# The Uniform Integrability of Martingales. On a Question by Alexander Cherny*

Johannes Ruf†
Department of Mathematics
University College London

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**Abstract**

Let $X$ be a progressively measurable, almost surely right-continuous stochastic process such that $X_\tau \in L^1$ and $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$ for each finite stopping time $\tau$. In 2006, Cherny showed that $X$ is then a uniformly integrable martingale provided that $X$ is additionally nonnegative. Cherny then posed the question whether this implication also holds even if $X$ is not necessarily nonnegative. We provide an example that illustrates that this implication is wrong, in general. If, however, an additional integrability assumption is made on the limit inferior of $|X|$ then the implication holds. Finally, we argue that this integrability assumption holds if the stopping times are allowed to be randomized in a suitable sense.

**Key words:** Stopping time; Uniform integrability
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## 1 Introduction

We fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ with expectation operator $\mathbb{E}[\cdot]$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ and $\mathcal{F}_\infty = \bigvee_{t\in[0,\infty)} \mathcal{F}_t \subset \mathcal{F}$. Furthermore, we fix a progressively measurable, almost surely right-continuous process $X$. We write $Z \in L^1$ if $\mathbb{E}[|Z|] < \infty$ for some random variable $Z$. For some $\mathbb{F}$–adapted process $Y$ and some stopping time $\eta$ we write $Y^\eta$ to denote the process $Y$ stopped at time $\eta$; to wit, $Y^\eta_t = Y_{\eta\wedge t}$ for each $t \in [0,\infty)$. All identifications and statements in the following are in the almost-sure sense.

We consider the following five statements:

(I) $X$ is a uniformly integrable martingale.

(II) $X_\infty = \lim_{t\uparrow\infty} X_t$ exists, $X_\tau \in L^1$ and $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$ for all stopping times $\tau$.

(III) $X_\tau \in L^1$ and $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$ for all finite stopping times $\tau$ and $\liminf_{t\uparrow\infty} |X_t| \in L^1$.

(IV) $X_\tau \in L^1$ and $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$ for all finite stopping times $\tau$.

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†E-Mail: j.ruf@ucl.ac.uk

(V) $X_t \in L^1$ and $\mathbb{E}[X_t] = \mathbb{E}[X_0]$ for all $t \in [0,\infty)$.

The optional sampling theorem yields the implications (I) $\Rightarrow$ (II) $\Rightarrow$ (III) $\Rightarrow$ (IV) $\Rightarrow$ (V). The implication (II) $\Rightarrow$ (I) follows from the following simple argument. Fix $s, t \in [0,\infty]$ with $s < t$ and $A \in \mathcal{F}_s$. Let $\tau_1 = s$ and $\tau_2 = s\mathbf{1}_{A^c} + t\mathbf{1}_A$ denote two stopping times, where $A^c = \Omega \setminus A$. Then, by assumption, $\mathbb{E}[X_{\tau_1}] = \mathbb{E}[X_{\tau_2}]$, which yields $\mathbb{E}[X_s \mathbf{1}_A] = \mathbb{E}[X_t \mathbf{1}_A]$. Thus we obtain the desired implication (II) $\Rightarrow$ (I). However, if only (III) or (IV) is assumed, this argument only yields the martingale property of $X$ but not its uniform integrability.

Cherny (2006) now asks the question whether also the implication (IV) $\Rightarrow$ (I) holds. Before answering this question and discussing the role of (III), let us first briefly consider the statement in (V). Hulley (2009) provides an example of a local martingale $X$, such that (V) is satisfied but $X$ is not a martingale. Alternatively, if $X$ denotes Brownian motion started in 0 then (V) holds but (IV) is not satisfied. To see this, we only need to let $\tau$ denote the first hitting time of level 1 by $X$. Thus, the implication (V) $\Rightarrow$ (IV) does *not* hold in general.

We now return to discuss the missing implications, namely whether (III) or, more generally (IV), imply (I) (or equivalently, (II)). Cherny (2006) proves that these implications hold if $X$ is nonnegative. The following theorem proves that the implication (III) $\Rightarrow$ (I) holds always, not only if $X$ is nonnegative. However, the example of the next section shows that (IV) does not necessarily imply any of the statements (I) – (III) if the nonnegativity assumption on $X$ is dropped. Yet, as proven in Section 3, if the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ allows for some additional randomization, then these implications hold. More precisely, if the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ allows for a $(0,1)$–uniformly distributed random variable, measurable with respect to $\mathcal{F}_\eta$ for some finite stopping time $\eta$, then the implications (IV) $\Rightarrow$ (I), (IV) $\Rightarrow$ (II), and (IV) $\Rightarrow$ (III) hold.

**Theorem 1.** *The statements* (I)*,* (II)*, and* (III) *are equivalent.*

*Proof.* We only need to show the implication (III) $\Rightarrow$ (I). We start by arguing that we may assume, without loss of generality, that $\mathbb{F}$ and $X(\omega)$ are right-continuous for each $\omega \in \Omega$. Towards this end, we set $\mathcal{F}_t^+ = \bigcap_{s>t} \mathcal{F}_s$ and $\mathbb{F}^+ = (\mathcal{F}_t^+)_{t\in[0,\infty)}$. Next, we observe that the $\mathbb{F}$–martingale $X$ is also an $\mathbb{F}^+$–martingale due to Exercise 1.5.8 in Stroock and Varadhan (2006). Now, Lemma 1.1 in Föllmer (1972) yields the existence of a right-continuous version of $X$, which we call again $X$. Next, we fix a finite $\mathbb{F}^+$–stopping time $\widehat{\sigma}$ and set $\sigma = \widehat{\sigma} + 1$, which is a finite $\mathbb{F}$–stopping time by Theorem IV.57 in Dellacherie and Meyer (1978). Then, $X^\sigma$ satisfies (II) and is therefore a uniformly integrable $\mathbb{F}^+$–martingale. The optional sampling theorem then also yields that $X_{\widehat{\sigma}} \in L^1$ and $\mathbb{E}[X_{\widehat{\sigma}}] = \mathbb{E}[X_0]$. Thus, (III) also holds for all finite $\mathbb{F}^+$–stopping times, and we shall assume from now on, throughout this proof, that $\mathbb{F}$ and $X(\omega)$ are right-continuous for each $\omega \in \Omega$.

We now construct a nondecreasing sequence $(T_n)_{n\in\mathbb{N}}$ of $[0,\infty)$–valued random variables such that $\lim_{n\uparrow\infty} |X_{T_n}| = \liminf_{t\uparrow\infty} |X_t| \in \mathcal{F}_\infty$. For example, we can choose $T_n$ as the first time that $||X| - \liminf_{t\uparrow\infty} |X_t|| \le 1/n$. Then $T_n$ is $\mathcal{F}_\infty$–measurable, due to the right-continuity of $X$, for each $n \in N$ (but not necessarily a stopping time). Now we set $Y = \liminf_{n\uparrow\infty} X_{T_n} \in \mathcal{F}_\infty$ and note that $|Y| = \liminf_{t\uparrow\infty} |X_t|$, thus $Y \in L^1$ by assumption.

Next, let us consider the martingale $M$ given by $M_t = X_t - \mathbb{E}[Y|\mathcal{F}_t]$ for each $t \in [0,\infty)$, where we may use a right-continuous modification of the conditional expectation process thanks to $\mathbb{F}$ being right-continuous; see again Lemma 1.1 in Föllmer (1972). Then (III) holds with $X$ replaced by $M$. We note that $\liminf_{t\uparrow\infty} |M_t| = 0$ since $\liminf_{n\uparrow\infty} M_{T_n} = \liminf_{n\uparrow\infty} X_{T_n} - Y = 0$, thanks to Lévy's martingale convergence theorem and the fact that $Y \in \mathcal{F}_\infty$.

It is sufficient to show that the martingale $M$ is uniformly integrable, or, equivalently that $M \equiv 0$. Towards this end, we assume that there exists $\varepsilon \in (0,1)$ such that $\mathbf{P}(\sup_{t\in[0,1/\varepsilon)} M_t > \varepsilon) > \varepsilon$. We then let $\sigma_1$ be the first time that $M$ is greater than or equal to $\varepsilon$ and $\sigma_2$ the first time after time $1/\varepsilon$ that $|M|$ is less than or equal to $\varepsilon^2/4$. Then $\sigma_2$ is finite since $\liminf |M_t| = 0$ and, with $\tau = \sigma_1 \wedge \sigma_2$, we may assume that $M_\tau \in L^1$ and obtain

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_{\sigma_1}\mathbf{1}_{\{\sigma_1 \leq \sigma_2\}}] + \mathbb{E}[M_{\sigma_2}\mathbf{1}_{\{\sigma_1 > \sigma_2\}}] \geq \varepsilon \mathbf{P}\left(\sigma_1 \leq \frac{1}{\varepsilon}\right) - \frac{\varepsilon^2}{4} \geq \frac{3\varepsilon^2}{4} > \frac{\varepsilon^2}{4} \geq \mathbb{E}[|M_{\sigma_2}|] \geq \mathbb{E}[M_{\sigma_2}],$$

which contradicts (III) with $X$ replaced by $M$. Thus $M \leq 0$ and in the same manner, we can show that $M \geq 0$, which yields the statement. □

We briefly remark that if $X$ is nonnegative, then (IV) yields that $X$ is a martingale, thus a nonnegative supermartingale and therefore $\liminf_{t\uparrow\infty} |X_t| \in L^1$. Theorem 1 then yields Cherny's result, namely the implication (IV) ⇒ (I) provided that $X$ is nonnegative.

## 2 A counterexample

We now construct a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ with a right-continuous martingale $X$ that satisfies (IV) and has a limit $X_\infty = \lim_{t\uparrow\infty} X_t$, but is not uniformly integrable.

*Example* 1. We let $\Omega = (\mathbb{N} \cup \{\infty\}) \times \{-1, 1\}$ and $\mathcal{F}$ the power set of $\Omega$. Next, we let $\mathbf{P}$ be the probability measure on $(\Omega, \mathcal{F})$ such that $\mathbf{P}((n, i)) = 1/(4n^2)$ for all $n \in \mathbb{N}$ and $i \in \{-1, 1\}$ and $\mathbf{P}((\infty, i)) = (1 - \pi^2/12)/2$ for all $i \in \{-1, 1\}$. Since $\sum_{n\in\mathbb{N}} 1/(2n^2) = \pi^2/12 \in (0, 1)$, this yields indeed a probability measure on $(\Omega, \mathcal{F})$.

We let $\sigma : \Omega \to [0, \infty]$, $(\omega_1, \omega_2) \mapsto \omega_1$ denote the first component and $D : \Omega \to \{-1, 1\}$, $(\omega_1, \omega_2) \mapsto \omega_2$ the second component of each scenario $(\omega_1, \omega_2) \in \Omega$. Then $\sigma$ and $D$ are independent and $\mathbf{P}(\sigma = x) = 1/(2x^2)\mathbf{1}_{x\in\mathbb{N}}$ for all $x \in [0, \infty]$, $\mathbf{P}(\sigma = \infty) = 1 - \pi^2/12$, and $\mathbf{P}(D = -1) = 1/2 = \mathbf{P}(D = 1)$.

We now set $X \equiv D\sigma^2\mathbf{1}_{[\![\sigma,\infty[\![}$ and let $\mathbb{F}$ denote the natural filtration of $X$. To wit, $X$ is a martingale that at time $\sigma$ jumps to either $\sigma^2$ or $-\sigma^2$ provided that $\sigma$ is finite. In particular, we have

$$X_\infty = \lim_{t\uparrow\infty} X_t = D\sigma^2\mathbf{1}_{\{\sigma<\infty\}}.$$

Next, we observe that

$$\mathbb{E}[|X_\infty|] = \mathbb{E}[\sigma^2\mathbf{1}_{\{\sigma<\infty\}}] = \sum_{n\in\mathbb{N}} n^2 \frac{1}{2n^2} = \infty.$$

Hence, $X_\infty \notin L^1$, and $X$ is not a uniformly integrable martingale.

Now, we let $\tau$ be an arbitrary finite stopping time and set $u = \tau((\infty, -1)) \vee \tau((\infty, 1)) \in [0, \infty)$. Thus, $\{\sigma = \infty\} \subset \{\tau \leq u\} \in \mathcal{F}_u$, which again yields $\{\sigma > u\} \subset \{\tau \leq u\}$ since $\{\sigma > u\}$ is the smallest event in $\mathcal{F}_u$ that contains $\{\sigma = \infty\}$. Thus, since

$$\{\tau \wedge \sigma \leq u\} = \{\sigma \leq u\} \cup \{\tau \leq u\} \supset \{\sigma \leq u\} \cup \{\sigma > u\} = \Omega,$$

the stopping time $\tau \wedge \sigma$ is uniformly bounded by $u$ and we obtain that $X_\tau = X_\tau^\sigma = X_{\tau\wedge\sigma} \in L^1$ and, by the optional sampling theorem again, that $\mathbb{E}[X_\tau] = \mathbb{E}[X_{\tau\wedge\sigma}] = 0 = \mathbb{E}[X_0]$. Hence, $X$ satisfies (IV) but not (I), and therefore neither (II) nor (III). □

We remark that Dellacherie (1970) discusses the filtration of a similar example.

In order to motivate the arguments in the next section, we now slightly modify Example 1 by extending the underlying filtration.

*Example* 2. We let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ be an arbitrary probability space that supports a right-continuous martingale $X = D\sigma^2 \mathbf{1}_{[\![\sigma,\infty[\![}$ with the same distribution as in Example 1 and an independent $\mathcal{F}_0$–measurable $(0,1)$–uniformly distributed random variable $U$.

Our goal is to construct a finite stopping time $\tau$ such that $X_\tau \notin L^1$. Thus, under this enlarged filtration, the previous example is not a counterexample for the implication (IV) $\Rightarrow$ (I). Indeed, we note that $\tau = 1/U$ is a finite stopping time and $|X_\tau| = \sigma^2 \mathbf{1}_{\{\sigma \leq 1/U\}}$. Therefore,

$$\mathbb{E}[|X_\tau|] = \mathbb{E}\left[\sigma^2 \mathbf{1}_{\{\sigma \leq 1/U\}}\right] = \mathbb{E}\left[\sum_{n=1}^{\lfloor 1/U \rfloor} n^2 \frac{1}{2n^2}\right] = \frac{1}{2}\mathbb{E}\left[\left\lfloor \frac{1}{U} \right\rfloor\right] \geq \mathbb{E}\left[\frac{1}{2U}\right] - \frac{1}{2} = \int_0^1 \frac{1}{2y}\mathrm{d}y - \frac{1}{2} = \infty,$$

where $\lfloor \cdot \rfloor$ denotes the Gauss brackets, by independence of $U$ and $X$. □

Example 2 indicates that if "randomized stopping" is possible, a non-uniformly integrable martingale $X$ will not satisfy (IV). In the next section, we will prove this assertion.

## 3 An additonal randomization

In this section, we show that the implication (IV) $\Rightarrow$ (I) holds if we may randomize stopping times. More precisely, we shall make the following assumption:

$$\begin{gathered}\text{There exists a } (0,1)\text{–uniformly distributed random variable } U \\ \text{and a finite stopping time } \eta, \text{ such that } U \text{ is } \mathcal{F}_\eta\text{–measurable.}\end{gathered} \tag{$\mathfrak{R}$}$$

We emphasize that $(\mathfrak{R})$ is an assumption on the underlying filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, and not on the stochastic process $X$. We recall that we already argued that $X$ is a martingale if (IV) holds. The conclusion that $X$ is also a uniformly integrable martingale, if additionally $(\mathfrak{R})$ holds, follows then from the following theorem.

**Theorem 2.** *If* (IV) *and* $(\mathfrak{R})$ *hold then so does* (I)*; to wit, $X$ is then a uniformly integrable martingale.*

*Proof.* Exactly as in the proof of Theorem 1, we may assume, without loss of generality, that $\mathbb{F}$ and $X(\omega)$ are right-continuous for each $\omega \in \Omega$. Lemmata 1 and 2 below then yield that $\liminf_{t\uparrow\infty} |X_t| \in L^1$ and the implication (III) $\Rightarrow$ (I), proven in Thereom 1, yields the assertion. □

**Lemma 1.** *Assume that $\mathbb{F}$ and $X(\omega)$ are right-continuous for each $\omega \in \Omega$. If $X_\tau \in L^1$ for all finite stopping times $\tau$ and* $(\mathfrak{R})$ *holds then* $\mathbb{E}[\liminf_{t\uparrow\infty} |X_t| | \mathcal{F}_\eta] < \infty$.

*Proof.* We define the event

$$A = \left\{ \mathbb{E}\left[\liminf_{t\uparrow\infty} |X_t| \middle| \mathcal{F}_\eta\right] = \infty \right\} \in \mathcal{F}_\eta.$$

We need to argue that $\mathbf{P}(A) = 0$. Towards this end, we assume that $\mathbf{P}(A) > 0$ and define the function $g : [0,1] \to [0,\infty]$ by $t \mapsto 1/\mathbf{P}(A \cap \{U \leq t\})$, where $U$ is the uniformly distributed random variable of $(\mathfrak{R})$. We note that the function $1/g$ is continuous and nondecreasing and set $t_\infty = \sup\{t \in [0,1] | g(t) = \infty\}$. Then we have $\mathbf{P}(A \cap \{U \leq t_\infty\}) = 0$ and $\mathbf{P}(A \cap \{U \leq t\}) > 0$ for all $t > t_\infty$, which yields that $\mathbf{1}_A g(U)$ (with $0 \times \infty = 0$) is finite (almost surely).

We now let $\sigma$ denote the first time $t$ after $\eta$ such that $\mathbb{E}[|X_t||\mathcal{F}_\eta]$ is greater than or equal to $g(U)$ and note that $\sigma$ is a stopping time. Then, Fatou's lemma yields that $\sigma$ is finite on $A$. We now set $\tau = \eta\mathbf{1}_{\Omega\setminus A} + \sigma\mathbf{1}_A$, which is again a finite stopping time, and observe

$$\mathbb{E}[|X_\tau|] \geq \mathbb{E}\left[\mathbf{1}_A|X_\sigma|\right] = \mathbb{E}\left[\mathbf{1}_A\mathbb{E}[|X_\sigma||\mathcal{F}_\eta]\right] \geq \mathbb{E}\left[\mathbf{1}_A g(U)\right] \geq \sum_{n\in\mathbb{N}} \mathbf{P}\left(A \cap \left\{\mathbf{P}(A\cap\{U \leq t\})_{t=U} \leq \frac{1}{n}\right\}\right) = \infty.$$

Here the last inequality follows from Tonelli's theorem and the last equality follows from the fact that for each $n \geq 1/\mathbf{P}(A)$ the corresponding term in the sum equals $1/n$. To see this, fix $n \geq 1/\mathbf{P}(A)$, let $t_n = \sup\{t \in [0,1] | g(t_n) \geq n\}$, and use the fact that $\mathbf{P}(A \cap \{U \leq t_n\}) = 1/n$. The last display contradicts the assumption and thus yields $\mathbf{P}(A) = 0$. $\square$

**Lemma 2.** *Assume that $\mathbb{F}$ and $X(\omega)$ are right-continuous for each $\omega \in \Omega$. If $X_\tau \in L^1$ for all finite stopping times $\tau$ and $\mathbb{E}[\liminf_{t\uparrow\infty}|X_t||\mathcal{F}_\eta] < \infty$ holds for some finite stopping time $\eta$ then $\liminf_{t\uparrow\infty}|X_t| \in L^1$.*

*Proof.* We let $Y = (Y_t)_{t\in[0,\infty)}$ denote the right-continuous modification of the finite-valued conditional expectation process

$$\left(\mathbb{E}\left[\liminf_{s\uparrow\infty}|X_s|\middle|\mathcal{F}_{\eta\vee t}\right]\right)_{t\in[0,\infty)}.$$

For each $\kappa > 0$ the process $(Y_t\mathbf{1}_{\{Y_0\leq\kappa\}})_{t\in[0,\infty)}$ is a uniformly integrable martingale under its natural filtration and Lévy's martingale convergence theorem yields that $\mathbf{1}_{\{Y_0\leq\kappa\}}\lim_{t\uparrow\infty}Y_t = \mathbf{1}_{\{Y_0\leq\kappa\}}\liminf_{s\uparrow\infty}|X_s|$, and thus $\lim_{t\uparrow\infty}Y_t = \liminf_{s\uparrow\infty}|X_s|$.

We now let $\tau$ denote the first time after time $\eta$ such that $|X|$ is greater than or equal to $Y - 1$, which is a stopping time. Moreover, $\tau$ is finite since $\liminf_{t\uparrow\infty}|X_t| > \lim_{t\uparrow\infty}Y_t - 1$. We then obtain

$$\mathbb{E}\left[\liminf_{t\uparrow\infty}|X_t|\right] = \mathbb{E}\left[Y_\tau\right] \leq 1 + \mathbb{E}\left[|X_\tau|\right] < \infty,$$

which yields the statement. $\square$

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