Discrete Mathematics **Lent 2009**

MA210

Solutions to Exercises 4

(1) Define sequence $(b_n)_{n\geq 1}$ by $b_n = \binom{n}{0} + \binom{n-1}{1} + \dots$, where we use $\binom{n}{k} = 0$ for $k > n$. Verify that $b_1 = 1$, $b_2 = 2$, and that, for every $n \geq 3$, we have $b_n = b_{n-1} + b_{n-2}$.

Solution. Using $\binom{n}{k} = 0$ for $k > n$, we have

$$b_1 = \binom{1}{0} + \binom{0}{1} + \cdots = \binom{1}{0} = 1$$

and

$$b_2 = \binom{2}{0} + \binom{1}{1} + \binom{0}{2} + \cdots = \binom{2}{0} + \binom{1}{1} = 1 + 1 = 2.$$

Every b_n has only finitely many non-zero summands and we can write it as

$$b_n = \sum_{k=0}^{n} \binom{n-k}{k}.$$

Then,

$$\begin{aligned}
b_{n-1} + b_{n-2} &= \sum_{k=0}^{n-1} \binom{n-1-k}{k} + \sum_{k=0}^{n-2} \binom{n-2-k}{k} \\
&= \binom{n-1}{0} + \sum_{k=1}^{n-1} \binom{n-1-k}{k} + \sum_{k=0}^{n-2} \binom{n-2-k}{k} \\
&= 1 + \sum_{k=1}^{n-1} \binom{n-1-k}{k} + \sum_{k=1}^{n-1} \binom{n-2-(k-1)}{k-1} \\
&= 1 + \sum_{k=1}^{n-1} \left(\binom{n-1-k}{k} + \binom{n-1-k}{k-1} \right).
\end{aligned}$$

We will use the identity $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$ with $m = n - k$ and, also, that $\binom{n-0}{0} = \binom{n}{0} = 1$ and $\binom{n-n}{n} = \binom{0}{n} = 0$:

$$\begin{aligned}
b_{n-1} + b_{n-2} &= 1 + \sum_{k=1}^{n-1} \left(\binom{n-1-k}{k} + \binom{n-1-k}{k-1} \right) + 0 \\
&= \binom{n-0}{0} + \sum_{k=1}^{n-1} \binom{n-k}{k} + \binom{n-n}{n}
\end{aligned}$$

$$= \sum_{k=0}^{n} \binom{n-k}{k} = b_n.$$

$\square$

(2) Let a_n denote the number of n-digit sequences in which each digit is either 0 or 1, and no two consecutive 0's are allowed.

(a) Show that $a_1 = 2$ and $a_2 = 3$. What would you say a_0 is?

Solution. Both 0 and 1 are valid 1-digit sequences, hence $a_1 = 2$. Similarly, 11, 10, 01 are all the valid 2-digit sequences (00 is prohibited), hence $a_2 = 3$. Empty sequence (the equivalent of $\emptyset$ for sets) contains no consecutive 0's, so $a_0 = 1$. $\square$

(b) Show that for $n \geq 3$ we have $a_n = a_{n-1} + a_{n-2}$.

Solution. Consider any valid n-digit sequence $x_1 x_2 \dots x_n$.

• If $x_1 = 1$, then $x_2 x_3 \dots x_n$ is a valid $n-1$-digit sequence, and we have a_{n-1} of these.

• If $x_1 = 0$, then x_2 must be 1 to avoid consecutive zeros. All the remaining $n-2$ digits do not contain consecutive zeros, so they form a valid $(n-2)$-digit sequence, and we have a_{n-2} of them.

Thus, the total number a_n of valid n-digit sequences satisfies $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. $\square$

(c) Give a closed form expression for a_n.

Solution. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of $(a_n)_{n \geq 1}$. We shall use that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Indeed, we have

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \\
&= a_0 + a_1 x + \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2}) x^n \\
&= a_0 + a_1 x + x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \\
&= a_0 + a_1 x + x \sum_{n=1}^{\infty} a_n x^n + x^2 \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + x(f(x) - a_0) + x^2 f(x) \\
&= 1 + 2x - x + (x + x^2) f(x).
\end{aligned}$$

Hence,

$$f(x) = \frac{1+x}{1-x-x^2}.$$

Equation $1-x-x^2$ has two solutions $\alpha = (-1+\sqrt{5})/2$ and $\beta = (-1-\sqrt{5})/2$. We rewrite $f(x)$ as

$$f(x) = \frac{-1-x}{(\alpha-x)(\beta-x)} = \frac{A}{\alpha-x} + \frac{B}{\beta-x} = \frac{A\beta+B\alpha-(A+B)x}{(\alpha-x)(\beta-x)}.$$

Consequently, we have $A+B=1$ and $A\beta+B\alpha=-1$. These two equations have solution

$$A = \frac{-\beta}{\alpha-\beta} = \frac{-\beta}{\sqrt{5}} \text{ and } B = \frac{\alpha}{\alpha-\beta} = \frac{\alpha}{\sqrt{5}},$$

therefore,

$$\begin{aligned} f(x) &= \frac{A}{\alpha-x} + \frac{B}{\beta-x} = \frac{A}{\alpha}\cdot\frac{1}{1-\frac{x}{\alpha}} + \frac{B}{\beta}\cdot\frac{1}{1-\frac{x}{\beta}} \\ &= \frac{A}{\alpha}\sum_{n=0}^{\infty}\left(\frac{x}{\alpha}\right)^n + \frac{B}{\beta}\sum_{n=0}^{\infty}\left(\frac{x}{\beta}\right)^n \\ &= \sum_{n=0}^{\infty}\left(\frac{A}{\alpha^{n+1}} + \frac{B}{\beta^{n+1}}\right)x^n. \end{aligned}$$

From this we deduce that

$$a_n = \frac{A}{\alpha^{n+1}} + \frac{B}{\beta^{n+1}}.$$

Since $\alpha\beta = -1$, this can be rewritten as

$$a_n = (-1)^{n+1}(B\alpha^{n+1} + A\beta^{n+1}) = \frac{(-1)^{n+1}}{\sqrt{5}}(\alpha^{n+2} - \beta^{n+2}).$$

(3) Let f_n denote the n-th Fibonacci number, i.e., $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Prove that for every $n \geq 2$ we have $f_n^2 - f_{n-1}\cdot f_{n+1} = (-1)^n$.

Solution. We proceed by induction on n. For $n=2$, we look at $f_2^2 - f_1 f_3$. Using the recurrence $f_n = f_{n-1} + f_{n-2}$, we have $f_2 = f_1 + f_0 = 2$ and $f_3 = f_2 + f_1 = 3$. So,

$$f_2^2 - f_1 f_3 = 2^2 - 1\cdot 3 = 1 = (-1)^2.$$

Suppose that $f_n^2 - f_{n-1}\cdot f_{n+1} = (-1)^n$. Then

$$\begin{aligned} f_{n+1}^2 - f_{(n+1)-1}\cdot f_{(n+1)+1} &= f_{n+1}^2 - f_n\cdot(f_{n+1}+f_n) \text{ using } f_{(n+1)+1} = f_{n+1} + f_n \\ &= f_{n+1}(f_{n+1} - f_n) - f_n^2 \end{aligned}$$

$$
\begin{aligned}
&= f_{n+1}f_{n-1} - f_n^2 \text{ using } f_{n+1} = f_n + f_{n-1} \\
&= -(-1)^n \text{ using induction assumption} \\
&= (-1)^{n+1}.
\end{aligned}
$$

Thus our assertion holds by induction. □

(4) Use generating functions to solve the following recurrence relation:

$$
\begin{aligned}
a_n &= 5a_{n-1} - 6a_{n-2} \qquad \text{for } n \geq 2, \\
a_0 &= 0; \\
a_1 &= 3.
\end{aligned}
$$

Solution. Using Theorem 3.5 (see the notes for Lectures 5 and 6), we need to solve

$$x^2 = 5x - 6, \text{ or } (x-2)(x-3) = 0.$$

Then, the general solution is given by

$$a_n = A \cdot 2^n + B \cdot 3^n.$$

Using the initial conditions $a_0 = 0$ and $a_1 = 3$, we obtain

$$0 = A \cdot 2^0 + B \cdot 3^0 = A + B \text{ and } 3 = A \cdot 2^1 + B \cdot 3^1 = 2A + 3B.$$

Thus, $A = -3$ and $B = 3$. Consequently,

$$a_n = 3(3^n - 2^n).$$

□

(5) Suppose that $f(x)$ generates the sequence $a_0, a_1, a_2, \ldots$. Give the expressions, in terms of f, for the generating functions of the following sequences:

(a) $0, a_0, 0, a_1, 0, a_2, 0, \ldots;$

Solution. For $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we have $f(x^2) = \sum_{n=0}^{\infty} a_n x^{2n}$ and $xf(x^2) = \sum_{n=0}^{\infty} a_n x^{2n+1} = 0 + a_0 x + 0x^2 + a_1 x^3 + 0x^4 + a_2 x^5 + 0x^6 + \ldots$. Hence, $xf(x^2)$ is the generating function of $0, a_0, 0, a_1, 0, a_2, 0, \ldots$. □

(b) $1, a_0, a_1, a_2, \ldots;$

Solution. For $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we have $1+xf(x) = 1+\sum_{n=0}^{\infty} a_n (x)^{n+1} = \sum_{n=0}^{\infty}(-1)^n a_n x^n =$ $1 + a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + \ldots$. Hence, $1 + xf(x)$ is the generating function of $1, a_0, a_1, a_2, a_3, a_4, \ldots$. □

(c) $a_0, -a_1, a_2, -a_3, a_4, \ldots$.

Solution. For $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we have $f(-x) = \sum_{n=0}^{\infty} a_n (-x)^n = \sum_{n=0}^{\infty}(-1)^n a_n x^n =$ $a_0 - a_1 x + a_2 x^2 - a_3 x^3 + a_4 x^4 - \ldots$. Hence, $f(-x)$ is the generating function of $a_0, -a_1, a_2, -a_3, a_4, \ldots$. □

(6) (a) Show that the generating function of the sequence $a_n = n$, $n \geq 0$, is $f(x) = \frac{x}{(1-x)^2}$.

Solution. We know that $(1-x)^{-1} = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. By differentiating both sides, we obtain that

$$\tfrac{-1}{(1-x)^2}(-1) = \sum_{n=1}^{\infty} n x^{n-1}$$

and, therefore,

$$f(x) = \tfrac{x}{(1-x)^2} = x \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^n = \sum_{n=0}^{\infty} n x^n.$$

□

(b) Find generating functions for the sequences $b_n = n^2$, $n \geq 0$, and $c_n = n^3$, $n \geq 0$.

Solution. Let $f(x)$ is the generating function of a sequence $(a_n)_{n \geq 1}$, i.e., $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$. So,

$$g(x) = xf'(x) = x \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^n$$

is the generating function of the sequence $(na_n)_{n \geq 1}$.

For $a_n = n$, $n \geq 1$, we proved in part (a) that its generating function is $f(x) = \frac{x}{(1-x)^2}$. We also notice that $b_n = n^2 = na_n$ for every $n \geq 1$. Therefore, by our earlier observation, $g(x) = xf'(x) = x(\frac{x}{(1-x)^2})' = \frac{x(1+x)}{(1-x)^3}$ is the generating function of $(b_n)_{n \geq 1}$.

For the last part, we have that $c_n = n^3 = nb_n$ for every $n \geq 1$. Therefore, by our earlier observation, $h(x) = xg'(x) = x(\frac{x(1+x)}{(1-x)^3})' = \frac{x(x^2+4x+1)}{(1-x)^4}$ is the generating function of $(c_n)_{n \geq 1}$. □

(7) Find the sequences generated by the following functions:

(a) $f(x) = \frac{x^3}{1+x}$;

Solution. We use that $\frac{1}{1-y} = \displaystyle\sum_{n=0}^{\infty} y^n$ with $y = -x$, hence $\frac{1}{1+x} = \frac{1}{1-(-x)} = \displaystyle\sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$. Consequently,

$$f(x) = \tfrac{x^3}{1+x} = x^3 \sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-1)^n x^{n+3} = \sum_{n=3}^{\infty}(-1)^{n-3}x^n.$$

Thus, $f(x)$ is the generating function of $0, 0, 0, 1, -1, 1, -1, \dots$. □

(b) $g(x) = \frac{x}{1-7x+12x^2}$;

Solution. We write

$$g(x) = \frac{x}{(1-3x)(1-4x)} = \frac{A}{1-3x} + \frac{B}{1-4x} = \frac{(A+B)+(-4A-3B)x}{(1-3x)(1-4x)}.$$

Hence, $A+B = 0$ and $-4A-3B = 1$, from which we obtain $A = -1$ and $B = 1$. Using $\frac{1}{1-y} = \displaystyle\sum_{n=0}^{\infty} y^n$ with $y = 3x$ and $y = 4x$, we have

$$g(x) = \frac{1}{1-4x} - \frac{1}{1-3x} = \sum_{n=0}^{\infty}(4x)^n - \sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty}(4^n - 3^n)x^n.$$

Hence, $g(x)$ is the generating function of $(4^n - 3^n)_{n\geq 0}$. □

(c) $h(x) = \frac{x^7}{2-x^7}$;

Solution. We see that

$$h(x) = \frac{x^7}{2-x^7} = -1 + \frac{2}{2-x^7} = -1 + \frac{1}{1-\frac{x^7}{2}} = -1 + \sum_{n=0}^{\infty}\left(\tfrac{x^7}{2}\right)^n = -1 + \sum_{n=0}^{\infty} 2^{-n}x^{7n}.$$

From this we deduce that $h(x)$ generates $(x_n)_{n\geq 1}$, where

$$x_n = \begin{cases} 2^{-n/7} & \text{if } n \geq 1 \text{ and } 7 \text{ divides } n, \\ 0 & \text{otherwise.} \end{cases}$$

□

(d) $k(x) = e^{2x}$.

Solution. We know that $e^y = \sum_{n=0}^{\infty} \frac{1}{n!}y^n$. By taking $y = 2x$, we have $k(x) = e^{2x} = \sum_{n=0}^{\infty} \frac{1}{n!}(2x)^n = \sum_{n=0}^{\infty} \frac{2^n}{n!}x^n$, so $k(x)$ is the generating function of $(\frac{2^n}{n!})_{n\geq 1}$.

□