Discrete Mathematics **Lent 2009**

MA210

Solutions to Exercises 5

(1) Let a_n be the number of n-letter words formed from the 26 letters of the alphabet, in which the five vowels A, E, I, O, U together occur an even number of times. (By a word we mean simply any string of letters.) For example, when $n = 8$, such a word is APQIITOW since four (an even number) of the positions contain vowels.

(a) Show that $a_1 = 21$ and that, for $n \geq 2$, we have

$$a_n = 16a_{n-1} + 5 \cdot 26^{n-1}.$$

What would you say that a_0 is?

Solution. Any single letter except a vowel is a valid word because it contains 0 vowels. Hence, $a_1 = 26 - 5 = 21$. An empty word (a word containing no letters) also contains 0 vowels, hence $a_0 = 1$.

Let $w_1 w_2 \dots w_n$ be a valid word, that is, with an even number of vowels.

If w_n is not a vowel (21 options for this), then $w_1 w_2 \dots w_{n-1}$ must contain the same even number of vowels as the original word, in other words, $w_1 w_2 \dots w_{n-1}$ is a valid word of length $n-1$ (there are a_{n-1} valid words of length $n-1$). Hence, we have $21a_{n-1}$ valid words of length n ending with a consonant.

If w_n is a vowel (5 options for this), then $w_1 w_2 \dots w_{n-1}$ must contain an odd number of vowels (one less than the original word $w_1 w_2 \dots w_n$). How many such words are there? There are 26^{n-1} words of length $n-1$ made of 26 letters with repetition allowed. a_{n-1} of them contain an even number of vowels. Hence, $26^{n-1} - a_{n-1}$ words $w_1 w_2 \dots w_{n-1}$ contain an odd number of vowels.. Hence, we have $5(26^{n-1} - a_{n-1})$ valid words of length n ending with a vowel.

Hence,

$$a_n = 21a_{n-1} + 5(26^{n-1} - a_{n-1}) = 16a_{n-1} + 5 \cdot 26^{n-1}.$$

□

(b) Find the generating function for the sequence $a_0, a_1, \dots$.

Solution. We have $f(x) = \sum_{n=0}^{\infty} a_n x^n$, therefore,

$$f(x) = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{\infty} (16a_{n-1} + 5 \cdot 26^{n-1}) x^n =$$

$$a_0 + a_1 x + 16x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 5x \sum_{n=2}^{\infty} 26^{n-1} x^{n-1} =$$

$$1 + 21x + 16x\sum_{n=1}^{\infty} a_n x^n + 5x\sum_{n=1}^{\infty} 26^n x^n =$$

$$1 + 21x + 16x(f(x) - 1) + 5x(-1 + \sum_{n=0}^{\infty}(26x)^n) = 1 + 16xf(x) + \frac{5x}{1-26x}.$$

Hence,

$$(1-16x)f(x) = 1 + \frac{5x}{1-26x},$$

that is,

$$f(x) = \frac{1-21x}{(1-16x)(1-26x)}.$$

□

(c) Use this generating function to find a closed form expression for a_n.
Solution. We write $f(x)$ as

$$\begin{aligned} f(x) &= \frac{1-21x}{(1-16x)(1-26x)} = \frac{\frac{1}{2}}{1-16x} + \frac{\frac{1}{2}}{1-26x} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(16x)^n + \frac{1}{2}\sum_{n=0}^{\infty}(26x)^n = \sum_{n=0}^{\infty}\tfrac{1}{2}(16^n + 26^n)x^n. \end{aligned}$$

So,

$$a_n = \tfrac{1}{2}(16^n + 26^n).$$

□

(2) Let $f(x)$ be the generating function for the sequence $a_0, a_1, \ldots$. Find the sequence whose generating function is $(1-x)f(x)$.

Solution. We have $f(x) = \sum_{n=0}^{\infty} a_n x^n$, therefore, $(1-x)f(x) = f(x) - xf(x) = \sum_{n=0}^{\infty} a_n x^n - x\sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = a_0 + \sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = a_0 + \sum_{n=1}^{\infty}(a_n - a_{n-1})x^n$. Hence, $(1-x)f(x)$ generates the sequence $(b_n)_{n\geq 0}$, where

$$b_n = \begin{cases} a_n & \text{for } n = 0, \\ a_n - a_{n-1} & \text{for } n > 0. \end{cases}$$

□

(3) (a) Suppose we role a normal dice. Let d_n be the number of possible ways to role a dice so that the outcome is n. Explain why the generating function of the sequence $d_0, d_1, \ldots$ is

$$f(x) = x + x^2 + x^3 + x^4 + x^5 + x^6.$$

Solution. There is one way to get $1, 2, 3, 4, 5$, or 6 and no way to get 0 or any number larger than 6. Hence,

$$d_n = \begin{cases} 1 & \text{for } 1 \le n \le 6, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, $f(x) = \sum_{n=0}^{\infty} d_n x^n = s + x^2 + x^3 + x^4 + x^5 + x^6$. □

(b) Suppose that we role 4 dices. Let a_n be the number of throws such that the sum of outcomes is equal to n. Explain why the generating function of the sequence $a_0, a_1, \dots$ is

$$g(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^4.$$

Solution. □

(c) Now let b_n be the number of throws with any number of dices such that the sum of outcomes is equal to n. Explain why the generating function of the sequence $b_0, b_1, \dots$ is

$$h(x) = \sum_{n=0}^{\infty} (x + x^2 + x^3 + x^4 + x^5 + x^6)^n.$$

Also prove that

$$h(x) = (1 - x - x^2 - x^3 - x^4 - x^5 - x^6)^{-1}.$$

Solution. □

(4) The British coin system has 1p, 2p, 5p, 10p, 20p, 50p, £1 = 100p, and £2 = 200p coins. Let a_n count the number of different ways that you can pay a sum of n pennies. Show that the generating function of $a_0, a_1, \dots$ is

$$h(x) = \frac{1}{1 - x - x^2 - x^5 - x^{10} - x^{20} - x^{50} - x^{100} - x^{200}}.$$

Solution. Just follow the previous problem. Start with finding the generating function of $(d_n)_{n \ge 0}$, where d_n is the number of ways to pay a sum of n pennies with one coin. □

(5) The language of Verwegistan has words consisting of the letters A,E,O,U,B,P, and X. Words are formed according to the following rules: the vowels (A,E,I,O,U) always appear in pairs of the form AA,EE,OO, or UU, and they appear in a word before all non-vowels (if any). For instance, AAEEPXP and AAAA are words, but UUUB, AAXBAAX, and AEXX are not.

Let a_n denote the number of words of length n.

(a) Show that $a_0 = 1$, $a_1 = 3$, and

$$a_n = 4a_{n-2} + 3^n, \qquad \text{for } n \ge 2.$$

Solution. The empty word satisfies the rules above, so $a_0 = 1$. The only valid one-letter words are consonants, i.e., B,P, and X, hence, $a_1 = 3$.
Take any n-letter word. If it starts wit a consonant (3 choices), then all the remaining letters must be also consonants because the vowels (A,E,I,O,U) always appear in a word before all non-vowels. Hence, we have 3^n such words.
If it starts wit a vowel (5 choices), then the second letter must be the same vowel (vowels come in pairs) and the remaining $n-2$ letters form a valid word again (we have a_{n-2} of them). Hence, we have $5a_{n-2}$ such n-letter words. So,

$$a_n = 4a_{n-2} + 3^n, \qquad \text{for } n \geq 2.$$

□

(b) Let $f(x)$ be the generating function of the sequence $a_0, a_1, \ldots$. Show that

$$f(x) = \frac{1}{(1-3x)(1-4x^2)}.$$

Solution. We have $f(x) = \sum_{n=0}^{\infty} a_n x^n$, therefore,

$$f(x) = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = 1 + 3x + \sum_{n=2}^{\infty}(4a_{n-2} + 3^n)x^n =$$

$$1 + 3x + 4x^2 \sum_{n=2}^{\infty} a_{n-2}x^{n-2} + \sum_{n=2}^{\infty} 3^n x^n =$$

$$4x^2 \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty}(3x)^n = 4x^2 f(x) + \frac{1}{1-3x}.$$

Hence,

$$(1-4x^2)f(x) = \frac{1}{1-3x},$$

that is,

$$f(x) = \frac{1}{(1-4x^2)(1-3x)} = \frac{1}{(1-2x)(1+2x)(1-3x)}.$$

□

(c) Use this generating function to find a general expression for a_n.
Solution. Since

$$f(x) = \frac{1}{(1-2x)(1+2x)(1-3x)} = \frac{\frac{1}{5}}{1+2x} - \frac{1}{1-2x} + \frac{\frac{9}{5}}{1-3x},$$

we have

$$f(x) = \frac{1}{5}\sum_{n=0}^{\infty}(-2x)^n - \sum_{n=0}^{\infty}(2x)^n + \frac{9}{5}\sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty}\left(\frac{1}{5}(-2)^n - 2^n + \frac{9}{5}3^n\right)x^n.$$

So,

$$a_n = \frac{1}{5}(-2)^n - 2^n + \frac{9}{5}3^n.$$

□

(6) Let d_n denote the number of selections of n letters from $\{a, b, c\}$, with repetitions allowed, in which the letter a is selected an even number of times. (Note that these selections are unordered.)

(a) Show that the total number of unordered selections of n letters from $\{a, b, c\}$ with repetitions allowed is $\binom{n+2}{2}$.

Solution. We know that the number of unordered selections of n objects from r possible is is $\binom{n+r-1}{r-1}$. In this case, we have $r = 3$. □

(b) Use the result in (a) to prove that for $n \geq 2$,

$$d_n = \binom{n+2}{2} - d_{n-1} = \frac{1}{2}(n+2)(n+1) - d_{n-1}.$$

Solution. By (a), there are $\binom{n+2}{2}$ possible unordered selections of n letters. How many of them have an odd number of a's? If we remove one a from such a selection, we obtain an unordered selection of $n-1$ letters with an even number of a's. Each time we get a different selection (Why?). But we know that the number of unordered selection of $n-1$ letters with an even number of a's is d_{n-1}.

Consequently, $d_n = \binom{n+2}{2} - d_{n-1}$. □

(c) Show that the sequence $d_0, d_1, \ldots$ has the generating function

$$f(x) = \frac{1}{(1-x^2)(1-x)^2} = \frac{1}{(1+x)(1-x)^3}.$$

Solution. First of all, we see that $d_0 = 1$ and $d_1 = 2$. (Why?) We also recall that for every positive integer r, we have

$$(1-x)^{-r} = \sum_{n=0}^{\infty} \binom{n+r-1}{n} x^n = \sum_{n=0}^{\infty} \binom{n+r-1}{r-1} x^n.$$

Let $f(x) = \sum_{n=0}^{\infty} d_n x^n$ be the generating function of $(d_n)_{n\geq 0}$. Then,

$$\begin{aligned}
f(x) &= d_0 + d_1 x + \sum_{n=2}^{\infty} d_n x^n = d_0 + d_1 x + \sum_{n=2}^{\infty} \left(\binom{n+2}{2} - d_{n-1} \right) x^n \\
&= 1 + 2x + \sum_{n=2}^{\infty} \binom{n+2}{2} x^n - x \sum_{n=2}^{\infty} d_{n-1} x^{n-1} \\
&= 1 + 2x + \sum_{n=0}^{\infty} \binom{n+2}{2} x^n - \binom{2}{2} - \binom{1+2}{2} x - x \left(\sum_{n=0}^{\infty} d_n x^n - d_0 \right) \\
&= -x + (1-x)^{-3} - x(f(x) - 1) = x f(x) + \frac{1}{(1-x)^3}.
\end{aligned}$$

Hence,

$$(1-x)f(x) = \frac{1}{(1-x)^3},$$

that is,

$$f(x) = \frac{1}{(1-x)^3(1+x)}.$$

□

(d) Use this generating function to prove that

$$f(x) = \begin{cases} \frac{1}{4}(n+2)^2 & \text{if } n \text{ is even,} \\ \frac{1}{4}(n+1)(n+3) & \text{if } n \text{ is odd.} \end{cases}$$

Solution. We use partial fractions and rewrite (work out the details!) $f(x)$ as

$$f(x) = \frac{1}{(1-x)^3(1+x)} = \frac{\frac{1}{8}}{1+x} + \frac{\frac{1}{8}}{1-x} + \frac{\frac{1}{4}}{(1-x)^2} + \frac{\frac{1}{2}}{(1-x)^3}.$$

Using

$$(1-x)^{-r} = \sum_{n=0}^{\infty} \binom{n+r-1}{n} x^n = \sum_{n=0}^{\infty} \binom{n+r-1}{r-1} x^n,$$

we have

$$f(x) = \frac{1}{8}\sum_{n=0}^{\infty}(-x)^n + \frac{1}{8}\sum_{n=0}^{\infty} x^n + \frac{1}{4}\sum_{n=0}^{\infty}\binom{n+2-1}{2-1}x^n + \frac{1}{2}\sum_{n=0}^{\infty}\binom{n+3-1}{3-1}x^n,$$

or,

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{1}{8}(-1)^n + \frac{1}{8} + \frac{1}{4}\binom{n+1}{1} + \frac{1}{2}\binom{n+2}{2}\right)x^n.$$

Thus,

$$\begin{aligned} d_n &= \frac{1}{8}(-1)^n + \frac{1}{8} + \frac{1}{4}\binom{n+1}{1} + \frac{1}{2}\binom{n+2}{2} \\ &= \frac{(-1)^n+1}{8} + \frac{n+1}{4} + \frac{(n+2)(n+1)}{4} = \frac{(-1)^n+1}{8} + \frac{(n+3)(n+1)}{4}. \end{aligned}$$

For n even,

$$d_n = \frac{(-1)^n+1}{8} + \frac{(n+3)(n+1)}{4} = \frac{2}{8} + \frac{n^2+4n+3}{4} = \frac{1+n^2+4n+3}{4} = \frac{(n+2)^2}{4}.$$

For n odd,

$$\frac{(-1)^n+1}{8} + \frac{(n+3)(n+1)}{4} = \frac{0}{8} + \frac{(n+3)(n+1)}{4} = \frac{(n+3)(n+1)}{4}.$$

□