Discrete Mathematics Lent 2009
MA 210

Notes for lectures 1and 2

1.1 Basic principles of counting

A non-empty set X has n elements, where n is a natural number, if there exists a bijection $f : X \to \{1, 2, \ldots, n\}$. Instead of finding the bijection f, this is typically shown by listing all distinct elements of X, that is, $X = \{x_1, \ldots, x_n\}$.

When the set X comes from a complex problem, we may need more sophisticated methods than finding a bijection in order to determine $|X|$. It is our goal to develop some of these methods.

Addition Rule (AR): *The number of objects in a set can be counted by splitting the set into disjoint subsets and then adding together the number of objects in each set.*

Exercise 1.1. *There are 9 women and 6 men in a room. How many people are in the room?*

Solution. The Addition Rule applies because the set of women is clearly disjoint from the set of men, hence there are $9 + 6 = 15$ people. □

Exercise 1.2. *There are 9 German speakers and 6 French speakers in a room. How many people are in the room?*

Solution. We can't tell. The Addition Rule does not apply in this case because there may be people in the room speaking both the languages (additionally, there may be people not speaking any of these languages). □

Theorem 1.3.

- *If A and B are two finite disjoint sets (i.e., $A \cap B = \emptyset$), then $|A \cup B| = |A| + |B|$.*
- *If $A_1, A_2, \ldots, A_n$ are finite pairwise disjoint sets (i.e., $A_i \cap A_j = \emptyset$ for all $i \neq j$), then $|\bigcup_{i=1}^{n} A_i| = |A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$.*

Exercise 1.4 (Pigeonhole Principle)**.** *If m objects are distributed into n boxes and $m > nk$ for some natural number k, then at least one box must contain at least $k + 1$ objects.*

Solution. Part (a) is an easy exercise. Part (b) is proved by induction on n. The case $n = 2$ is actually a part (a). Consider pairwise disjoint sets $A_1, \ldots, A_n, A_{n+1}$. Set $A = A_1 \cup \cdots \cup A_n$ and $B = A_{n+1}$ and apply part (a), then use the induction assumption. □

Exercise 1.5. *Show that in any group of six people there are either three mutual friends or three mutual strangers.*

Hint. Fix one person α and divide the remaining five people into two groups: friends of α and strangers to α. Then apply Pigeonhole Principle with $m = 5$, $n = 2$ and $k = 2$. □

Multiplication Rule (MR): *If a counting problem can be split into a number of stages, each of which involves choosing one of a number of options, then the total number of possibilities can be found by multiplying together the number of options available at each stage.*

Exercise 1.6. *Given two non-empty finite sets X and Y, what is the number of functions $f : X \to Y$? And how many of these are 1-to-1 (injections)?*

1.2 Ordered selections

Basic problem: Given n distinct objects, how many ways are there to choose r of these objects when the order in which they are chosen is important and

- repetition is allowed;

or

- repetition is not allowed?

The first choice is made in r stages and in each of the stages we have n objects to choose from because repetition is allowed. Hence, by the Multiplication Rule, the total number of choices is

$$\underbrace{n \times \cdots \times n}_{r\text{-times}} = n^r.$$

The second choice is again made in r stages, but at the stage i we have only $n - i + i$ objects to choose from because we must exclude $i - 1$ objects previously chosen. By the Multiplication Rule, the number of choices is

$$(n)_r := n \times (n-1) \times \cdots \times (n - r + 1).$$

Definition 1.7. *For every natural number n we define $n!$ (n factorial) as*

$$n! = 1 \times 2 \times \cdots \times n.$$

We also set $0! = 1$.

Consequently,

$$(n)_r := n(n-1)\ldots(n-r+1) = \frac{n \times (n-1) \times \ldots (n-r+1) \times (n-r) \cdots \times 2 \times 1}{(n-r) \cdots \times 2 \times 1} = \frac{n!}{(n-r)!}.$$

Exercise 1.8. *Find the number of bijections from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$.*

Exercise 1.9. *How many 7-digit numbers are with all digits from the set $\{0, 1, \ldots, 0\}$? How many of these numbers have all theirs digits distinct?*

1.3 Unordered selections

Basic problem: Given n distinct objects, how many ways are there to choose r of these objects when the order in which they are chosen is irrelevant and

- repetition is not allowed;

or

- repetition is allowed?

If repetition is not allowed , then we simply want to find the number of r-element subsets if an n-set and this is the *binomial number* $\binom{n}{r}$.

Fact 1.10. *For every natural number n,*

1. $\binom{n}{0} = \binom{n}{n} = 1$;
2. $\binom{n}{r} = 0$ *for every* $r > n$;
3. $\binom{n}{r} = \binom{n}{n-r}$.

Theorem 1.11. *For every n and r,*

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

Theorem 1.12. *For every n and r,*

$$\binom{n}{r} = \frac{(n)_r}{r!} = \frac{n!}{r!(n-r)!}.$$

Exercise 1.13. *15 faculty members and 60 students need to form a committee of 5. In how many ways can this be done if*

1. *the committee should consist of 3 students and 2 faculty members;*
2. *the committee should have at least 3 faculty members;*
3. *the committee should have at least 2 students and at least 1 faculty member?*