**Discrete Mathematics** **Lent 2009**

**MA210**

# Exercises 8

(1) Suppose that $G$ is a graph in which every vertex has degree at least $k$, where $k \geq 1$, and in which every cycle contains at least 4 vertices.
  (a) Show that $G$ contains a path of length at least $2k - 1$.
  (b) For each $k \geq 1$, give an example of a graph in which every vertex has degree at least $k$, every cycle contains at least 4 vertices, but which does not contain a path of length $2k$.

(2) How many non-isomorphic trees with five vertices are there? Let $V = \{1, 2, 3, 4, 5\}$. How many different trees with vertex set $V$ are there?
  (Hint: do not try to list all the trees on $V = \{1, 2, 3, 4, 5\}$. Find all the non-isomorphic trees first and then count in how many different ways you can assign vertices labels from $\{1, 2, 3, 4, 5\}$.)

(3) Prove that if $G$ is a connected graph with $n$ vertices and $n - 1$ edges, then $G$ is a tree.

(4) Let $G$ be a graph. Prove that $G$ is a tree if and only if for every pair of vertices $u$ and $v$, there is a unique path between $u$ and $v$.

(5) Suppose that $G$ is a forest with $n$ vertices and $c$ components. Prove that $G$ has $n - c$ edges.

(6) Prove by induction that every tree is a bipartite graph. (Do not use the theorem about the characterization of bipartite graphs from lectures. This problem is easy to prove directly.)

(7) (a) How many spanning trees does the graph $P_n$ have?
  (b) How many spanning trees does the graph $C_n$ have?
  (c) How many spanning trees does the graph $K_4$ have?

(8) Let the graph $K_n$ have vertices $\{1, 2, \ldots, n\}$ and suppose that for each $u, v \in \{1, 2, \ldots, n\}$, the edge $uv$ has weight $c_{uv} = u + v$. Determine the minimal spanning tree of this graph. What is the total cost for this minimal spanning tree?

You must justify the answers to all problems!

These exercises are to be handed in **before 13.55pm on March 10, 2009**.