Discrete Mathematics **Lent 2009**

MA210

Exercises 9

(1) There are 5 cities. The cost of building a road directly between i and j is the entry $a_{i,j}$ in the matrix below. An indefinite entry indicates that the road cannot be built. Determine the least cost of making all the cities reachable from each other.

$$\begin{pmatrix} 0 & 3 & 5 & 11 & 9 \\ 3 & 0 & 3 & 9 & 8 \\ 5 & 3 & 0 & \infty & 10 \\ 11 & 9 & \infty & 0 & 7 \\ 9 & 8 & 10 & 7 & 0 \end{pmatrix}$$

(2) For natural numbers n and p, let $G(n,p)$ be the complete graph with vertex set $\{1,2,\ldots,n\}$, and let the weight of the edge ij be given by $c_{ij} = |i-j| \mod p$. (So $c_{ij} \in \{0,1,\ldots,p-1\}$.) For every n and p, determine the minimum weight of a spanning tree in $G(n,p)$.

(Do not expect to be able to write down the answer; **try** it for **a few small values** of n and p to see what is going on. The **answer** also **depends on** which of n and p is larger.)

(3) (a) What is the chromatic number of the complete graph K_n?
   (b) What is the chromatic number of the path P_n?
   (c) What is the chromatic number of the cycle C_n?

(4) Prove or disprove:
   (a) Every k-chromatic graph G has a proper k-colouring in which some colour class has $\alpha(G)$ vertices.

(b) For every n-vertex graph G, $\chi(G) \leq n - \alpha(G) + 1$.

(c) For every two vertex disjoint graphs G and H, $\chi(G+H) = \max\{\chi(G), \chi(H)\}$. Here, $G + H$ is defined as follows: Let $G = (V, E)$ and $H = (V', E')$ be two graphs with disjoint vertex sets, i.e., $V \cap V' = 0$. The disjoint union of G and H, denoted by $G + H$, is the graph with vertex set $V \cup V'$ and edge set $E \cup E'$.

(5) Let G be a graph. Prove that there exists some ordering of the vertices of G such that the greedy algorithm uses exactly $\chi(G)$ colours.

(6) Find the minimum distance for the following codes:

(a) $C_1 = \{10000, 01010, 00001\}$;

(b) $C_2 = \{0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111\}$;

(c) $C_3 = \{000000, 101010, 010101\}$.

Suppose we want to add extra codewords to the codes above. For which of the three of them is that possible without altering the minimum distance?

(7) Prove the triangle inequality: for all $\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{z}} \in \{0, 1\}^n$,

$$d_H(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}) + d_H(\bar{\boldsymbol{z}}, \bar{\boldsymbol{y}}) \geq d_H(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}).$$

(8) Construct a binary code C of length 6 such that $|C| = 5$ and C is 1-error-correcting.

You must justify the answers to all problems!

These exercises are to be handed in **before 13.55pm on March 17, 2009**.