Discrete Mathematics **Lent 2009**

MA210

Exercises 7

(1) The *complete bipartite graph* $K_{m,n}$ is defined by taking two disjoint sets, V_1 of size m and V_2 of size n, and putting an edge between u and v whenever $u \in V_1$ and $v \in V_2$.

(a) How many edges does $K_{m,n}$ have?

(b) What is the degree sequence of $K_{m,n}$?

(c) Which complete bipartite graphs $K_{m,n}$ are connected?

(d) Which complete bipartite graphs $K_{m,n}$ have an Euler tour?

(e) Which complete bipartite graphs $K_{m,n}$ have a Hamilton cycle?

Justify your answers.

(2) The *cube graph* Q_n was defined in lectures: the vertices of Q_n are all sequences of lenght n with entries from $\{0, 1\}$ and two sequences are joined by an edge if they differ in exactly one position.

(a) How many edges does Q_n have?

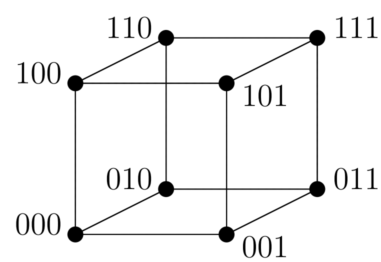
(b) What is the degree sequence of Q_n?

(c) Which cube graphs Q_n are connected?

(d) Which cube graphs Q_n have an Euler tour?

(e) Which cube graphs Q_n have a Hamilton cycle?

Justify your answers.



The cube graph Q_3

(3) Suppose that G is a graph in which every vertex has degree at least k, where $k \geq 1$, and in which every cycle contains at least 4 vertices.

(a) Show that G contains a path of length at least $2k - 1$.

(b) For each $k \geq 1$, give an example of a graph in which every vertex has degree at least k, every cycle contains at least 4 vertices, but which does not contain a path of length $2k$.

(4) Show that the cube graph Q_n is bipartite.

(5) We call a graph *tree* if it is connected and contains no cycles. Prove that if G is a connected graph with n vertices and $n-1$ edges, then G is a tree.

(6) Recall that the complement of a graph $G = (V, E)$ is the graph $\bar{G}$ with the same vertex V and for every two vertices $u, v \in V$, uv is an edge in $\bar{G}$ if and only if uv is not and edge of G.

Suppose that G is a graph on n vertices such that G is isomorphic to its own complement $\bar{G}$. Prove that $n \equiv 0 (\mod 4)$ or $n \equiv 1 (\mod 4)$.

(7) A mouse intends to eat a $3 \times 3 \times 3$ cube of cheese. Being tidy-minded, it begins at a corner and eats the whole of a $1 \times 1 \times 1$ cube, before going on to an adjacent one.

Can the mouse end in the centre ?

You must justify the answers to all problems!

These exercises are to be handed in **before 13.55pm on March 3, 2009**.