Discrete Mathematics **Lent 2009**

MA210

Solutions to Exercises 3

(1) Prove the identity

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

in the following two ways.

(a) Apply the Binomial Theorem to both sides of the identity

$$(1+x)^n \cdot (1+x)^n = (1+x)^{2n},$$

and look at the coefficient of x^n.

Solution. By the Binomial Theorem, we have

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

and

$$(1+x)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} x^i.$$

We rewrite

$$(1+x)^n \cdot (1+x)^n = (1+x)^{2n},$$

as

$$\sum_{i=0}^{n} \binom{n}{i} x^i \cdot \sum_{j=0}^{n} \binom{n}{j} x^j = \sum_{k=0}^{2n} \binom{2n}{k} x^k.$$

From the right-hand side of this equality, we have that the coefficient of x^n is $\binom{2n}{n}$. To the left-hand side, we apply the Convolution Theorem and obtain that the coefficient of x^n is

$$\sum_{i=0}^{n} \binom{n}{i} \cdot \binom{n}{n-i}.$$

Since $\binom{n}{i} = \binom{n}{n-i}$, we have that

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i} \cdot \binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}^2.$$

$\square$

(b) Consider two disjoint sets A and B, each of size n, and count the number of subsets of $A \cup B$ with n elements.

Solution. $A \cup B$ has $2n$ elements and, therefore, it has $\binom{2n}{n}$ subsets of size n. How can we count these subsets another way?

Each subset of $A \cup B$ has k elements from A and $n-k$ elements from B for some k, $0 \le k \le n$. We can choose k elements from A in $\binom{n}{k}$ ways and $n-k$ elements from B in $\binom{n}{n-k} = \binom{n}{k}$ ways. So, we have $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$ subsets of $A \cup B$ with k elements from A and $n-k$ elements from B. Summing over all k, we get all the subsets of $A \cup B$ of size n, that is,

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

$\square$

(2) Downtown Metropolis consists of a rectangular grid of streets. It has k blocks from west to east, and m blocks from north to south. You stand at the south-west (lower left) corner and want to go to your flat, which is at the north-east (top right) corner.

It is clear that if you want to take a shortest path to your flat, then you only walk in eastward or northward directions. Moreover, you can only move a whole number of blocks each time. Show that if you are interested in shortest paths only, you still have $\binom{k+m}{k}$ possibilities to reach your flat.

Solution. The shortest way home is given by a sequence of one-block moves eastwards or northwards. We must make $m+k$ moves: m northwards and k eastwards. Once we decide positions for eastwards moves, the remaining ones must be northwards. Each such a sequence gives a different path home. There are $\binom{m+k}{k}$ ways to chose positions for eastwards moves, so there are $\binom{m+k}{k}$ paths home. $\square$

(3) In an experiment on the effects of fertiliser on 27 plots of new breed of tomatoes, 8 plots are given nitrogen, phosphorus and potash fertiliser; 12 plots are given

at least nitrogen and phosphorus, 12 plots are given at least phosphorus and potash; and 12 plots are given at least nitrogen and potash. Also, 18 plots receive nitrogen; 18 plots receive phosphorus; and 18 plots receive potash. How many plots were left unfertilised?

Solution. Let A_N be the set of plots that get nitrogen, let A_F be the set of plots that get phosphorus, and let A_P be the set of plots that receive potash. From the statement of the problem, we have that: $|A_F| = |A_N| = |A_P| = 18$, $|A_F \cap A_N| = |A_N \cap A_P| = |A_P \cap A_F| = 12$, and $|A_F \cap A_N \cap A_P| = 8$. Using the Inclusion-Exclusion Principle, we obtain

$$\begin{aligned} |A_F \cup A_N \cup A_P| &= |A_F| + |A_N| + |A_P| - |A_F \cap A_N| - |A_N \cap A_P| - |A_P \cap A_F| + |A_F \cap A_N \cap A_P| \\ &= 3 \cdot 18 - 3 \cdot 12 + 8 = 26. \end{aligned}$$

Hence, the number of unfertilised plots is $27 - |A_F \cup A_N \cup A_P| = 27 - 26 = 1$. $\square$

(4) Solve the following recurrence relation:

$$\begin{aligned} a_n &= 4a_{n-1} - 4a_{n-2} \quad \text{for } n \geq 2, \\ a_0 &= 1; \\ a_1 &= 3. \end{aligned}$$

Solution. We use Theorem 3.5 from the lecture notes: first, we must find the roots of $x^2 = 4x - 4$, i.e., solve $x^2 - 4x + 4 = 0$. Since $x^2 - 4x + 4 = (x-2)^2$, we have one double root $r = 2$. Hence, the general solution is

$$a_n = (k_1 + k_2 n)2^n.$$

Then, we obtain $1 = a_0 = k_1 2^0 = k_1$ and $3 = a_1 = (k_1 + k_2)2^1 = 2k_1 + 2k_2$. From these two equations we obtain $k_1 = 1$ and $k_2 = \frac{1}{2}$. So,

$$a_n = \left(1 + \frac{n}{2}\right) 2^n.$$

$\square$

(5) Solve the following recurrence relation:

$$\begin{aligned} b_n &= b_{n-1} + 6b_{n-2} \quad \text{for } n \geq 2, \\ b_0 &= 1; \\ b_1 &= 1. \end{aligned}$$

Solution. We again use Theorem 3.5 from the lecture notes: first, we must find the roots of $x^2 = x + 6$, i.e., solve $x^2 - x - 6 = 0$. Since $x^2 - x - 6 = (x-3)(x+2)$, we have roots $r_1 = -2$ and $r_2 = 3$. Hence, the general solution is

$$b_n = k_1(-2)^n + k_2 3^n.$$

Then, we obtain $1 = b_0 = k_1 + k_2$ and $1 = b_1 = k_1(-2)^1 + k_2 3^1 = -2k_1 + 3k_2$. From these two equations we obtain $k_1 = \frac{2}{5}$ and $k_2 = \frac{3}{5}$. So,

$$b_n = \frac{2}{5}(-2)^n + \frac{3}{5}3^n.$$

□

(6) Let a_n denote the number of n-digit sequences in which each digit is 0, 1 or -1, and no two consecutive 1's or two consecutive -1's are allowed.

(a) Show that $a_n = 2a_{n-1} + a_{n-2}$ for $n \geq 3$.

Solution. We are going to show that

$$a_n = 2(a_{n-1} - a_{n-2}) + 3a_{n-2} = 2a_{n-1} + a_{n-2}.$$

Let $x_1, x_2, \ldots, x_n$ be a valid sequence of length n, that is, a sequence with $x_i \in \{-1, 0, 1\}$ for every $i = 1, 2, \ldots, n$, and with no repeated 1's or -1's. Then, $x_1, x_2, \ldots, x_{n-1}$ is also a valid sequence of length $n-1$ and we have a_{n-1} of them. How many of these sequences have $x_{n-1} = 0$?
Again, $x_1, x_2, \ldots, x_{n-2}$ is a valid sequence of length $n-2$ and there are no additional restrictions on x_{n-2} because $x_{n-1} = 0$. (Note: if we considered $x_{n-1} = 1$, then we would have to guarantee that $x_{n-2} \neq 1$.) Since there are a_{n-2} valid sequences of length $n-2$, we also have a_{n-2} valid sequences

of length $n-1$ that ends with 0. Consequently, we have $a_{n-1} - a_{n-2}$ valid sequences of length $n-1$ that ends with 1 or -1.

In the case when $x_{n-1} = 0$, x_n can be any one of $-1, 0, 1$. So, we have $3a_{n-2}$ valid sequences of length n of this type.

When $x_{n-1} \neq 0$, we have only two options for x_n: either $-1, 0$ (when $x_{n-1} = 1$) or $0, 1$ (when $x_{n-1} = -1$). So, we have $2(a_{n-1} - a_{n-2})$ valid sequences of length n of this type.

Altogether, we have that

$$a_n = 2(a_{n-1} - a_{n-2}) + 3a_{n-2} = 2a_{n-1} + a_{n-2}.$$

□

(b) Determine a_1 and a_2.

Solution. Since 0, -1, 1 are all valid sequences of length 1, we have $a_1 = 3$. There are $3^2 = 9$ sequence of length 2 with entries from $\{-1, 0, 1\}$. Only two of them are not valid: $1, 1$ and $-1, -1$. Hence, $a_2 = 9 - 2 = 7$. □

(c) Find a closed form expression for a_n.

Solution. We find the roots of $x^2 = 2x + 1$, i.e., $x^2 - 2x - 1 = 0$. Using the usual formula for the roots of quadratic equations, we obtain two roots: $1 + \sqrt{2}$ and $1 - \sqrt{2}$. Hence, $a_n = k_1(1 + \sqrt{2})^n + k_2(1 - \sqrt{2})^n$.

Using (b), we get

$$3 = a_1 = k_1(1 + \sqrt{2}) + k_2(1 - \sqrt{2})$$

and

$$7 = a_2 = k_1(1 + \sqrt{2})^2 + k_2(1 - \sqrt{2})^2.$$

Solving these two equations, we obtain that $k_1 = \frac{1}{2}(1 + \sqrt{2})$ and $k_2 = \frac{1}{2}(1 - \sqrt{2})$. Hence,

$$a_n = \frac{1}{2}(1 + \sqrt{2})^{n+1} + \frac{1}{2}(1 - \sqrt{2})^{n+1}.$$

□

(7) On working through a problem, a student is said to be at the n-th stage if she or he is n steps from the solution. At any stage the student has five choices how to proceed. Two of these choices result in the student going to the $(n-1)$-th stage, and the remaining three of them are better and they take her or him directly to the $(n-2)$-th stage.

Let s_n be the number of ways the student can reach the solution if she or he starts from the n-th stage.

(a) If $s_1 = 2$, verify that $s_2 = 7$.

Solution. There are two choices for the student to go to stage 1, from which there are 2 ways to get to the solution. And there are 3 choices that get the student directly to the solution (stage 0), so $s_2 = 2s_1 + 3 = 7$. □

(b) Give a recurrence relation for s_n.

Solution. There are two choices for the student to go to stage $n-1$, from which there are s_{n-1} ways to get to the solution. And there are 3 choices that get the student to stage $n-2$, from which there are s_{n-2} ways to get to the solution. So,

$$s_n = 2s_{n-1} + 3s_{n-2}.$$

□

(c) Deduce that $s_n = \frac{1}{4}(3^{n+1} + (-1)^n)$.

Solution. We find the roots of $x^2 = 2x+3$, i.e., $x^2-2x-3 = (x-3)(x+1) = 0$. Thus we have two roots: 3 and -1. Hence, $s_n = k_1 3^n + k_2(-1)^n$. Using $2 = s_1 = 3k_1 + (-1)k_2 = 3k_1 - k_2$ and $7 = s_2 = 3^2 k_1 + (-1)^2 k_2 = 9k_1 + k_2$, we obtain $k_1 = \frac{3}{4}$ and $k_2 = \frac{1}{4}$. So,

$$s_n = k_1 3^n + k_2(-1)^n = \frac{3}{4}3^n + \frac{1}{4}(-1)^n = \frac{1}{4}(3^{n+1} + (-1)^n).$$

□