# Discrete Mathematics — Lent 2009
# MA 210

## Notes for lectures 15 and 16

## 3 Introduction to Graph Theory

### 3.1 Basic definitions

**Definition 3.1.** *A graph $G = (V(G), E(G))$ is a set of $V(G)$ of vertices together with a set $E(G)$ of edges, where $E(G)$ is a subset of*

$$\binom{V(G)}{2} = \{A \subset V(G) \mid |A| = 2\}.$$

In this course we always assume that $V(G)$ is finite. This, of course, means that $E(G)$ is also finite. If there is no danger of confusion, we write $G = (V, E)$ instead of $G = (V(G), E(G))$.

An edge consists of two vertices, its *end vertices*. In general, we denote an edge $\{u, v\}$ by $uv$, and hence $vu$ denotes the same edge. If $uv \in E(G)$, then we say that $u$ and $v$ are *adjacent* or *neighbours* (in $G$).

If $u \in V(G)$ and $uv \in E(G)$, then we say that the vertex $U$ and the edge $uv$ are *incident*.

If $u \in V(G)$, then the *degree* of $u$, denoted by $d(u)$, is the number of vertices in $V(G)$ adjacent to $u$.

The *degree sequence* of a graph is a list of the degree of all the vertices, written in non-increasing order.

**Theorem 3.2** (Hand-shaking Lemma)**.** *For any finite graph $G$, we have*

$$\sum_{u \in V(G)} d(u) = 2|E(G)|.$$

**Corollary 3.3.** *In any finite graph, the number of vertices of odd degree is even.*

We say that a graph is *$k$-regular* if its every vertex has degree $k$.

**Exercise 3.4.** *How many edges does a $k$-regular graph on $n$ vertices have?*

### 3.2 Isomorphism of graphs

Two graphs $G$ and $H$ are the *same* if $V(G) = V(H)$ and $E(G) = E(H)$.

**Definition 3.5.** *Two graphs $G$ and $H$ are isomorphic if there exists a bijection $\phi : V(G) \to V(H)$ such that*

$$uv \in E(G) \text{ if and only if } \phi(u)\phi(v) \in E(H).$$

Isomorphic graphs must have the same number of vertices, edges, the same degree sequences (why?) and many other structural properties.

However: it is not true that two graphs with the same number of vertices, edges and with the same degree sequences must necessarily be isomorphic!

There are many ways in which we can represent a graph $G$. Some of these are:

- a picture of the graph;
- a list of all its vertices and all its edges;
- an *adjacency matrix*: this is an $|V(G)| \times |V(G)|$ matrix in which the rows and columns are indexed by the elements of $V(G)$ and we put 1 in the entry for row $u$ and column $v$ if $uv \in E(G)$ and 0 otherwise;
- an *adjacency list*, i.e., for each vertex we list all the other vertices it is adjacent to.

## 3.3 Walks, paths, connectivity,tours, cycles

A *walk* in a graph $G$ is a sequence of vertices $v_1, v_2, \ldots, v_m$ such that $v_j$ is adjacent to $v_{j+1}$ for all $j = 1, 2, \ldots, m-1$. The *length* of the walk is in this case equal to $m-1$, that is, equal to the number of edges in the walk.

If the edges $vjv_{j+1}$, $j = 1, \ldots, m-1$, of a walk are all distinct, then we talk about a *trail*. If, in addition, all the vertices $v_j$ are distinct, we have a *path*.

Thus, every path is also a trail, and every trail is also a walk.

**Definition 3.6.** *We say that a graph is connected if for every two vertices $u$ and $v$, we can find a walk $v_1, v_2, \ldots, v_m$ such that $v_1 = u$ and $v_m = v$.*

Yous should be able to convince yourself that this definition does not change if we replace the walk by a trail or a path.

A walk in a graph is *closed* if it has positive length and the first and the last vertex are the same. A closed trail is called a *tour*. If, in addition, all the vertices of a trail, except the first and the last, are distinct, then we have a *cycle*.

Tours and cycles are often denoted by $v_1, v_2, \ldots, v_m, v_1$. The length of this tour or cycle is equal to $m$.

An *Euler trail* in a graph is a trail in which every edge of the graph appears exactly once. An *Euler tour* is a tour with this property.

A *Hamilton cycle* is a cycle in which every vertex of a graph appears exactly once in the cycle.

## 3.4 The first theorem in graph theory

**Lemma 3.7.** *Let a graph have at least one edge and no vertex of degree one. Then the graph contains a cycle.*

**Theorem 3.8** (Euler, 1736)**.**

1. *A connected graph contains an Euler tour if and only if every vertex has even degree.*

2. *A connected graph contains an Euler trail if and only if the graph has exactly two vertices of odd degree.*