Larsson, Ramdas, and Ruf's contribution to the Discussion of Safe Testing by Grünwald, de Heide, and Koolen

Martin Larsson[*] Aaditya Ramdas[†] Johannes Ruf[‡]

April 29, 2024

We congratulate the authors on a stimulating article that has inspired many follow-up works. We discuss the central concept of growth-rate optimality (GRO). For a point null and alternative, the GRO e-variable is their likelihood ratio (Shafer, 2021). The generalization to composite nulls takes center stage in Grünwald et al. (2024). Their use of the reverse information projection (RIPr) builds on Li (1999), and has been recently extended in Lardy et al. (2023). Inspired by their centrality, we report on our efforts to significantly generalize these results.

Our preprint (Larsson et al., 2024) considers testing a composite null $\mathcal{P}$ against a point alternative $\mathbb{Q}$. We establish a powerful result: under no conditions whatsoever (e.g., reference/dominating measure), there always exists a unique "optimal" e-variable that we call the numeraire X^*: for every other e-variable X, we have $\mathbb{E}_{\mathbb{Q}}[X/X^*] \leq 1$. In particular, X^* also satisfies $\mathbb{E}_{\mathbb{Q}}[\log(X/X^*)] \leq 0$ (log-optimality), as well as $\mathbb{E}_{\mathbb{Q}}[1/X^*] \leq 1$.

X^* also identifies a particular measure $\mathbb{P}^*$ by the definition $d\mathbb{P}^*/d\mathbb{Q} = 1/X^*$. In general, $\mathbb{P}^*$ is a subprobability measure in the "bipolar" of $\mathcal{P}$. In particular, $X^* = d\mathbb{Q}/d\mathbb{P}^*$ is a generalized likelihood ratio of $\mathbb{Q}$ against $\mathcal{P}$, and is the only e-variable with such a representation. Further, $\mathbb{P}^*$ coincides with the reverse information projection (RIPr) when additional assumptions are made that are required for the latter to exist (such as $\mathbb{Q} \ll \mathbb{P}$ for some $\mathbb{P} \in \mathcal{P}$, σ-convexity of $\mathcal{P}$, and finite maximum description gain in Lardy et al. (2023)). Thus, we contend that going forward, absent any assumptions on $\mathbb{Q}, \mathcal{P}$, the above should be the default definition of the reverse information projection (RIPr). We also present a more general optimality theory that studies general concave utilities (beyond the logarithm), exemplified by a reverse Rényi projection, which also exists under no conditions.

Surprisingly, our results imply that the universal inference e-variable (Wasserman et al., 2020) is (essentially) always inadmissible. Practically, the numeraire is now on a somewhat equal footing with universal inference. While the latter seems easy to apply, it is has only

[*]Department of Mathematical Sciences, Carnegie Mellon University, larsson@cmu.edu
[†]Departments of Statistics and ML, Carnegie Mellon University, aramdas@cmu.edu
[‡]Department of Mathematics, London School of Economics, j.ruf@lse.ac.uk

been applicable in a few nonparametric situations thus far (like testing log-concavity). In contrast, we explicitly calculate the numeraire and RIPr in many nonparametric settings where there is no reference measure, and universal inference does not apply; we explore several such cases in the paper (testing symmetry, subGaussian means, bounded means).

We are currently working to extend these ideas to composite alternatives using standard techniques like the method of mixtures, and to sequential settings using the mixture and plug-in methods; see Ramdas et al. (2023) for details.

References

Peter Grünwald, Rianne de Heide, and Wouter M Koolen. Safe testing. *Journal of the Royal Statistical Society, Series B (Methodology), with discussion*, 2024. 1

Tyron Lardy, Peter Grünwald, and Peter Harremoës. Universal reverse information projections and optimal e-statistics. *arXiv:2306.16646*, 2023. 1

Martin Larsson, Aaditya Ramdas, and Johannes Ruf. The numeraire e-variable and reverse information projection. *arXiv:2402.18810*, 2024. 1

Qiang Jonathan Li. *Estimation of Mixture Models*. PhD thesis, 1999. 1

Aaditya Ramdas, Peter Grünwald, Vladimir Vovk, and Glenn Shafer. Game-theoretic statistics and safe anytime-valid inference. *Statistical Science*, 38(4):576–601, 2023. 2

Glenn Shafer. Testing by betting: A strategy for statistical and scientific communication. *Journal of the Royal Statistical Society Series A: Statistics in Society, with discussion*, 184(2):407–431, 2021. 1

Larry Wasserman, Aaditya Ramdas, and Sivaraman Balakrishnan. Universal inference. *Proceedings of the National Academy of Sciences*, 117(29):16880–16890, 2020. 1