Discrete Mathematics **Lent 2009**

MA210

Exercises 3

(1) Prove the identity

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

in the following two ways.

(a) Apply the Binomial Theorem to both sides of the identity

$$(1+x)^n \cdot (1+x)^n = (1+x)^{2n},$$

and look at the coefficient of x^n.

(b) Consider two disjoint sets A and B, each of size n, and count the number of subsets of $A \cup B$ with n elements. (Hint: remember that $\binom{n}{i} = \binom{n}{n-i}$.)

(2) Downtown Metropolis consists of a rectangular grid of streets. It has k blocks from west to east, and m blocks from north to south. You stand at the south-west (lower left) corner and want to go to your flat, which is at the north-east (top right) corner.

It is clear that if you want to take a shortest path to your flat, then you only walk in eastward or northward directions. Moreover, you can only move a whole number of blocks each time. Show that if you are interested in shortest paths only, you still have $\binom{k+m}{k}$ possibilities to reach your flat.

(3) In an experiment on the effects of fertiliser on 27 plots of new breed of tomatoes, 8 plots are given nitrogen, phosphorus and potash fertiliser; 12 plots are given at least nitrogen and phosphorus, 12 plots are given at least phosphorus and potash; and 12 plots are given at least nitrogen and potash. Also, 18 plots receive nitrogen; 18 plots receive phosphorus; and 18 plots receive potash. How many plots were left unfertilised?

(4) Solve the following recurrence relation:

$$\begin{aligned} a_n &= 4a_{n-1} - 4a_{n-2} \quad \text{for } n \geq 2, \\ a_0 &= 1; \\ a_1 &= 3. \end{aligned}$$

(5) Solve the following recurrence relation:

$$\begin{aligned} b_n &= b_{n-1} + 6b_{n-2} \quad \text{for } n \geq 2, \\ b_0 &= 1; \\ b_1 &= 1. \end{aligned}$$

(6) Let a_n denote the number of n-digit sequences in which each digit is 0, 1 or -1, and no two consecutive 1's or two consecutive -1's are allowed.
   (a) Show that $a_n = 2a_{n-1} + a_{n-2}$ for $n \geq 3$.
   (b) Determine a_1 and a_2.
   (c) Find a closed form expression for a_n.

(7) On working through a problem, a student is said to be at the n-th stage if she or he is n steps from the solution. At any stage the student has five choices how to proceed. Two of these choices result in the student going to the $(n-1)$-th stage, and the remaining three of them are better and they take her or him directly to the $(n-2)$-th stage.
   Let s_n be the number of ways the student can reach the solution if she or he starts from the n-th stage.
   (a) If $s_1 = 2$, verify that $s_2 = 7$.
   (b) Give a recurrence relation for s_n.
   (c) Deduce that $s_n = \frac{1}{4}(3^{n+1} + (-1)^n)$.

You must justify the answers to all problems!

These exercises are to be handed in **before 4.55pm on February 2, 2009**.