Discrete Mathematics — Lent 2009

MA 210

Notes for lectures 5 and 6

2.2 Inclusion-Exclusion Principle (continued)

We want to prove the Inclusion Exclusion Principle:

Theorem 2.4. *Let $A_1, \dots, A_n$ be finite sets. For $X \subset \{1, \dots, n\}$, define*

$$N(X) = \left| \bigcap_{i \in X} A_i \right|$$

and, for $i, 1 \le i \le n$, define

$$\alpha_i = \sum_{X \subset \{1,\dots,n\}, |X| = i} N(X).$$

Then

$$\begin{aligned} (1) \qquad |A_1 \cup A_2 \cup \dots \cup A_n| &= \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \dots + (-1)^{n-1}\alpha_n \\ &= \sum_{i=1}^{n} \sum_{X \subset \{1,\dots,n\}, |X|=i} (-1)^i \left| \bigcap_{i \in X} A_i \right|. \end{aligned}$$

One proof is by induction. For $n = 2$, the statement follows from our earlier observation that $|A \cup B| = |A| + |B| - |A \cap B|$.

If $A_1, A_2, \dots, A_n, A_{n+1}$ are finite sets, then we use $A = A_1 \cup A_2 \cup \ldots \cup A_n$ and $B = A_{n+1}$ to conclude that

$$\begin{aligned} |A_1 \cup \dots \cup A_n \cup A_{n+1}| &= |A \cup B| \\ &= |A| + |B| - |A \cap B| \\ &= |A_1 \cup \dots \cup A_n| + |A_{n+1}| - |(A_1 \cup \dots \cup A_n) \cap A_{n+1}|. \end{aligned}$$

It is known that

$$(A_1 \cup \dots \cup A_n) \cap A_{n+1} = (A_1 \cap A_{n+1}) \cup \dots \cup (A_n \cap A_{n+1}),$$

hence

$$(2) \qquad |A_1 \cup \dots \cup A_n \cup A_{n+1}| = |A_1 \cup \dots \cup A_n| + |A_{n+1}| - |(A_1 \cap A_{n+1}) \cup \dots \cup (A_n \cap A_{n+1})|.$$

At this moment we apply induction assumption to obtain an expression for $|A_1 \cup \dots \cup A_n|$ and, also, we find a similar expression for $|(A_1 \cap A_{n+1}) \cup \dots \cup (A_n \cap A_{n+1})|$. A short calculation (left as an exercise) gives (1) the desired formula for $n + 1$ sets. $\square$

Authors: Jozef Skokan

Another proof considers any $x \in A_1 \cup A_2 \cup \cdots \cup A_n$ and finds its contribution to the both sides of (1). Clearly, x contributes 1 to the left-hand side of (1).

Suppose that x is in exactly k sets, i.e., there is $Y \subset \{1, \ldots, n\}$. $|Y| = k$, such that $x \in A_i$ if and only if $i \in Y$. Clearly, contribution of x to $\alpha_1 = |A_1| + \cdots + |A_n|$ is k because x is in k sets A_i, $i \in Y$.

Now let us look at $\alpha_2 = \sum_{i<j} |A_i \cap A_j|$. Observe that x is in precisely those intersections $A_i \cap A_j$ for which $i \in Y$ and $j \in Y$. Since $|Y| = k$, x contributes $\binom{k}{2}$ to α_2.

A similar reasoning (try it yourself!) shows that x contributes $\binom{k}{i}$ to α_i. Notice that for $i > k$ this contribution is 0, which is right because x can be in no intersection of more than k sets.

Hence, x contributes $\binom{k}{1} - \binom{k}{2} + \cdots + (-1)^{k-1}\binom{k}{k}$ to the right-hand side of (1). Using exercise 2.3, this contribution is 1. $\square$

2.3 Multinomial numbers and Multinomial Theorem

Exercise 2.5. *How many different arrangements are there of the letters of the word MATHEMATICS?*

Basic problem: Suppose we have an n-element set X whose elements come in k different types. We assume there are r_1 elements of type 1, r_2 elements of type 2,…, r_k elements of type k, so $\sum_{i=1}^{k} r_i = n$. How many ways are there to order the elements of X?

The answer is given by *multinomial number* $\binom{n}{r_1, r_2, \ldots, r_k}$.

Definition 2.6. *The multinomial number* $\binom{n}{r_1, r_2, \ldots, r_k}$ *is defined as*

$$\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! \cdot r_2! \cdot \cdots \cdot r_k!}.$$

If all elements are distinct, i.e., $k = n$ and $r_1 = \cdots = r_n = 1$, we have

$$\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{1! \ldots 1!} = n!,$$

as expected.

When $k = 2$, we have $n = r_1 + r_2$ and

$$\binom{n}{r_1, r_2} = \frac{n!}{r_1! \cdot r_2!} = \frac{n!}{r_1! \cdot (n - r_1)!} = \binom{n}{r_1}.$$

This is not surprising because we just need to choose r_1 positions out of n to place the objects of the first type, the position of the objects of the second type are then uniquely determined.

Notice that

$$\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! \cdot r_2! \cdot \cdots \cdot r_k!} = \binom{n}{r_1}\binom{n - r_1}{r_2}\binom{n - r_1 - r_2}{r_3} \cdots \binom{r_{k-1} + r_k}{r_{k-1}}\binom{r_k}{r_k}.$$

Theorem 2.7 (Multinomial Theorem). *For all (complex) numbers x_1, x_2,…, x_k and for all natural numbers n, we have*

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{r_1 + r_2 + \cdots + r_k = n \\ r_1, r_2, \ldots, r_k \geq 0}} \binom{n}{r_1, r_2, \ldots, r_k} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}.$$

3 Introduction to Recurrence Relations

3.1 Recurrence Relations

A *sequence* is a function f that maps natural numbers (or non-negative integers) to the set of real numbers. Instead of working with f itself, we set $a_n = f(n)$ and use the notation $(a_n)_{n\geq 1}$ or $(a_n)_{n\geq 0}$.

In many problems we do not know the function f, but we are able to write a_n as a function of preceeding elements in the sequence. Such relations are called *recurrence relations*. For example,

$$a_n = a_{n-1} + a_{n-2} \text{ or } b_n = b_{n-1}^2 + (n-3)b_{n-3}$$

are recurrence relations.

In order to find all values of a_n (or b_n), we need to know some initial values. For instance, the recurrence relations above will not tell us anything about $(a_n)_{n\geq 0}$ or $(b_n)_{n\geq 1}$ unless we know a_0 and a_1 or b_1, b_2, and b_3.

Definition 3.1. *A recurrence relation is a sequence $(a_n)_{n\geq 0}$ together with a relation*

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_0)$$

which holds for a certain function f and for all $n \geq N$ for some N, and with values for the initial terms $a_0, a_1, \ldots, a_{N-1}$.

A solution *or a* closed form *of a recurrence relation is a function $F(n)$, only depending on n, such that for all n, $a_n = F(n)$.*

Exercise 3.2. *Show that the recurrence relation $a_n = 2a_{n-1}$, $n \geq 1, a_0 = 1$ has a closed form $a_n = F(n) = 2^n$.*

3.2 Homogenuous linear recurrsions of order 1 or 2

A recurrence relation

$$a_n = \alpha a_{n-1}, \quad n \geq 1,$$
$$a_0 = \beta,$$

is called a homogenuous linear recurrsion of order 1 with constant coefficients.

Theorem 3.3. *A homogenuous linear recurrsion of order 1 with constant coefficients $a_n = \alpha a_{n-1}$, $n \geq 1$, $a_0 = \beta$, has the solution*

$$a_n = \beta \cdot \alpha^n.$$

Exercise 3.4. *Find a solution of the recurrence relation $a_n = 5a_{n-1} - 3$, $n \geq 1$, $a_0 = 1$.*

Hint: *set $a_n = b_n + x$ and choose a suitable value for x.*

A recurrence relation

$$a_n = \alpha a_{n-1} + \beta a_{n-2}, \quad n \geq 2,$$
$$a_0 = c_0,$$
$$a_1 = c_1,$$

is called a homogenuous linear recurrsion of order 2 with constant coefficients.

Theorem 3.5. *Suppose we have a homogenuous linear recurrsion of order 2 with constant coefficients $a_n = \alpha a_{n-1} + \beta a_{n-2}$, $n \geq 2$, $a_0 = c_0$ and $a_1 = c_1$, where $\beta \neq 0$.*
Let r_1, r_2 be the roots of the equation $x^2 = \alpha x + \beta$.

1. *If these roots are distinct ($r_1 \neq r_2$), then the recurrence relation has the solution*

$$a_n = k_1 \cdot r_1^n + k_2 \cdot r_2^n,$$

where k_1 and k_2 are constants depending on the initial conditions:

$$\begin{aligned} c_0 &= a_0 = k_1 \cdot r_1^0 + k_2 \cdot r_2^0 = k_1 + k_2, \\ c_1 &= a_1 = k_1 \cdot r_1^1 + k_2 \cdot r_2^1 = k_1 r_1 + k_2 r_2. \end{aligned}$$

2. *If these roots are equal ($r_1 = r_2 = r$), then the recurrence relation has the solution*

$$a_n = (k_1 + k_2 \cdot n) r^n,$$

where k_1 and k_2 are constants depending on the initial conditions:

$$\begin{aligned} c_0 &= a_0 = (k_1 + k_2 \cdot 0) r^0 = k_1, \\ c_1 &= a_1 = (k_1 + k_2 \cdot 1) r^1 = (k_1 + k_2) r. \end{aligned}$$

Exercise 3.6. Find a solution of the recurrence relation $a_n = a_{n-1} + b_{n-2}$, $n \geq 2$, $a_0 = a_1 = 1$.