Discrete Mathematics **Lent 2009**

MA210

Exercises 5

(1) Let a_n be the number of n-letter words formed from the 26 letters of the alphabet, in which the five vowels A, E, I, O, U together occur an even number of times. (By a word we mean simply any string of letters.) For example, when $n = 8$, such a word is APQIITOW since four (an even number) of the positions contain vowels.

(a) Show that $a_1 = 21$ and that, for $n \geq 2$, we have

$$a_n = 16a_{n-1} + 5 \cdot 26^{n-1}.$$

What would you say that a_0 is?

(b) Find the generating function for the sequence $a_0, a_1, \ldots$.

(c) Use this generating function to find a closed form expression for a_n.

(2) Let $f(x)$ be the generating function for the sequence $a_0, a_1, \ldots$. Find the sequence whose generating function is $(1-x)f(x)$.

(3) (a) Suppose we role a normal dice. Let d_n be the number of possible ways to role a dice so that the outcome is n. Explain why the generating function of the sequence $d_0, d_1, \ldots$ is

$$f(x) = x + x^2 + x^3 + x^4 + x^5 + x^6.$$

(b) Suppose that we role 4 dices. Let a_n be the number of throws such that the sum of outcomes is equal to n. Explain why the generating function of the sequence $a_0, a_1, \ldots$ is

$$g(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^4.$$

(c) Now let b_n be the number of throws with any number of dices such that the sum of outcomes is equal to n. Explain why the generating function of the sequence $b_0, b_1, \ldots$ is

$$h(x) = \sum_{n=0}^{\infty}(x + x^2 + x^3 + x^4 + x^5 + x^6)^n.$$

Also prove that

$$h(x) = (1 - x - x^2 - x^3 - x^4 - x^5 - x^6)^{-1}.$$

(4) The British coin system has 1p, 2p, 5p, 10p, 20p, 50p, £1 = 100p, and £2 = 200p coins. Let a_n count the number of different ways that ypu can pay a sum of n pennies. Show that the generating function of $a_0, a_1, \ldots$ is

$$h(x) = \frac{1}{1 - x - x^2 - x^5 - x^{10} - x^{20} - x^{50} - x^{100} - x^{200}}.$$

Hint: compare to the previous question.

(5) The language of Verwegistan has words consisting of the letters A,E,O,U,B,P, and X. Words are formed according to the following rules: the vowels (A,E,I,O,U) always appear in pairs of the form AA,EE,OO, or UU, and they appear in a word before all non-vowels (if any). For instance, AAEEPXP and AAAA are words, but UUUB, AAXBAAX, and AEXX are not.

Let a_n denote the number of words of length n.

(a) Show that $a_0 = 1$, $a_1 = 3$, and
$$a_n = 4a_{n-2} + 3^n, \qquad \text{for } n \geq 2.$$

(b) Let $f(x)$ be the generating function of the sequence $a_0, a_1, \ldots$. Show that
$$f(x) = \frac{1}{(1-3x)(1-4x^2)}.$$

(c) Use this generating function to find a general expression for a_n.

(6) Let d_n denote the number of selections of n letters from $\{a, b, c\}$, with repetitions allowed, in which the letter a is selected an even number of times. (Note that these selections are unordered.)

(a) Show that the total number of unordered selections of n letters from $\{a, b, c\}$ with repetitions allowed is $\binom{n+2}{2}$.

(b) Use the result in (a) to prove that for $n \geq 2$,
$$d_n = \binom{n+2}{2} - d_{n-1} = \frac{1}{2}(n+2)(n+1) - d_{n-1}.$$

(c) Show that the sequence $d_0, d_1, \ldots$ has the generating function
$$f(x) = \frac{1}{(1-x^2)(1-x)^2} = \frac{1}{(1+x)(1-x)^3}.$$

(d) Use this generating function to prove that
$$f(x) = \begin{cases} \frac{1}{4}(n+2)^2 & \text{if } n \text{ is even,} \\ \frac{1}{4}(n+1)(n+3) & \text{if } n \text{ is odd.} \end{cases}$$

You must justify the answers to all problems!

These exercises are to be handed in **before 4.55pm on February 16, 2009**.