Discrete Mathematics **Lent 2009**

MA210

Exercises 4

(1) Define sequence $(b_n)_{n\geq 1}$ by $b_n = \binom{n}{0} + \binom{n-1}{1} + \ldots$, where we use $\binom{n}{k} = 0$ for $k > n$. Verify that $b_1 = 1$, $b_2 = 2$, and that, for every $n \geq 3$, we have $b_n = b_{n-1} + b_{n-2}$.

(2) Let a_n denote the number of n-digit sequences in which each digit is either 0 or 1, and no two consecutive 0's are allowed.
(a) Show that $a_1 = 2$ and $a_2 = 3$. What would you say a_0 is?
(b) Show that for $n \geq 3$ we have $a_n = a_{n-1} + a_{n-2}$.
(c) Give a closed form expression for a_n.

(3) Let f_n denote the n-th Fibonacci number, i.e., $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Prove that for every $n \geq 2$ we have $f_n^2 - f_{n-1} \cdot f_{n+1} = (-1)^n$.

(4) <u>Use generating functions</u> to solve the following recurrence relation:

$$\begin{aligned} a_n &= 5a_{n-1} - 6a_{n-2} \quad \text{for } n \geq 2, \\ a_0 &= 0; \\ a_1 &= 3. \end{aligned}$$

(5) Suppose that $f(x)$ generates the sequence $a_0, a_1, a_2, \ldots$. Give the expressions, in terms of f, for the generating functions of the following sequences:
(a) $0, a_0, 0, a_1, 0, a_2, 0, \ldots;$
(b) $1, a_0, a_1, a_2, \ldots;$
(c) $a_0, -a_1, a_2, -a_3, a_4, \ldots.$

(6) (a) Show that the generating function of the sequence $a_n = n$, $n \geq 0$, is $f(x) = \frac{x}{(1-x)^2}$.
(b) Find generating functions for the sequences $b_n = n^2$, $n \geq 0$, and $c_n = n^3$, $n \geq 0$.
Hint: differentiation.

(7) Find the sequences generated by the following functions:
(a) $f(x) = \frac{x^3}{1+x};$
(b) $g(x) = \frac{x}{1-7x+12x^2};$
(c) $h(x) = \frac{x^7}{2-x^7};$
(d) $k(x) = e^{2x}.$

You must justify the answers to all problems!

These exercises are to be handed in **before 4.55pm on February 9, 2009**.