# Discrete Mathematics MA 210 Lent 2009

## Notes for lectures 3 and 4

### 1.3 Unordered selections (continued)

Suppose that $n$ distinct objects are given. We have already observed that if we want to choose $r$ of these objects, the order in which they are chosen is irrelevant and repetition is not allowed, then this can be done in $\binom{n}{r}$ ways.

What if repetition is allowed? Imagine $n+r-1$ slots in a row into which we place $n-1$ markers (no two markers can be placed into the same slot). The number of empty slots before the first marker will be the number of times the first object is selected. The number of empty slots between the first and second marker will be the number of times the second object is selected. In general, the number of empty slots between $(i-1)-th$ and $i$-th marker will be the number of times the $i$-th object is selected. So, the number of ways to choose $r$ objects from $n$ distinct objects, the order in which they are chosen is irrelevant and repetition of which is allowed, is equal to the number of ways to place $n-1$ markers into $n+r-1$ slots: $\binom{n+r-1}{n-1} = \binom{n+r-1}{r}$.

**Exercise 1.14.** *Show that $x_1 + x_2 + \cdots + x_n = r$ has $\binom{n+r-1}{n-1}$ integer, non-negative solutions.*

**Exercise 1.15.** *In how many ways can we list $p$ 0's and $q$ 1's so that no consecutive 0's appear?*

### 2.1 Binomial Theorem

**Theorem 2.1** (Binomial Theorem). *For all (complex) numbers $a$, $b$, and for all natural numbers $n$, we have*

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b^1 + \cdots + \underbrace{\binom{n}{i} a^{n-i} b^i}_{i\text{-}th\ term} + \cdots + \binom{n}{n} b^n.$$

*Proof.* This is an easy induction on $n$ using the fact that for every $n$ and $i$,

$$\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}.$$

□

**Exercise 2.2.** *If we choose $a = b = 1$, we have that $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{0}$, in other words, the set $\{1, \ldots, n\}$ has $2^n$ subsets.*

**Exercise 2.3.** *If we choose* $a = 1, b = -1$, *we have that* $0 = 0^n = \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{0}$, *in other words, the number of even-element subsets of* $\{1, \ldots, n\}$ *is the same as the number of odd-element subsets of* $\{1, \ldots, n\}$.

To prove identities we often choose suitable $a$ and $b$ or use that, for example, $(a+b)^{m+n} = (a+b)^n(a+b)^m$ or $(a+b)^{mn} = [(a+b)^m]^n$.
One should be creative: for example, take $f(t) = (1+t)^n = \sum_{i=0}^{n} \binom{n}{i} t^i$. We have that

$$f'(t) = n(1+t)^{n-1} = \sum_{i=0}^{n-1} n \binom{n-1}{i} t^i$$

and

$$f'(t) = \sum_{i=1}^{n} i \binom{n}{i} t^{i-1} = \sum_{i=0}^{n-1} (i+1) \binom{n}{i+1} t^i.$$

Comparing the coefficients of $t^i$ yields

$$(i+1)\binom{n}{i+1} = n\binom{n-1}{i}.$$

## 2.2 Inclusion-Exclusion Principle

We have seen that if $A$ and $B$ are two disjoint sets, then $|A \cup B| = |A| + |B|$. If $A$ and $B$ are not disjoint, then we obtain

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

If we have more than two sets, then it can be generalized as follows.

**Theorem 2.4.** *Let* $A_1, \ldots, A_n$ *be finite sets. For* $X \subset \{1, \ldots, n\}$, *define*

$$N(X) = \left| \bigcap_{i \in X} A_i \right|$$

*and, for* $i, 1 \leq i \leq n$, *define*

$$\alpha_i = \sum_{X \subset \{1,\ldots,n\}, |X| = i} N(X).$$

*Then*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \cdots + (-1)^{n-1}\alpha_n. \tag{1}$$

**Exercise 2.5.** *How many numbers between 1 and 1000 are divisible by none of 3,5,7 ?*