbf{x}}, \underline{\mathbf{n}} - \underline{\mathbf{m}}) \ge d(\underline{\mathbf{x}}, \underline{\mathbf{n}})$. In a similar way we can prove $d(\underline{\mathbf{x}}, \underline{\mathbf{m}} - \underline{\mathbf{n}}) \ge d(\underline{\mathbf{x}}, \underline{\mathbf{m}})$. The lemma follows from (5.1). ■

6 Proofs of the results in Section 3

6.1 Definition

Let G be a finite group and $S \subseteq G$, $e \notin S$. The *Cayley digraph* $D = D(G;S)$ is defined by

$$V(D) = G \qquad \text{and} \qquad A(D) = \{\,(g, gs) \mid g \in G,\ s \in S\,\}.$$

A *hamiltonian path* in a digraph D is a directed path containing all vertices of D.

Instead of giving the vertices of a hamiltonian path in a digraph, we can also describe the first vertex and the arcs of the path. For a Cayley digraph this means that it is enough to prescribe the first vertex and the sequence of elements from S that form the arcs. Moreover, since the same collection of arcs give a hamiltonian path from any starting vertex, we only need to give the arcs of the path.

The question which Cayley graphs and digraphs contain a hamiltonian path (or cycle) has a long history. See, e.g., [3, 16] for a survey. For our purposes we only need the following special, easy result.

6.2 Lemma
Let G be a finite abelian group and $S \subseteq G$, $0 \notin S$. Then $D(G;S)$ contains a hamiltonian path if and only if S generates G.

Proof Since $D = D(G;S)$ is connected if and only if S generates the group G, it is obvious that the condition is necessary.

Now let S be a generating set of G. We use induction on $|S|$ to show that D contains a hamiltonian path. If $|S| = 1$, say $S = \{s\}$, then $(|G| - 1) * s$ is the collection of arcs of a hamiltonian path in D. Now assume $S = \{s_1, s_2, \ldots, s_k\}$, $k > 1$. Let G' be the abelian group generated by $S' = \{s_1, s_2, \ldots, s_{k-1}\}$. By induction we know that $D(G';S')$ contains a hamiltonian path. Say $H = s^{(1)} s^{(2)} \cdots$ is the sequence of arcs on such a path. Since G is abelian, G' is a normal subgroup of G. Set $m = |G/G'|$, the order of the quotient group. Now it is straightforward to check that $H, s_k, H, s_k, \ldots, s_k, H$, where we take m copies of H, is a hamiltonian path in D. ■

Now we can give the proof of Theorem 3.4.

Proof of Theorem 3.4 From the definition of a strict tiling it follows that if there exists one strict $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$-labelling of Λ satisfying (d_0), then in fact every strict $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$-labelling of Λ satisfies (d_0). Let $B \subseteq \Lambda$ satisfy the conditions in the theorem. Set $n = |\Lambda/\Lambda^*|$ and $S = \{\, \underline{\mathbf{b}} + \Lambda^* \mid \underline{\mathbf{b}} \in B \,\}$. Then by Lemma 6.2 and condition (1) in the theorem, there exists a hamiltonian path in the Cayley digraph $D(\Lambda/\Lambda^*;S)$. Suppose the arcs of this hamiltonian path are $\underline{\mathbf{b}}_1 + \Lambda^*, \underline{\mathbf{b}}_2 + \Lambda^*, \ldots, \underline{\mathbf{b}}_{n-1} + \Lambda^*$, for certain $\underline{\mathbf{b}}_i \in B$. Next define the vectors $\underline{\mathbf{c}}_0, \underline{\mathbf{c}}_1, \ldots, \underline{\mathbf{c}}_{n-1}$ by

$$\underline{\mathbf{c}}_0 = \underline{\mathbf{0}} \qquad \text{and} \qquad \underline{\mathbf{c}}_j = \sum_{i=1}^{j} \underline{\mathbf{b}}_i \quad \text{for } j = 1, \ldots, n-1.$$

Then the cosets $\underline{\mathbf{c}}_0 + \Lambda^*, \underline{\mathbf{c}}_1 + \Lambda^*, \ldots, \underline{\mathbf{c}}_{n-1} + \Lambda^*$ form a partition of Λ (the cosets are the vertices of the hamiltonian path in $D(\Lambda/\Lambda^*;S)$). So if we define $\varphi : \Lambda \to \{0, \ldots, n-1\}$ by

$$\varphi(\underline{\mathbf{x}}) = j \qquad \text{if } \underline{\mathbf{x}} \in \underline{\mathbf{c}}_j + \Lambda^*, \tag{6.1}$$

then φ is a strict $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$-labelling of Λ with $n = |\Lambda/\Lambda^*|$ channels. It remains to show that this labelling satisfies the constraints $(d_0, \ldots, d_k)$.

For this, let $\mathrm{conv}(B)$ denote the convex hull of B. By condition (2) in the theorem we have

$$d(\underline{\mathbf{a}}, \frac{1}{i}\,\Lambda^*) \;\geq\; \frac{1}{i}\,d_i \qquad \text{for all } \underline{\mathbf{a}} \in \mathrm{conv}(B) \text{ and } i = 1, \ldots, k.$$

This is equivalent to

$$d(\underline{\mathbf{a}}', \Lambda^*) \;\geq\; d_i \qquad \text{for all } \underline{\mathbf{a}}' \in i \cdot \mathrm{conv}(B) \text{ and } i = 1, \ldots, k. \tag{6.2}$$

Now look again at the sequence $\underline{\mathbf{b}}_1, \ldots, \underline{\mathbf{b}}_{n-1} \in B \subseteq \mathrm{conv}(B)$. If we take a subsequence $\underline{\mathbf{b}}_{p+1}, \underline{\mathbf{b}}_{p+2}, \ldots, \underline{\mathbf{b}}_{p+q}$ of q consecutive elements of this sequence ($0 \leq p$, $1 \leq q \leq k$, $p + q \leq n - 1$), then

$$\underline{\mathbf{b}}_{p+1} + \cdots + \underline{\mathbf{b}}_{p+q} \;=\; q \cdot \big(\frac{1}{q}\,\underline{\mathbf{b}}_{p+1} + \cdots + \frac{1}{q}\,\underline{\mathbf{b}}_{p+q}\big) \;\in\; q \cdot \mathrm{conv}(B).$$

The last step follows since $\mathrm{conv}(B)$ is a convex set, hence $\sum_{i=1}^{r} \lambda_i\,\underline{\mathbf{a}}_i \in \mathrm{conv}(B)$ for any choice of r, $\underline{\mathbf{a}}_i \in \mathrm{conv}(B)$ and $\lambda_i \in \mathbb{R}$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{r} \lambda_i = 1$. In particular we find

$$\underline{\mathbf{c}}_{p+q} - \underline{\mathbf{c}}_p \;=\; \underline{\mathbf{b}}_{p+1} + \cdots + \underline{\mathbf{b}}_{p+q} \;\in\; q \cdot \mathrm{conv}(B).$$

Combining this with (6.2) we get that $d(\underline{\mathbf{c}}_{p+q} - \underline{\mathbf{c}}_p, \Lambda^*) \geq d_q$, which is equivalent to

$$d(\underline{\mathbf{v}}, \underline{\mathbf{w}}) \;\geq\; d_q \qquad \text{for all } \underline{\mathbf{v}} \in \underline{\mathbf{c}}_{p+q} + \Lambda^* \text{ and } \underline{\mathbf{w}} \in \underline{\mathbf{c}}_p + \Lambda^*.$$

From this last inequality, which holds for all p, q with $0 \leq p$, $1 \leq q \leq k$, $p+q \leq n-1$, and from the remark in the first sentence of this proof, it follows that the labelling φ defined in (6.1) satisfies the constraints $(d_0, d_1, \ldots, d_k)$. This completes the proof of the theorem. ■

In the proof of Theorem 3.5 the following is essential.

6.3 Lemma
Let $\Lambda = L(\underline{\mathbf{m}}, \underline{\mathbf{n}})$ be a lattice, where $\underline{\mathbf{m}}, \underline{\mathbf{n}}$ form a minimal basis of Λ, and $\underline{\mathbf{z}} \in \mathbb{R}^2$. Let $D = \{\,\underline{\mathbf{x}} \in \mathbb{R}^2 \mid d(\underline{\mathbf{x}}, \underline{\mathbf{z}}) \leq \gamma_\Lambda\,\}$ be the disk with centre $\underline{\mathbf{z}}$ and radius γ_Λ. Then there exist $\underline{\mathbf{b}}_1, \underline{\mathbf{b}}_2, \underline{\mathbf{b}}_3 \in \Lambda \cap D$ with $\underline{\mathbf{b}}_2 - \underline{\mathbf{b}}_1 = \underline{\mathbf{m}}$ and $\underline{\mathbf{b}}_3 - \underline{\mathbf{b}}_1 = \underline{\mathbf{n}}$, or $\underline{\mathbf{b}}_2 - \underline{\mathbf{b}}_1 = -\underline{\mathbf{m}}$ and $\underline{\mathbf{b}}_3 - \underline{\mathbf{b}}_1 = -\underline{\mathbf{n}}$.

Proof Since $\underline{\mathbf{m}}, \underline{\mathbf{n}}$ form a basis of $\mathbb{R}^2$ we can write $\underline{\mathbf{z}} = z_1\,\underline{\mathbf{m}} + z_2\,\underline{\mathbf{n}}$. Following Lemma 2.7, set $\underline{\mathbf{z}}_\Lambda = \lfloor z_1 \rfloor\,\underline{\mathbf{m}} + \lfloor z_2 \rfloor\,\underline{\mathbf{n}} \in \Lambda$, and set $z_1' = z_1 - \lfloor z_1 \rfloor$, $z_2' = z_2 - \lfloor z_2 \rfloor$, $\underline{\mathbf{z}}' = z_1'\,\underline{\mathbf{m}} + z_2'\,\underline{\mathbf{n}}$. Then $0 \leq z_1', z_2' < 1$. First suppose $0 \leq z_1' + z_2' < 1$. Then $\underline{\mathbf{z}}'$ lies in the triangle with vertices $\underline{\mathbf{0}}$, $\underline{\mathbf{m}}$ and $\underline{\mathbf{n}}$. Since the lengths of the edges of this triangle are $\|\underline{\mathbf{m}}\|$, $\|\underline{\mathbf{n}}\|$ and $\|\underline{\mathbf{n}} - \underline{\mathbf{m}}\|$, of which $\|\underline{\mathbf{n}} - \underline{\mathbf{m}}\|$ is the longest, it follows that all vertices of the triangle lie within the disk D' with centre $\underline{\mathbf{z}}'$ and radius $\|\underline{\mathbf{n}} - \underline{\mathbf{m}}\| = \gamma_\Lambda$. Hence the points $\underline{\mathbf{b}}_1 = \underline{\mathbf{z}}_\Lambda$, $\underline{\mathbf{b}}_2 = \underline{\mathbf{z}}_\Lambda + \underline{\mathbf{m}}$ and $\underline{\mathbf{b}}_3 = \underline{\mathbf{z}}_\Lambda + \underline{\mathbf{n}}$ satisfy the requirements in the lemma.

If $1 \leq z_1' + z_2' < 2$, then we can prove similarly that the points $\underline{\mathbf{b}}_1 = \underline{\mathbf{z}}_\Lambda + \underline{\mathbf{m}} + \underline{\mathbf{n}}$, $\underline{\mathbf{b}}_2 = \underline{\mathbf{z}}_\Lambda + \underline{\mathbf{n}}$ and $\underline{\mathbf{b}}_3 = \underline{\mathbf{z}}_\Lambda + \underline{\mathbf{m}}$ satisfy the requirements in the lemma. ■

Proof of Theorem 3.5 Let the vector $\underline{\mathbf{z}} \in \mathbb{R}^2$ satisfy the condition in the theorem. We will show that this means that there exists a set $B = \{\underline{\mathbf{b}}_1, \underline{\mathbf{b}}_2, \underline{\mathbf{b}}_3\} \subseteq \Lambda$ satisfying conditions (1) and (2) of Theorem 3.4. Using Theorem 3.4 this will yield a proof of the theorem.

Define $D = \{\, \underline{\mathbf{x}} \in \mathbb{R}^2 \mid d(\underline{\mathbf{x}}, \underline{\mathbf{z}}) \leq \gamma_\Lambda \,\}$. Then by the previous lemma there exist $\underline{\mathbf{b}}_1, \underline{\mathbf{b}}_2, \underline{\mathbf{b}}_3 \in \Lambda \cap D$ such that $\underline{\mathbf{b}}_2 - \underline{\mathbf{b}}_1 = \underline{\mathbf{m}}$ and $\underline{\mathbf{b}}_3 - \underline{\mathbf{b}}_1 = \underline{\mathbf{n}}$, or $\underline{\mathbf{b}}_2 - \underline{\mathbf{b}}_1 = -\underline{\mathbf{m}}$ and $\underline{\mathbf{b}}_3 - \underline{\mathbf{b}}_1 = -\underline{\mathbf{n}}$. Set $B = \{\underline{\mathbf{b}}_1, \underline{\mathbf{b}}_2, \underline{\mathbf{b}}_3\}$. It follows that both $\underline{\mathbf{m}} + \Lambda^*$ and $\underline{\mathbf{n}} + \Lambda^*$ are in the group generated by $\{\underline{\mathbf{b}}_1 + \Lambda^*, \underline{\mathbf{b}}_2 + \Lambda^*, \underline{\mathbf{b}}_3 + \Lambda^*\}$. Since $\{\underline{\mathbf{m}}, \underline{\mathbf{n}}\}$ generates the group Λ, $\{\underline{\mathbf{m}} + \Lambda^*, \underline{\mathbf{n}} + \Lambda^*\}$ generates the quotient group Λ/Λ^*, hence $\{\underline{\mathbf{b}}_1 + \Lambda^*, \underline{\mathbf{b}}_2 + \Lambda^*, \underline{\mathbf{b}}_3 + \Lambda^*\}$ generates the group Λ/Λ^*. This proves that B satisfies condition (1) in Theorem 3.4.

If $\underline{\mathbf{a}} \in D$, then $d(\underline{\mathbf{z}}, \underline{\mathbf{a}}) \leq \gamma_\Lambda$ and hence

$$d(\underline{\mathbf{a}}, \tfrac{1}{i}\Lambda^*) \;\geq\; d(\underline{\mathbf{z}}, \tfrac{1}{i}\Lambda^*) - \gamma_\Lambda \;\geq\; \tfrac{1}{i}\, d_i \qquad \text{for all } i = 1, \ldots, k.$$

Note that the tree vectors $\underline{\mathbf{b}}_1, \underline{\mathbf{b}}_2, \underline{\mathbf{b}}_3$ lie within the disk D. Hence also the convex hull $\mathrm{conv}(B)$ lies within this disk. So certainly for every $\underline{\mathbf{a}} \in \mathrm{conv}(B)$ the inequality above holds, which means that B satisfies condition (2) in Theorem 3.4. ■

Proof of Theorem 3.6 Choose $\underline{\mathbf{s}}, \underline{\mathbf{t}} \in \Lambda$ such that there exists a strict $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$-labelling of Λ with $|\Lambda/\Lambda^*| = sp_T(\Lambda;\, d_0)$, where $\Lambda^* = L(\underline{\mathbf{s}}, \underline{\mathbf{t}})$. This means $\|\underline{\mathbf{s}}\|, \|\underline{\mathbf{t}}\| \geq d_0$. We assume that $\underline{\mathbf{s}}, \underline{\mathbf{t}}$ form a minimal basis of Λ^*. Take $\underline{\mathbf{w}} = \underline{\mathbf{t}} - \dfrac{\underline{\mathbf{s}} \cdot \underline{\mathbf{t}}}{\|\underline{\mathbf{s}}\|^2}\,\underline{\mathbf{s}}$, i.e., $\underline{\mathbf{w}}$ is the projection of $\underline{\mathbf{t}}$ on the line perpendicular to $\underline{\mathbf{s}}$, and set $\underline{\mathbf{z}} = \frac{1}{2}\,\underline{\mathbf{s}} + \frac{1}{3}\,\underline{\mathbf{w}}$ (see Figure 6.1 for a sketch of the situation). Then

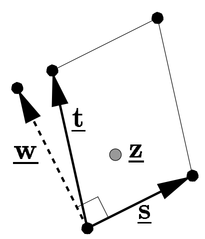

Figure 6.1. The essential points from the proof of Theorem 3.6 (a).

it is easy to show that

$$d(\underline{\mathbf{z}}, \underline{\mathbf{0}})^2 \;=\; d(\underline{\mathbf{z}}, \underline{\mathbf{s}})^2 \;=\; \tfrac{1}{4}\,\|\underline{\mathbf{s}}\|^2 + \tfrac{1}{9}\,\|\underline{\mathbf{t}}\|^2 - \tfrac{1}{9}\,\frac{(\underline{\mathbf{s}} \cdot \underline{\mathbf{t}})^2}{\|\underline{\mathbf{s}}\|^2}.$$

Since $\|\underline{\mathbf{t}}\| \geq \|\underline{\mathbf{s}}\|$ and $\underline{\mathbf{s}} \cdot \underline{\mathbf{t}} \leq \frac{1}{2}\,\|\underline{\mathbf{s}}\|^2$, by Lemma 5.1, this gives $d(\underline{\mathbf{z}}, \underline{\mathbf{0}})^2 = d(\underline{\mathbf{z}}, \underline{\mathbf{s}})^2 \geq \frac{1}{3}\,\|\underline{\mathbf{s}}\|^2$, hence

$$d(\underline{\mathbf{z}}, \underline{\mathbf{0}}) \;=\; d(\underline{\mathbf{z}}, \underline{\mathbf{s}}) \;\geq\; \tfrac{1}{3}\sqrt{3}\,\|\underline{\mathbf{s}}\| \;\geq\; \tfrac{1}{3}\sqrt{3}\,d_0 \;\geq\; d_1 + \gamma_\Lambda. \tag{6.3}$$

Next notice that

$$(\tfrac{2}{3}\,\|\underline{\mathbf{w}}\|)^2 \;=\; \tfrac{4}{9}\,\|\underline{\mathbf{t}}\|^2 - \tfrac{4}{9}\,\frac{(\underline{\mathbf{s}} \cdot \underline{\mathbf{t}})^2}{\|\underline{\mathbf{s}}\|^2} \;\geq\; \tfrac{1}{3}\,\|\underline{\mathbf{s}}\|^2.$$

Since $\underline{\mathbf{w}}$ is the projection of $\underline{\mathbf{t}}$ on the line perpendicular to $\underline{\mathbf{s}}$ we get

$$d(\underline{\mathbf{z}},\underline{\mathbf{t}}) \;=\; d(\tfrac{1}{2}\,\underline{\mathbf{s}}+\tfrac{1}{3}\,\underline{\mathbf{w}},\underline{\mathbf{t}}) \;\ge\; d(\tfrac{1}{3}\,\underline{\mathbf{w}},\underline{\mathbf{w}}) \;=\; \tfrac{2}{3}\,\|\underline{\mathbf{w}}\| \;\ge\; \tfrac{1}{3}\sqrt{3}\,\|\underline{\mathbf{s}}\| \;\ge\; \tfrac{1}{3}\sqrt{3}\,d_0 \;\ge\; d_1+\gamma_\Lambda,$$

and, similarly, $d(\underline{\mathbf{z}},\underline{\mathbf{s}}+\underline{\mathbf{t}}) \ge d_1+\gamma_\Lambda$. Together with (6.3) and Lemma 2.7 we find that $d(\underline{\mathbf{z}},\Lambda^*) \ge d_1+\gamma_\Lambda$. By Theorem 3.5 this completes the proof of (a).

In order to prove (b) we take a closer look at the possibilities we have for $\underline{\mathbf{s}},\underline{\mathbf{t}} \in \Lambda$ such that there exists a strict $(\underline{\mathbf{s}},\underline{\mathbf{t}})$-labelling of Λ with $|\Lambda/\Lambda^*| = sp_T(\Lambda;\, d_0)$, where $\Lambda^* = L(\underline{\mathbf{s}},\underline{\mathbf{t}})$. We can always assume that $\underline{\mathbf{s}},\underline{\mathbf{t}}$ form a minimal basis of Λ^*. Define $\underline{\mathbf{w}}$ as above. Since $\underline{\mathbf{s}}\cdot\underline{\mathbf{t}} \le \frac{1}{2}\,\|\underline{\mathbf{s}}\|^2$ and $\|\underline{\mathbf{t}}\| \ge \|\underline{\mathbf{s}}\|$ this gives

$$\|\underline{\mathbf{w}}\|^2 \;=\; \|\underline{\mathbf{t}}\|^2 - \frac{(\underline{\mathbf{s}}\cdot\underline{\mathbf{t}})^2}{\|\underline{\mathbf{s}}\|^2} \;\ge\; \|\underline{\mathbf{t}}\|^2 - \tfrac{1}{4}\,\|\underline{\mathbf{s}}\|^2 \;\ge\; \tfrac{3}{4}\,\|\underline{\mathbf{s}}\|^2. \tag{6.4}$$

Now the number of points from Λ in the parallelogram $\{\,\lambda_1\,\underline{\mathbf{s}}+\lambda_2\,\underline{\mathbf{t}} \mid 0 \le \lambda_i < 1\,\}$, which is equal to the number of channels in a strict $\underline{\mathbf{s}},\underline{\mathbf{t}}$-labelling of Λ, is equal to the area of the parallelogram divided by the cell area $\rho(\Lambda)$. Since the area of the parallelogram is equal to $\|\underline{\mathbf{s}}\|\cdot\|\underline{\mathbf{w}}\|$, we can use (6.4) and the fact that $\|\underline{\mathbf{s}}\| \ge d_0$ to estimate

$$sp_T(\Lambda;\, d_0) \;=\; |\Lambda/\Lambda^*| \;=\; \frac{\|\underline{\mathbf{s}}\|\cdot\|\underline{\mathbf{w}}\|}{\rho(\Lambda)} \;\ge\; \frac{\sqrt{\frac{3}{4}}\,\|\underline{\mathbf{s}}\|^2}{\rho(\Lambda)} \;\ge\; \tfrac{1}{2}\sqrt{3}\,\frac{d_0^2}{\rho(\Lambda)}. \tag{6.5}$$

On the other hand, we know from Theorem 3.2 that $sp_T(\Lambda;\, d_0) \le \frac{1}{2}\sqrt{3}\,\dfrac{d_0^2}{\rho(\Lambda)} + \mathcal{O}(d_0)$ ($d_0 \to \infty$). Looking back at all the inequalities we used to obtain (6.5), this means

$$\begin{aligned} \|\underline{\mathbf{s}}\| \;&=\; (1+o(1))\,d_0 \\ \|\underline{\mathbf{t}}\| \;&=\; (1+o(1))\,\|\underline{\mathbf{s}}\| \;=\; (1+o(1))\,d_0 \qquad\qquad (\,d_0\to\infty\,). \\ \underline{\mathbf{s}}\cdot\underline{\mathbf{t}} \;&=\; (\tfrac{1}{2}+o(1))\,\|\underline{\mathbf{s}}\|^2 \;=\; (\tfrac{1}{2}+o(1))\,d_0^2 \end{aligned} \tag{6.6}$$

Now suppose we have a strict $\underline{\mathbf{s}},\underline{\mathbf{t}}$-labelling of Λ satisfying (d_0,d_1). Without loss of generality we can assume that the points in Λ^* receive label 0. This means that the points receiving label 1 have distance at least d_1 from the points in Λ^*. The situation is sketched in Figure 6.2, where the "forbidden" area for a point to have label 1 is indicated by the grey disks of radius d_1. So label 1 must appear in the triangle-shaped black areas between the disks. By (6.6) the triangles in Figure 6.2 spanned by $\underline{\mathbf{0}},\underline{\mathbf{s}},\underline{\mathbf{t}}$ and by $\underline{\mathbf{s}},\underline{\mathbf{t}},\underline{\mathbf{s}}+\underline{\mathbf{t}}$ (which are congruent) have all edges equal to $(1+o(1))\,d_0$ ($d_0\to\infty$). It follows that if $d_1 > (\frac{1}{3}\sqrt{3}+\varepsilon)\,d_0$ for a fixed ε, then for d_0 large enough the grey disks cover the whole area, hence there is no point that can get label 1. So in this situation we must find $sp_T(\Lambda;\, d_0,d_1) > |\Lambda/\Lambda^*| = sp_T(\Lambda;\, d_0)$. ■

Proof of Theorem 3.7 Choose $\underline{\mathbf{s}},\underline{\mathbf{t}} \in \Lambda$ such that there exists a strict $(\underline{\mathbf{s}},\underline{\mathbf{t}})$-labelling of Λ with $|\Lambda/\Lambda^*| = sp_T(\Lambda;\, d_0)$, where $\Lambda^* = L(\underline{\mathbf{s}},\underline{\mathbf{t}})$. This means $\|\underline{\mathbf{s}}\|,\|\underline{\mathbf{t}}\| \ge d_0$. We assume that $\underline{\mathbf{s}},\underline{\mathbf{t}}$ form a minimal basis of Λ^*. By the definition of a minimal basis and Lemma 5.1 this means $0 \le \underline{\mathbf{s}}\cdot\underline{\mathbf{t}} \le \frac{1}{2}\,\|\underline{\mathbf{s}}\|^2$. Define the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T\binom{d_0}{0} = \underline{\mathbf{s}}$ and

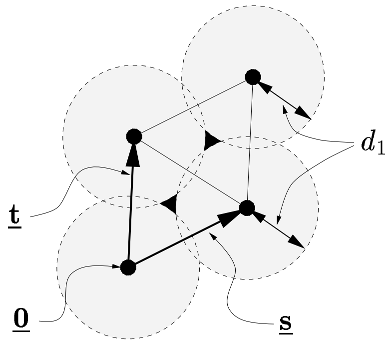

Figure 6.2. The grey area is "forbidden" for label 1 if the points of Λ^* have label 0.

$T\begin{pmatrix} \frac{1}{2}\, d_0 \\ \frac{1}{2}\, d_0\, \sqrt{3} \end{pmatrix} = \underline{\mathbf{t}}$. We want to show that for all $\underline{\mathbf{x}} \in \mathbb{R}^2$, $\underline{\mathbf{x}} \neq \underline{\mathbf{0}}$, $\dfrac{\|T\underline{\mathbf{x}}\|}{\|\underline{\mathbf{x}}\|} \geq \sqrt{\frac{2}{3}}$. For this, take $\underline{\mathbf{x}} \in \mathbb{R}^2$, $\underline{\mathbf{x}} \neq \underline{\mathbf{0}}$, and set $\underline{\mathbf{x}} = x_1 \begin{pmatrix} d_0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} \frac{1}{2}\, d_0 \\ \frac{1}{2}\, d_0\, \sqrt{3} \end{pmatrix}$. Then $\|\underline{\mathbf{x}}\|^2 = (x_1^2 + x_1\, x_2 + x_2^2)\, d_0^2$. Also, $T\underline{\mathbf{x}} = x_1\, \underline{\mathbf{s}} + x_2\, \underline{\mathbf{t}}$, hence

$$\|T\underline{\mathbf{x}}\|^2 \;=\; x_1^2\, \|\underline{\mathbf{s}}\|^2 + 2\, x_1\, x_2\, \underline{\mathbf{s}} \cdot \underline{\mathbf{t}} + x_2^2\, \|\underline{\mathbf{t}}\|^2. \tag{6.7}$$

First consider the case $x_1\, x_2 \leq 0$. Then since $\underline{\mathbf{s}} \cdot \underline{\mathbf{t}} \leq \frac{1}{2}\, \|\underline{\mathbf{s}}\|^2 \leq \frac{1}{2}\, d_0^2$, we find $2\, x_1\, x_2\, \underline{\mathbf{s}} \cdot \underline{\mathbf{t}} \geq x_1\, x_2\, d_0^2$. Using that $\|\underline{\mathbf{s}}\|, \|\underline{\mathbf{t}}\| \geq d_0$ we obtain from (6.7) that $\|T\underline{\mathbf{x}}\|^2 \geq x_1^2\, d_0^2 + x_1\, x_2\, d_0^2 + x_2^2\, d_0^2 = \|\underline{\mathbf{x}}\|^2$, hence $\dfrac{\|T\underline{\mathbf{x}}\|}{\|\underline{\mathbf{x}}\|} \geq 1 > \sqrt{\frac{2}{3}}$.

Next consider the case $x_1\, x_2 \geq 0$. This time we use $\underline{\mathbf{s}} \cdot \underline{\mathbf{t}} \geq 0$ to obtain $\|T\underline{\mathbf{x}}\|^2 \geq x_1^2\, d_0^2 + x_2^2\, d_0^2$. This means $\dfrac{\|T\underline{\mathbf{x}}\|^2}{\|\underline{\mathbf{x}}\|^2} \geq \dfrac{x_1^2 + x_2^2}{x_1^2 + x_1\, x_2 + x_2^2}$. Standard calculus shows that the function $f(x_1, x_2) = \dfrac{x_1^2 + x_2^2}{x_1^2 + x_1\, x_2 + x_2^2}$ has a minimum for $x_1 = x_2$ for which the function value is $f(x_1, x_1) = \frac{2}{3}$. This means that in this case we have $\dfrac{\|T\underline{\mathbf{x}}\|}{\|\underline{\mathbf{x}}\|} \geq \sqrt{\frac{2}{3}}$ as well.

By the condition in the theorem there exists a $\underline{\mathbf{z}} \in \mathbb{R}^2$ such that for all $i = 1, \ldots, k$ and $\underline{\mathbf{x}} \in \dfrac{1}{i}\, \Lambda_0$, $\|\underline{\mathbf{z}} - \underline{\mathbf{x}}\| = d(\underline{\mathbf{z}}, \underline{\mathbf{x}}) \geq \frac{1}{2}\, \sqrt{6}\, \big(\dfrac{1}{i}\, d_i + \gamma_\Lambda\big)$. This means that for all $\underline{\mathbf{x}} \in \dfrac{1}{i}\, \Lambda_0$ we have

$$\|T\underline{\mathbf{z}} - T\underline{\mathbf{x}}\| \;=\; \|T(\underline{\mathbf{z}} - \underline{\mathbf{x}})\| \;\geq\; \sqrt{\tfrac{2}{3}}\, \|\underline{\mathbf{z}} - \underline{\mathbf{x}}\| \;=\; \sqrt{\tfrac{2}{3}} \cdot \tfrac{1}{2}\, \sqrt{6}\, \big(\frac{1}{i}\, d_i + \gamma_\Lambda\big) \;=\; \frac{1}{i}\, d_i + \gamma_\Lambda.$$

Since $T\big(\dfrac{1}{i}\, \Lambda_0\big) = \dfrac{1}{i}\, \Lambda^*$, we find that $d(T\underline{\mathbf{z}}, \underline{\mathbf{y}}) \geq \dfrac{1}{i}\, d_i + \gamma_\Lambda$ for all $i = 1, \ldots, k$ and $\underline{\mathbf{y}} \in \dfrac{1}{i}\, \Lambda^*$ and we are done by Theorem 3.5. ■

Proof of Corollary 3.8 We can rewrite the conditions in the corollary as $d_0 \geq \frac{3}{2}\, \sqrt{2}\, (d_1 + \gamma_\Lambda)$ and $d_0 \geq 3\, \sqrt{2}\, (\frac{1}{2}\, d_2 + \gamma_\Lambda)$. Define Λ_0 as in Theorem 3.7 and set $\underline{\mathbf{z}} = \begin{pmatrix} \frac{1}{2}\, d_0 \\ \frac{1}{6}\, \sqrt{3}\, d_0 \end{pmatrix}$. Then straightforward calculus shows that

$$d(\underline{\mathbf{z}}, \Lambda_0) \;=\; \tfrac{1}{3}\sqrt{3}\,d_0 \;\geq\; \tfrac{1}{3}\sqrt{3}\cdot\tfrac{3}{2}\sqrt{2}\,(d_1+\gamma_\Lambda) \;=\; \tfrac{1}{2}\sqrt{6}\,(d_1+\gamma_\Lambda)$$

and

$$d(\underline{\mathbf{z}}, \tfrac{1}{2}\Lambda_0) \;=\; \tfrac{1}{6}\sqrt{3}\,d_0 \;\geq\; \tfrac{1}{6}\sqrt{3}\cdot 3\sqrt{2}\,(\tfrac{1}{2}d_2+\gamma_\Lambda) \;=\; \tfrac{1}{2}\sqrt{6}\,(\tfrac{1}{2}d_2+\gamma_\Lambda).$$

The result follows from Theorem 3.7. ∎

7 Proofs of the results in Section 4

Throughout this section we will fix the minimal basis $\underline{\mathbf{m}} = \binom{1}{0}$ and $\underline{\mathbf{n}} = \binom{\frac{1}{2}}{\frac{1}{2}\sqrt{3}}$. Before proving the results in Section 4, we take a closer look at Theorem 4.1. Let $d_0 > 0$ and let $\underline{\mathbf{s}}, \underline{\mathbf{t}} \in \Delta$ be such that there exists an $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$-labelling of Δ with $|\Delta/\Delta^*| = sp_T(\Delta;\, d_0)$, where $\Delta^* = \Lambda(\underline{\mathbf{s}}, \underline{\mathbf{t}})$. It follows from the proof of Theorem 4.1 in [15] that there exist integers p, q such that

$$\begin{aligned} d_0^+ &= \sqrt{p^2+p\,q+q^2}, \\ \underline{\mathbf{s}} &= p\,\underline{\mathbf{m}} + q\,\underline{\mathbf{n}} &&= \begin{pmatrix} p+\frac{1}{2}q \\ \frac{1}{2}q\sqrt{3} \end{pmatrix}, \\ \underline{\mathbf{t}} &= q\,(\underline{\mathbf{n}}-\underline{\mathbf{m}}) + p\,\underline{\mathbf{n}} &&= \begin{pmatrix} \frac{1}{2}p-\frac{1}{2}q \\ \frac{1}{2}(p+q)\sqrt{3} \end{pmatrix}. \end{aligned} \tag{7.1}$$

Without loss of generality we can assume

$$p \;\geq\; q \;\geq\; 0. \tag{7.2}$$

Finally, recall that $\gamma_\Delta = 1$.

Proof of Theorem 4.2 Part (a) follows immediately from Theorem 3.6 (a).

For (b), choose $\underline{\mathbf{s}}, \underline{\mathbf{t}} \in \Delta$ such that there exists an $(\underline{\mathbf{s}}, \underline{\mathbf{t}})$-labelling of Δ with $|\Delta/\Delta^*| = sp_T(\Delta;\, d_0)$, where $\Delta^* = \Lambda(\underline{\mathbf{s}}, \underline{\mathbf{t}})$. Since $d_0 > \sqrt{3}$, we are guaranteed that $sp_T(\Delta;\, d_0) \geq 4$. Choose p, q satisfying (7.1) and (7.2). This means that $\|\underline{\mathbf{s}}\| = \|\underline{\mathbf{t}}\| = d_0^+$ and $\underline{\mathbf{s}}\cdot\underline{\mathbf{t}} = \frac{1}{2}(d_0^+)^2$. Following the proof of Theorem 3.6 (b) we can conclude that if $d_1 > \frac{1}{3}\sqrt{3}\,d_0^+$, then $sp_T(\Delta;\, d_0, d_1) > sp_T(\Delta;\, d_0)$.

So we are left with the case $d_1 = \frac{1}{3}\sqrt{3}\,d_0^+$. Again following the proof of Theorem 3.6 (b), we can assume without loss of generality that all points in Δ^* have label 0. Then the only points that have distance at least d_1 from Δ^* are the points in $(\frac{1}{3}\underline{\mathbf{s}} + \frac{1}{3}\underline{\mathbf{t}}) + \Delta^*$ and the points in $(\frac{2}{3}\underline{\mathbf{s}} + \frac{2}{3}\underline{\mathbf{t}}) + \Delta^*$. Without loss of generality, we can choose the points in $(\frac{1}{3}\underline{\mathbf{s}} + \frac{1}{3}\underline{\mathbf{t}}) + \Delta^*$ to receive label 1. Then the points at distance at least d_1 from $(\frac{1}{3}\underline{\mathbf{s}} + \frac{1}{3}\underline{\mathbf{t}}) + \Delta^*$ are the points in Δ^* and in $(\frac{2}{3}\underline{\mathbf{s}} + \frac{2}{3}\underline{\mathbf{t}}) + \Delta^*$. Since the points in Δ^* already have label 0, we must assign label 2 to the points in $(\frac{2}{3}\underline{\mathbf{s}} + \frac{2}{3}\underline{\mathbf{t}}) + \Delta^*$. But now the points at distance at least d_1 from $(\frac{2}{3}\underline{\mathbf{s}} + \frac{2}{3}\underline{\mathbf{t}}) + \Delta^*$ are the points in Δ^* and those in $(\frac{1}{3}\underline{\mathbf{s}} + \frac{1}{3}\underline{\mathbf{t}}) + \Delta^*$. All points in these sets have label 0 or 1, hence there are no points to which we can assign label 3 to. Since we must assign as least 4 labels, we can only conclude $sp_T(\Delta;\, d_0, d_1) > sp_T(\Delta;\, d_0)$. ∎

Proof of Theorem 4.3 We can follow the proof of Theorem 3.7, using $\underline{\mathbf{s}}, \underline{\mathbf{t}}$ defined in the beginning of this section. Considering the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{pmatrix} d_0 \\ 0 \end{pmatrix} = \underline{\mathbf{s}}$ and $T\begin{pmatrix} \frac{1}{2}\, d_0 \\ \frac{1}{2}\, d_0 \sqrt{3} \end{pmatrix} = \underline{\mathbf{t}}$, we find that for all $\underline{\mathbf{x}} \in \mathbb{R}^2$, $\|T\underline{\mathbf{x}}\| \geq \underline{\mathbf{x}}$. This gives that for all $\underline{\mathbf{x}} \in \frac{1}{i}\, \Lambda_0$,

$$d(T\underline{\mathbf{z}}, T\underline{\mathbf{x}}) \;=\; \|T\underline{\mathbf{z}} - T\underline{\mathbf{x}}\| \;=\; \|T(\underline{\mathbf{z}} - \underline{\mathbf{x}})\| \;\geq\; \|\underline{\mathbf{z}} - \underline{\mathbf{x}}\| \;=\; d(\underline{\mathbf{z}}, \underline{\mathbf{x}}) \;\geq\; d_i + 1.$$

Since $T\big(\frac{1}{i}\, \Lambda_0\big) = \frac{1}{i}\, \Delta^*$, we are done by Theorem 3.5. ■

Proof of Corollary 4.4 The corollary follows from Theorem 4.3 by considering the point $\underline{\mathbf{z}} = \begin{pmatrix} \frac{1}{2}\, d_0 \\ \frac{1}{6}\, d_0 \sqrt{3} \end{pmatrix}$. ■

Proof of Theorem 4.7 Let $0 \leq A < \sqrt{\frac{1}{2}\sqrt{3}}$ and choose k and d_0 such that

$$\frac{k-1}{k+1}\, A^2 + \frac{1 + \frac{1}{2}\sqrt{3}}{\sqrt{k+1}}\, A + \frac{2\,k}{d_0} \;\leq\; \tfrac{1}{2}\sqrt{3}. \tag{7.3}$$

Assume $d \leq A\, \dfrac{d_0}{\sqrt{k+1}}$. Define Λ_0 as in Theorem 4.3. We will show that the point $\underline{\mathbf{z}} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ with

$$z_1 \;=\; d + 1 \qquad \text{and} \qquad z_2 \;=\; \frac{d_0 + d^2 + d}{d_0 - d}$$

satisfies the conditions of Theorem 4.3.

For $i = 1, \ldots, k$ and $p, q \in \mathbb{Z}$, define

$$x_i(p,q) \;=\; \frac{p}{i}\, d_0 + \tfrac{1}{2}\, \frac{q}{i}\, d_0, \qquad y_i(p,q) \;=\; \tfrac{1}{2}\, \frac{q}{i}\, d_0 \sqrt{3} \qquad \text{and} \qquad \underline{\mathbf{x}}_i(p,q) \;=\; \begin{pmatrix} x_i(p,q) \\ y_i(p,q) \end{pmatrix}.$$

This means that the $\underline{\mathbf{x}}_i(p,q)$ are precisely the points in $\frac{1}{i}\, \Lambda_0$. So we need to show that $d(\underline{\mathbf{z}}, \underline{\mathbf{x}}_i(p,q)) \;\geq\; \frac{1}{i}\, d + 1$ for all i, p, q. Using Lemma 2.7, it is more than enough if we do this for $p, q \geq 0$.

First observe that $z_1 = d + 1 \geq \frac{1}{i}\, d + 1$, hence $d(\underline{\mathbf{z}}, \underline{\mathbf{x}}_i(0,0)) \geq \frac{1}{i}\, d + 1$ for all i. We now consider the case $q = 0$ and $p \geq 1$. If $i \leq \dfrac{d_0 - d}{d + 2}$, then $\frac{p}{i}\, d_0 \geq d + 2$ and we find

$$\begin{aligned} d(\underline{\mathbf{z}}, \underline{\mathbf{x}}_i(p,0)) - \frac{1}{i}\, d \;&\geq\; |z_1 - x_i(p,0)| - \frac{1}{i}\, d \\ &=\; \frac{p}{i}\, d_0 - (d+1) - \frac{1}{i}\, d \;=\; \frac{p\, d_0 - d}{i} - (d+1) \\ &\geq\; \frac{(p\, d_0 - d)\,(d+2)}{d_0 - d} - (d+1) \;\geq\; 1. \end{aligned}$$

Whereas if $i \geq \dfrac{d_0 - d}{d+2}$, we can argue

$$\begin{aligned} d(\underline{\mathbf{z}}, \underline{\mathbf{x}}_i(p,0)) - \frac{1}{i}\,d \;&\geq\; |z_2 - y_i(p,0)| - \frac{1}{i}\,d \;=\; \frac{d_0 + d^2 + d}{d_0 - d} - 0 - \frac{1}{i}\,d \\ &\geq\; \frac{d_0 + d^2 + d}{d_0 - d} - \frac{d+2}{d_0 - d}\,d \;=\; 1. \end{aligned}$$

It follows that $d(\underline{\mathbf{z}}, \underline{\mathbf{x}}_i(p,0)) \geq \dfrac{1}{i}\,d + 1$ for all i and $p \geq 1$.

Next consider the case $q \geq 1$. From (7.3) and the fact that $d \leq A\,\dfrac{d_0}{\sqrt{k+1}}$ we get that

$$(k-1)\,\frac{d^2}{d_0^2} + (\tfrac{1}{2}\sqrt{3} + 1)\,\frac{d}{d_0} + \frac{2\,k}{d_0} \;\leq\; \tfrac{1}{2}\sqrt{3}.$$

This last inequality is equivalent to $\dfrac{1}{k}\,(\tfrac{1}{2}\,d_0\sqrt{3} - d) - \dfrac{d_0 + d^2 + d}{d_0 - d} \geq 1$. This means that for all $q \geq 1$ we have $\tfrac{1}{2}\,\dfrac{q}{i}\,d_0\sqrt{3} \geq \tfrac{1}{2}\,\dfrac{1}{k}\,d_0\sqrt{3} > \dfrac{d_0 + d^2 + d}{d_0 - d}$ and hence

$$\begin{aligned} d(\underline{\mathbf{z}}, \underline{\mathbf{x}}_i(p,q)) - \frac{1}{i}\,d \;&\geq\; |z_2 - y_i(p,q)| - \frac{1}{i}\,d \\ &=\; \tfrac{1}{2}\,\frac{q}{i}\,d_0\sqrt{3} - \frac{d_0 + d^2 + d}{d_0 - d} - \frac{1}{i}\,d \\ &=\; \frac{1}{i}\,(\tfrac{1}{2}\,q\,d_0\sqrt{3} - d) - \frac{d_0 + d^2 + d}{d_0 - d} \\ &\geq\; \frac{1}{k}\,(\tfrac{1}{2}\,d_0\sqrt{3} - d) - \frac{d_0 + d^2 + d}{d_0 - d} \;\geq\; 1. \end{aligned}$$

We find that $d(\underline{\mathbf{z}}, \underline{\mathbf{x}}_i(p,q)) \geq \dfrac{1}{i}\,d + 1$ for all i, p and $q \geq 1$. This completes the proof of the theorem. ■

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