On the integrability of the supremum of stochastic volatility models and other martingales

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Abstract

We propose a method to determine the expectation of the supremum of the price process in stochastic volatility models. It can be applied to the rough Bergomi model, avoiding the need to discuss finiteness of higher moments. Our motivation stems from the theory of American option pricing, as an integrable supremum implies the existence of an optimal stopping time for any linearly bounded payoff. Moreover, we survey the literature on martingales with non-integrable supremum, and give a new construction that yields uniformly integrable martingales with this property.

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1 Introduction

This paper deals mainly with the question whether a non-negative martingale $S = (S_t)_{0 \le t \le T}$, where $T \in (0, \infty]$ is deterministic, satisfies

$$\mathbb{E}\left[\sup_{t \in [0,T]} S_t \right] < \infty. \tag{1.1}$$

From the viewpoint of mathematical finance, this property is of interest in the theory of American option pricing. If S is the price process of the underlying and the expectation is under the chosen pricing measure, condition (1.1) implies the existence of an optimal exercise time for all American payoffs of at most linear growth [25, Theorem D.12]. A standard example is the straddle [5], with payoff $|S_t - K|$. In the European case, its price decomposes into a call price plus a put price, where both put and call have strike K. Under American exercise rights, such a decomposition does not hold. Thus, (1.1) serves as theoretical basis of any convergence analysis regarding approximate optimal exercise strategies, as it guarantees the existence of the limiting object.

If $T < \infty$ and $\mathbb{E}[S_T^p] < \infty$ for some $p > 1$, which may depend on T, then (1.1) follows from Doob's L^p inequality. A convenient way of verifying finiteness of moments $\mathbb{E}[S_T^p]$ is to study the domain of the characteristic function, and so the existence of higher moments is well understood for affine models [26] and affine Volterra models [15, 27]. Further examples of stochastic volatility models with known characteristic function can be found in [19, Chapter 4]. The 3/2 model admits a particularly simple statement in this regard, which has not been made explicit in the literature: If the asset and variance processes are negatively correlated, any moment $\mathbb{E}[S_T^p]$ with exponent $p \in [1, u_+)$ is finite, irrespective of the value of T. We refer to [16] for the definition of $u_+ > 1$ in terms of the model parameters, and for refined tail asymptotics of the marginal density. For models that do not feature an explicit characteristic function, a tool that could be used to establish (1.1) is [30, Theorem 4.1], which gives sufficient conditions for the finiteness of the exponential Orlicz norm of the maximum of a continuous local martingale.

For rough volatility models outside the affine Volterra class, it seems that there are only results on the *non*-existence of moments so far [14, 20], which motivates studying (1.1) directly. We give a sufficient condition for (1.1), and show that it is satisfied for the rough Bergomi model, for which there are so far no positive results on moments with $p > 1$. Our approach is based on Doob's L^1 inequality and a change of measure. We also show that it is applicable to generic stochastic volatility models driven by solutions of stochastic Volterra integral equations, under certain regularity assumptions.

We note in passing that the stronger condition that the supremum of the expectation in (1.1) over all equivalent martingale measures be finite is of interest as well. Under this assumption, *all* arbitrage free price processes of a linearly bounded American payoff can be represented by Snell envelopes with respect to some equivalent martingale measure. This follows from [23, Theorem 13.2.9], by specializing from game options to American options. The statement is not found in many textbooks, while the European counterpart of the theorem, viz. that all arbitrage free price processes arise from equivalent martingale measures, is standard material in mathematical finance courses – see [13, Theorem 5.29] for this (the European) statement in discrete time, which avoids technicalities.

It is not obvious whether there actually are martingales that do *not* satisfy (1.1). We discuss this question in Section 4. We survey the literature and show how known results easily yield explicit examples of such martingales. Moreover, we present a new construction that gives plenty of examples, by appropriately stopping a strict local martingale.

As usual, we define the function $\log^+$ on $[0,\infty)$ by $\log^+(x) = (\log x)^+$. For a given probability space, the class $L \log L$ consists of the random variables X with $\mathbb{E}[|X| \log^+|X|] < \infty$.

2 Integrability of the supremum in stochastic volatility models

The following lemma states a version of Doob's L^1 inequality [4, 17].

Lemma 2.1. *Let $T > 0$ and suppose that the process $(X_t)_{t\in[0,T]}$ is a strictly positive right-continuous submartingale. Then we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} X_t\right] \le \frac{e}{e-1}\Big(\mathbb{E}\left[X_T \log(X_T)\right] + \mathbb{E}\left[X_0(1-\log(X_0))\right]\Big).$$

Doob's original statement has $\log^+$ instead of log, which is slightly less convenient for our application. While some sources present the inequality in discrete time, it is clear that we can easily pass to continuous time, by means of the monotone convergence theorem. We note in passing that an application of the inequality to pathwise hedging is presented in [3, Section 5]. We will apply this lemma to a generic stochastic volatility model introduced below. In particular, the modelling framework includes the rough Bergomi model (see Section 3).

Assumption 2.2. *Let $f : [0, \infty) \times \mathbb{R} \to [0, \infty)$ be a continuous non-negative function. Fix $y_0 \in \mathbb{R}$ such that $f(0, y_0) > 0$, fix $\rho \in [-1, 1]$, and consider measurable functions*

$$b : [0, \infty) \times \mathbb{R} \to \mathbb{R}, \quad \sigma : [0, \infty) \times \mathbb{R} \to \mathbb{R},$$

and $K : \Delta \to [0, \infty)$, where $\Delta = \{(t, s) \in [0, \infty)^2 : t \geq s\}$. We assume that for any filtered probability space, satisfying the usual conditions and supporting a one-dimensional Brownian motion $(B_t)_{t\geq 0}$, the stochastic Volterra equations

$$Y_t = y_0 + \int_0^t K(t, s)b(s, Y_s)\mathrm{d}s + \int_0^t K(t, s)\sigma(s, Y_s)\mathrm{d}B_s \tag{2.1}$$

and

$$\tilde{Y}_t = y_0 + \int_0^t K(t, s)\Big(b(s, \tilde{Y}_s) + \rho\sqrt{f(s, \tilde{Y}_s)}\sigma(s, \tilde{Y}_s)\Big)\mathrm{d}s + \int_0^t K(t, s)\sigma(s, \tilde{Y}_s)\mathrm{d}B_s \tag{2.2}$$

have unique continuous adapted solution processes $(Y_t)_{t\geq 0}$ and $(\tilde{Y}_t)_{t\geq 0}$, respectively. For the given solution process $\tilde{Y}$ of (2.2), we define the process $\tilde{v}$ via

$$\tilde{v}_t = f(t, \tilde{Y}_t), \quad t \geq 0. \tag{2.3}$$

We do not impose any further regularity conditions on the functions K, B, σ; our strong existence assumptions tacitly include the well-definedness of all occurring integrals. We refer to the introduction of [8] for an up-to-date survey of the solution theory of stochastic Volterra integral equations.

Definition 2.3. *Under Assumption 2.2, and for a filtered probability space supporting a two-dimensional Brownian motion $(W, \bar{W})$, we define the* associated stochastic volatility model *with initial value $S_0 > 0$ by*

$$\begin{aligned} \mathrm{d}S_t &= S_t\sqrt{v_t}\mathrm{d}W_t, \\ v_t &= f(t, Y_t). \end{aligned}$$

Here, the process Y (see (2.1)) is driven by the Brownian motion $B = \rho W + \sqrt{1-\rho^2}\bar{W}$.

Lemma 2.4. *Fix $T > 0$ and assume that S, from the preceding definition, is a martingale on $[0, T]$. Then, we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} S_t\right] \leq \frac{e}{e-1}\left(S_0 + S_0\mathbb{E}\left[\int_0^T \sqrt{\tilde{v}_s}\mathrm{d}W_s + \frac{1}{2}\int_0^T \tilde{v}_s\mathrm{d}s\right]\right),$$

where $\tilde{v}$ is the process defined in (2.3)*. Additionally, we have the upper bound*

$$\mathbb{E}\left[\sup_{t\in[0,T]} S_t\right] \leq \alpha + \beta\int_0^T \mathbb{E}\left[\tilde{v}_s\right]\mathrm{d}s, \tag{2.4}$$

where α and β are some positive constants.

Proof. Assume w.l.o.g. that $S_0 = 1$. By assumption, S is a martingale on $[0, T]$, which allows us to define the measure $\mathrm{d}\hat{\mathbb{P}} = S_T\mathrm{d}\mathbb{P}$, sometimes called the share measure, and apply Girsanov's theorem to see that

$$\begin{pmatrix} W^{(1)} \\ W^{(2)} \end{pmatrix} = \begin{pmatrix} W - \int_0^{\cdot\wedge T}\sqrt{v_s}\mathrm{d}s \\ \bar{W} \end{pmatrix}$$

is a Brownian motion under $\hat{\mathbb{P}}$. An application of Lemma 2.1 and the definition of $W^{(1)}$ yield

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} S_t\right] &\leq \frac{e}{e-1}\left(1 + \mathbb{E}\left[S_T\left(\int_0^T \sqrt{v_s}\mathrm{d}W_s - \frac{1}{2}\int_0^T v_s\mathrm{d}s\right)\right]\right) \\ &= \frac{e}{e-1}\left(1 + \mathbb{E}_{\hat{\mathbb{P}}}\left[\int_0^T \sqrt{v_s}\mathrm{d}W_s - \frac{1}{2}\int_0^T v_s\mathrm{d}s\right]\right) \\ &= \frac{e}{e-1}\left(1 + \mathbb{E}_{\hat{\mathbb{P}}}\left[\int_0^T \sqrt{v_s}dW_s^{(1)} + \frac{1}{2}\int_0^T v_s\mathrm{d}s\right]\right).\end{aligned}$$

Next, using $v_t = f(t, Y_t)$, we observe

$$\begin{aligned}Y_t &= y_0 + \int_0^t K(t,s)b(s,Y_s)\mathrm{d}s + \int_0^t K(t,s)\sigma(s,Y_s)\mathrm{d}(\rho W_s + \sqrt{1-\rho^2}\bar{W}_s) \\ &= y_0 + \int_0^t K(t,s)\left(b(s,Y_s) + \rho\sqrt{f(s,Y_s)}\sigma(s,Y_s)\right)\mathrm{d}s \\ &\qquad + \int_0^t K(t,s)\sigma(s,Y_s)\mathrm{d}(\rho W_s^{(1)} + \sqrt{1-\rho^2}W_s^{(2)}),\end{aligned}$$

for $t \in [0, T]$. Since $\rho W^{(1)} + \sqrt{1-\rho^2}W^{(2)}$ is a Brownian motion under $\hat{\mathbb{P}}$, by the uniqueness and existence assumptions made in Assumption 2.2, the law of Y under the measure $\hat{\mathbb{P}}$ must agree with the law of $\tilde{Y}$ under the measure $\mathbb{P}$. Whence, the same is true for v and $\tilde{v}$, which yields, when changing back to the measure $\mathbb{P}$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} S_t\right] \leq \frac{e}{e-1}\left(1+\mathbb{E}\left[\int_0^T \sqrt{\tilde{v}_s}\mathrm{d}W_s + \frac{1}{2}\int_0^T \tilde{v}_s\mathrm{d}s\right]\right), \tag{2.5}$$

which proves our claim. For the second inequality, note that the Burkholder-Davis-Gundy inequality [24, Theorem 3.3.28] yields for some $C>0$ that

$$\mathbb{E}\left[\int_0^T \sqrt{\tilde{v}_s}\mathrm{d}W_s\right] \leq 1+\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \sqrt{\tilde{v}_s}\mathrm{d}W_s\right|^2\right] \leq 1+C\mathbb{E}\left[\int_0^T \tilde{v}_s\mathrm{d}s\right].$$

Combining this with (2.5) yields the existence of positive α and β such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} S_t\right] \leq \alpha+\beta\int_0^T \mathbb{E}\left[\tilde{v}_s\right]\mathrm{d}s,$$

where we used that $\tilde{v}$ is non-negative. This completes the proof. □

Note that the dynamics of the two auxiliary processes (2.1) and (2.2) are similar, and so results yielding strong existence and uniqueness for the former process could well be applicable to the latter as well. Additionally, note that the right hand side of (2.4) does not depend on the stock price process. Hence, we only have to bound the process $\tilde{v}$ appropriately in order to conclude integrability of the supremum of S. While our main example, the rough Bergomi model, satisfies Assumption 2.2, we show in Appendix A that the assumption can be weakened.

Under suitable regularity assumptions, it is a straightforward matter to apply Lemma 2.4 to stochastic volatility models driven by solutions of generic stochastic Volterra integral equations. Concerning the kernel, we quote the following assumption, which is condition (H_0) in [1]:

Assumption 2.5. *The kernel K is of the form $K(t,s)=\tilde{K}(t-s)$, where $\tilde{K}\in L^2_{\mathrm{loc}}([0,\infty),\mathbb{R})$, and there is $\gamma>0$ with*

$$\int_0^h \tilde{K}(t)^2\mathrm{d}t = O(h^\gamma), \quad h\downarrow 0,$$

and

$$\int_0^T (\tilde{K}(t+h)-\tilde{K}(t))\mathrm{d}t = O(h^\gamma), \quad h\downarrow 0, \quad \textit{for all}\ \ T>0.$$

We refer to [2, Example 3.3] for examples of such kernels.

Theorem 2.6. *Let K be a kernel satisfying Assumption 2.5. Fix $T > 0$ and suppose that*

$$\sup_{t\in[0,T]} |f(t, y)| \leq c(1 + |y|), \quad y \in \mathbb{R}, \tag{2.6}$$

for some $c > 0$, and that the coefficients b, σ and $b+\rho\sqrt{f}\sigma$ in (2.1) *and* (2.2) *are of at most linear growth, in the sense of* (2.6)*, and globally Lipschitz continuous on* $[0, T] \times \mathbb{R}$*. Then S from Definition 2.3 satisfies* (1.1)*.*

Proof. We apply [1, Theorem A.1] to infer strong existence and uniqueness for (2.1) and (2.2). Note that the theorem covers multi-dimensional solutions of time-homogeneous stochastic Volterra integral equations, and thus yields solution processes (t, Y_t) and $(t, \tilde{Y}_t)$, as well as the bound

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left|\tilde{Y}_t\right|\right] < \infty.$$

Thus, by (2.3) and (2.6), the statement follows from Lemma 2.4. □

In the following section we apply Lemma 2.4 to the rough Bergomi model. For the rough Heston model, another well-known rough volatility model, Assumption 2.2 amounts to a well-known open problem, viz. strong existence for stochastic Volterra integral equations under Hölder assumptions. In Appendix A, we show how Assumption 2.2 can be weakened. In the case of the rough Heston model, this serves illustrative purposes only: By [15] there is some $p > 1$ such that $\mathbb{E}[S_T^p] < \infty$, which allows us to simply use Doob's L^p inequality to conclude (1.1).

3 Application to the rough Bergomi model

After introducing the rough Bergomi model below, we apply Lemma 2.4 to conclude that (1.1) holds in this model. We do not consider positive correlation, since it is known (see [14]) that the price process S is a true martingale if and only if the instantaneous correlation of the two driving Brownian motions is non-positive, i.e., $\rho \leq 0$. Indeed, besides giving difficulties concerning option pricing [9], a non-negative strict local martingale cannot satisfy (1.1), by the dominated convergence theorem. For the definition of the rough Bergomi model, we use the kernel

$$K_{\alpha,\eta}(t, s) = \eta\sqrt{2\alpha - 1}(t - s)^{\alpha-1}, \quad 0 \leq s < t, \tag{3.1}$$

where $\alpha > 1/2$ and $\eta > 0$.

Definition 3.1 (Rough Bergomi model [6]). *The rough Bergomi model, where S denotes the stock price and v is the instantaneous variance process, is defined as*

$$\begin{aligned} \mathrm{d}S_t &= S_t\sqrt{v_t}\mathrm{d}W_t, \quad S_0 > 0, \\ v_t &= v_0 \exp\left(Z_t - \frac{\eta^2}{2}t^{2\alpha-1}\right), \quad v_0 > 0, \end{aligned} \tag{3.2}$$

where Y is the Riemann-Liouville process given by $Y_t = \int_0^t K_{\alpha,\eta}(t,s)\mathrm{d}B_s$ for $t \geq 0$. Here B is a Brownian motion defined by $B = \rho W + \sqrt{1-\rho^2}\bar{W}$ for $\rho \in [-1, 0]$ and some standard Brownian motion $\bar{W}$ independent of W.

Remark 3.2. *In* (3.2)*, we assume a flat initial forward variance curve $\xi_0(t) = \mathbb{E}[v_t] = v_0$ for simplicity of notation. Clearly, in what follows, v_0 could be generalized to a continuous, non-negative, and bounded function $\xi_0(\cdot)$.*

Note that Theorem 2.6 does not cover the rough Bergomi model, as the exponential in (3.3) violates the linear growth assumption. Still, the rough Bergomi model fits into Definition 2.3. Indeed, let

$$f(t, y) = v_0 \exp\left(y - \frac{\eta^2}{2}t^{2\alpha-1}\right),$$

use the Volterra kernel $K_{\alpha,\eta}$ defined in (3.1), and set $y_0 = 0$, $b = 0$, and $\sigma = 1$. The equation for the process $\tilde{Y}$ is given by

$$\tilde{Y}_t = \int_0^t K_{\alpha,\eta}(t,s)\rho\sqrt{v_0}\exp\left(\frac{1}{2}\tilde{Y}_s - \frac{\eta^2}{4}s^{2\alpha-1}\right)\mathrm{d}s + \int_0^t K_{\alpha,\eta}(t,s)\mathrm{d}B_s. \tag{3.3}$$

The main issue in the application of Lemma 2.4 to the rough Bergomi model is to establish strong existence and uniqueness for this equation.

Corollary 3.3. *For any filtered probability space, satisfying the usual conditions and supporting a one-dimensional Brownian motion $(B_t)_{t\geq 0}$, the stochastic Volterra equation* (3.3) *admits a unique and adapted global solution $\tilde{Y}$.*

Proof. Define the Riemann-Liouville process $Y = \int_0^\cdot K_{\alpha,\eta}(t,s)\mathrm{d}B_s$ and note that it is continuous, adapted, and starts in zero. Further note that the kernel $K_{\alpha,\eta}$ is a Volterra kernel of continuous type (see Definition B.1). Define next the function

$$g(t, y) = |\rho|\sqrt{v_0}\exp\left(\frac{1}{2}y - \frac{\eta^2}{4}t^{2\alpha-1}\right).$$

Now Theorem B.3 yields the existence of a unique, adapted, and continuous solution $\tilde{Y}$ to (3.3), which concludes the proof. □

Hence, by existence of a unique global solution, and combining (2.3) with the fact that f is non-decreasing in y, we can use the a-priori estimate

$$0 \leq \tilde{v}_t \leq v_0 \exp\Big(Y_t - \frac{\eta^2}{2} t^{2\alpha-1}\Big), \tag{3.4}$$

for $t \geq 0$. Note that the right-hand side of the above inequality has a log-normal distribution with constant expectation v_0. Hence, an application of Lemma 2.4 and the upper bound (3.4) show for the rough Bergomi model that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} S_t\Big] \leq \alpha + \beta \int_0^T \mathbb{E}\left[\tilde{v}_s\right] \mathrm{d}s \leq \alpha + \beta \int_0^T v_0 \mathrm{d}s < \infty,$$

for some positive α and β. We summarize the above analysis for the rough Bergomi model in the following theorem.

Theorem 3.4. *For the rough Bergomi model with $\rho \leq 0$, see Definition 3.1, we have*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} S_t\Big] < \infty.$$

4 Martingales with non-integrable supremum

4.1 General remarks

By the Burkholder-Davis-Gundy inequalities, the set of local martingales with integrable supremum coincides with $\mathcal{H}^1$, the space of local martingales $(M_t)_{t\geq 0}$ for which

$$\|M\|_1 = \mathbb{E}\big[[M]_\infty^{1/2}\big]$$

is finite. By dominated convergence, "local" can be dropped from this statement. Clearly, every element of $\mathcal{H}^1$ is uniformly integrable. For details and further interesting properties of $\mathcal{H}^1$, such as its duality to the space of BMO martingales, we refer to Section IV.4 in [31].

A family of *discrete time* examples of martingales with *non*-integrable supremum is provided by Theorem 2 in [7]. In this section, we give three different approaches that yield examples in continuous time. More examples can be found in [28]. First, recall that any continuous non-negative martingale $(X_t)_{t\geq 0}$ with $X_0 = 1$ satisfies the reverse L^1 inequality

$$\mathbb{E}\Big[\sup_{t\geq 0} X_t\Big] \geq 1 + \mathbb{E}[X_\infty \log^+(X_\infty)]. \tag{4.1}$$

For a proof, we refer to p. 149 in [11]; see also [21, 22] for earlier versions of similar inequalities.

For the mere existence of martingales with non-integrable supremum, the following observation suffices: For $T \in (0, \infty]$, suppose that a probability space with filtration $(\mathcal{F}_t)_{0 \le t \le T}$ supports a non-negative $\mathcal{F}_T$-measurable random variable $X_T \in L^1$ which is not in the class $L \log L$. If $X_t = \mathbb{E}[X_T | \mathcal{F}_t]$ is continuous, then it is a uniformly integrable martingale which is not in $\mathcal{H}^1$.

4.2 A construction by Dubins and Gilat

The second construction is due to [10] and appears also in [28, Example 1]. In our exposition, we add some details. In contrast to the above argument, the process will be explicitly defined, and is a martingale w.r.t. its own filtration. Unlike the rest of the paper, we drop the assumption of non-negativity, and aim at finding a real-valued martingale $(X_t)_{0 \le t \le 1}$ with $\mathbb{E}[\sup_{t \in [0,1]} X_t] = \infty$. For convenience, the following assumption is in force throughout this subsection.

Assumption 4.1. *Let F be the cumulative distribution function of some distribution on the real line with finite first moment, with full support and a continuous density.*

The function

$$H_F(t) = \frac{1}{1-t} \int_t^1 F^{-1}(s) \mathrm{d}s, \quad 0 < t < 1, \tag{4.2}$$

is known as the Hardy–Littlewood maximal function of the distribution. For the significance of this function concerning analytic and martingale inequalities, we refer to [17] and the references therein. Note that, in financial terms, the Hardy–Littlewood maximal function is the tail value at risk, or expected shortfall, of the distribution defined by F. In economics, the closely related function $t \mapsto \int_0^t F^{-1}(s) \mathrm{d}s / \int_0^1 F^{-1}(s) \mathrm{d}s$ is known as the Lorenz curve. It has been shown in [7] that the distribution of H_F, viewed as a random variable on the probability space

$$\big((0,1), \mathcal{B}(0,1), \mathrm{Leb}\big), \tag{4.3}$$

is an upper bound for the set

$$\{\nu : \nu \text{ is the distribution of the supremum of a martingale closed} \\ \text{by a random variable with cumulative distribution function } F\}$$

with respect to stochastic order. In [10] it was shown that the distribution of H_F is also a member of this set. This is achieved by defining a martingale X on the probability space (4.3) such that its closing element has cumulative distribution function F and the distribution of the supremum of X equals the distribution of H_F. If $H_F \notin L^1$, then X has the desired properties. See Proposition 4.4 for details.

Remark 4.2. *(i) It is clear that only the upper tail of the supremum of a martingale can cause non-integrability, since*

$$\left|\sup_{t\in[0,1]} X_t\right| \mathbf{1}_{\{\sup_{t\in[0,1]} X_t<0\}} \leq |X_0|.$$

(ii) Integrability of the Hardy–Littlewood maximal function (4.2) *can only fail because of a blowup as t tends to* 1. *Indeed,* $\lim_{t\downarrow 0} H_F(t) = \int_0^1 |F^{-1}(s)|\mathrm{d}s$ *is finite, by existence of the first moment of* F. *Thus, for integrability criteria concerning* H_F, *it does not matter if* $|F^{-1}(s)|$ *is used in the definition* (4.2) *instead of* $F^{-1}(s)$, *as is the case in [32].*

(iii) Concerning notation: The non-decreasing function f from [10] is our F^{-1}. *On the other hand,* f^* *from [17] is assumed to be non-increasing, and equals* $F^{-1}(1-\cdot)$ *in our notation. As a consequence, when reading [10, 17] it is important to note that the Hardy–Littlewood maximal functions H from [10] (which is our* H_F*) and* F^* *from [17] are not equal, but related by* $F^*(1-t) = H(t)$, *where* $0 < t < 1$.

Lemma 4.3. *If*

$$\int_{\mathbb{R}} |x| \log^+|x| F(\mathrm{d}x) = \infty, \tag{4.4}$$

then $H_F \notin L^1$.

Proof. This follows from a classical theorem due to Stein [32], which asserts that $H_F \notin L^1$ is equivalent to $\int_0^1 |F^{-1}(s)| \log^+ |F^{-1}(s)|\mathrm{d}s = \infty$, by substitution. □

Examples of cumulative distribution functions with finite first moment, but satisfying (4.4), can be easily found; e.g., assume that the density satisfies

$$F'(x) \leq \frac{c_1}{x^2(\log|x|)^\alpha}, \quad |x| \text{ large},$$

and

$$F'(x) \geq \frac{c_2}{(x\log x)^2}, \quad x > 0 \text{ large},$$

for some constants $\alpha \in (1, 2]$ and $c_1, c_2 > 0$.

Proposition 4.4. *Recall Assumption 4.1, and suppose that F satisfies* (4.4). *Then the process $(X_t)_{0\le t\le 1}$ defined by*

$$X_t(s) = \begin{cases} H_F(t) & \text{for} \quad t \le s, \\ F^{-1}(s) & \text{for} \quad t > s \end{cases}$$

is a martingale on the probability space (4.3), *with respect to its own filtration, and its supremum is not integrable.*

Proof. According to the proof of [10, Lemma 2], the process $(X_t)_{0\le t\le 1}$ is a martingale. As this is not shown in [10], we give a proof for the reader's convenience. Let $U(s) = s$, so that U is a standard uniformly distributed random variable on the probability space (4.3). It is easy to see that $\mathcal{F}_t = \sigma(U\mathbf{1}_{\{U<t\}})$ is the filtration generated by X. Define $\tilde{X}_t = \mathbb{E}[F^{-1}(U)|\mathcal{F}_t]$ for $t \in [0,1]$. It is readily verified that we have $\tilde{X}_t = F^{-1}(U)$ on the event $\{U < t\} \in \mathcal{F}_t$, while on the event $\{U \ge t\} \in \mathcal{F}_t$ we have $\tilde{X}_t = H_F(t)$. Thus, $X = \tilde{X}$. Finally, we have

$$\sup_{0\le t\le 1} X_t(s) = H_F(s), \quad 0 < s < 1.$$

By Lemma 4.3, this is not integrable with respect to Lebesgue measure on $(0,1)$. □

4.3 A new approach

Our third construction starts with an arbitrary non-negative local martingale M that is not a uniformly integrable martingale. In particular, M may be any non-negative strict local martingale. See, e.g., Exercise 3.3.36 in [24] for a standard example. We present the arguments for $T = \infty$, but they trivially apply to finite T as well. Observe that

$$\mathbb{E}\Big[\sup_{t\ge 0} M_t\Big] = \int_0^\infty \mathbb{P}\Big[\sup_{t\ge 0} M_t > u\Big]\mathrm{d}u$$

and

$$\sum_{n=1}^\infty \mathbb{P}\Big[\sup_{t\ge 0} M_t > n\Big] \le \int_0^\infty \mathbb{P}\Big[\sup_{t\ge 0} M_t > u\Big]\mathrm{d}u \le 1 + \sum_{n=1}^\infty \mathbb{P}\Big[\sup_{t\ge 0} M_t > n\Big].$$

As mentioned above, we have $M \in \mathcal{H}^1$ if and only if $\mathbb{E}[\sup_{t\ge 0} M_t] < \infty$, and so it follows that $M \in \mathcal{H}^1$ if and only if

$$\sum_{n=1}^\infty \mathbb{P}\Big[\sup_{t\ge 0} M_t > n\Big] < \infty. \tag{4.5}$$

This condition plays a key role in our construction.

Theorem 4.5. *Let $(M_t)_{t\geq 0}$ be a non-negative local martingale that is not a uniformly integrable martingale. Then there is an extended filtered probability space with a stopping time τ such that*

(i) The lift of M to the extended probability space, again denoted by M, has the same properties.

(ii) The stopped process M^τ is a uniformly integrable martingale which is not in $\mathcal{H}^1$, i.e., $\mathbb{E}[\sup_{t\geq 0} M_t^\tau] = \infty$.

Proof. We assume that $T = \infty$; the construction clearly works for finite T, too. Define the non-decreasing sequence $(c_n)_{n\in\mathbb{N}}$, by setting

$$c_n = \log\left(e + \sum_{k=1}^{n} \mathbb{P}\Big[\sup_{t\geq 0} M_t > k\Big]\right) > 1, \tag{4.6}$$

for each $n \in \mathbb{N}$. Since $M \notin \mathcal{H}^1$, it follows that $\lim_{n\uparrow\infty} c_n = \infty$. Without loss of generality, the original probability space, with filtration $(\mathcal{F}_t)_{t\geq 0}$, accommodates an $\mathbb{N}$-valued and $\mathcal{F}_0$-measurable random variable Y that is independent of M, and whose distribution satisfies $\mathbb{P}[Y > n] = 1/c_n$, for each $n \in \mathbb{N}$. Let

$$\sigma = \inf\{t \geq 0 \,|\, M_t > Y\} \tag{4.7}$$

denote the first time M exceeds Y. It follows that

$$\mathbb{P}\Big[\sup_{t\geq 0} M_t^\sigma > n\Big] \geq \mathbb{P}\Big[\sup_{t\geq 0} M_t > n\Big]\mathbb{P}[Y > n] = \frac{1}{c_n}\mathbb{P}\Big[\sup_{t\geq 0} M_t > n\Big],$$

for each $n \in \mathbb{N}$. Consequently,

$$\sum_{n=1}^{\infty} \mathbb{P}\Big[\sup_{t\geq 0} M_t^\sigma > n\Big] \geq \lim_{m\uparrow\infty} \frac{1}{c_m} \sum_{n=1}^{m} \mathbb{P}\Big[\sup_{t\geq 0} M_t > n\Big] = \lim_{m\uparrow\infty} \frac{e^{c_m} - e}{c_m} = \infty,$$

since $(c_n)_{n\in\mathbb{N}}$ is non-decreasing and $\lim_{n\uparrow\infty} c_n = \infty$. This implies that $M^\sigma \notin \mathcal{H}^1$. On the other hand, the almost sure limit $M_\infty^\sigma = M_{\infty-}^\sigma \in [0,\infty)$ exists, since M^σ is a non-negative local martingale, and hence also a non-negative supermartingale. Moreover,

$$\begin{aligned}
\mathbb{E}[M_\infty^\sigma] &= \sum_{n=1}^{\infty} \mathbb{E}[M_\infty^\sigma \,|\, Y = n]\mathbb{P}[Y = n] \\
&= \sum_{n=1}^{\infty} \mathbb{E}[M_\infty^{\tau_n} \,|\, Y = n]\mathbb{P}[Y = n] \\
&= \sum_{n=1}^{\infty} \mathbb{E}[M_\infty^{\tau_n}]\mathbb{P}[Y = n] = \sum_{n=1}^{\infty} \mathbb{E}[M_0]\mathbb{P}[Y = n] = \mathbb{E}[M_0^\sigma],
\end{aligned}$$

where

$$\tau_n = \inf\{t \geq 0 \,|\, M_t > n\}, \quad n \in \mathbb{N}.$$

Here, the penultimate equality follows from the fact that M^{τ_n} is uniformly integrable, for each $n \in \mathbb{N}$. Consequently, M^σ is uniformly integrable. □

Recall from the beginning of this section that the existence of martingales not in $\mathcal{H}^1$ is an immediate consequence of (4.1). If M is continuous, then the process M^σ from the proof of Theorem 4.5 is of the kind mentioned there, i.e., $M^\sigma_\infty \notin L\log L$. Besides accommodating càdlàg processes M, we make the following pedagogical remark: Theorem 4.5 and its proof require only material from an introductory course on stochastic calculus, and not the reverse L^1 inequality. Moreover, the natural way to prove $M^\sigma \notin \mathcal{H}^1$ is to show that (4.5) fails, and not to verify $M^\sigma_\infty \notin L\log L$.

Example 4.6. *Consider a non-negative local martingale M that belongs to Class $(\mathcal{C}_0)$, according to the terminology of [29], and suppose that $M_0 = 1$. In that case, M is a strictly positive local martingale without any positive jumps, for which $M_\infty = M_{\infty-} = 0$. The construction from Theorem 4.5 is then applicable, since $\mathbb{E}[M_\infty] = 0 < 1 = \mathbb{E}[M_0]$ implies that M is not uniformly integrable. Moreover, an application of Doob's maximal identity [29, Lemma 2.1] provides the following concrete representation for the non-decreasing sequence $(c_n)_{n\in\mathbb{N}}$, defined by* (4.6)*:*

$$c_n = \log\left(e + \sum_{k=1}^{n} \frac{1}{k}\right), \quad n \in \mathbb{N}.$$

As the harmonic numbers tend to infinity, we have $\lim_{n\uparrow\infty} c_n = \infty$, which is the crucial ingredient for showing that $M^\sigma \notin \mathcal{H}^1$, where the stopping time σ is given by (4.7)*.*

A Assuming only weak existence in Section 2

We generalize the results from Section 2, by weakening the assumption on strong existence and uniqueness for equations (2.1) and (2.2). Indeed, we remove the assumption on strong existence and uniqueness entirely and only require weak existence and uniqueness of (2.1). This of course increases the scope of models that fulfill these assumptions, but at the same time weakens the assertion of the corresponding Lemma A.3.

Assumption A.1. *Let f, y_0, ρ, b, σ, and K be as in Assumption 2.2. We assume weak existence and uniqueness in law for the stochastic Volterra* *equation* (2.1). *That is, there is some filtered probability space satisfying the usual conditions supporting a one-dimensional Brownian motion* $(B_t)_{t\geq 0}$ *and a continuous process* $(Y_t)_{t\geq 0}$ *such that the pair* (Y, B) *satisfies equation* (2.1). *Further, any such solution pair* (Y, B) *induces the same law on* $C([0,\infty),\mathbb{R}) \times C([0,\infty),\mathbb{R})$.

This yields a similar model definition compared to Definition 2.3.

Definition A.2. *Under Assumption A.1, let* (Y, B) *be a solution pair to* (2.1) *defined on some, from now on fixed, filtered probability space. By appropriately extending this space, we may assume the existence of another Brownian motion* $\tilde{W}$*, independent of* (Y, B). *The* associated stochastic volatility model *with initial value* $S_0 > 0$ *is then defined by*

$$\begin{aligned} \mathrm{d}S_t &= S_t\sqrt{v_t}\mathrm{d}W_t, \\ v_t &= f(t, Y_t), \end{aligned}$$

where the spot price process S *is driven by the Brownian motion* $W = \rho B + \sqrt{1-\rho^2}\tilde{W}$.

Using this definition, we can formulate a similar but weaker result than Lemma 2.4. If one can find a sufficiently integrable and uniform bound on $f(s, \tilde{Y}_s)$ for every weak solution of (2.2), then the supremum is still integrable.

Lemma A.3. *Fix* $T > 0$ *and assume that* S*, from Definition A.2, is a martingale on* $[0, T]$. *Then there exists a weak solution to Equation* (2.2) *and we have the upper bound*

$$\mathbb{E}\left[\sup_{t\in[0,T]} S_t\right] \leq \alpha + \beta \sup \int_0^T \mathbb{E}_{\tilde{\mathbb{P}}}[f(s, \tilde{Y}_s)]\mathrm{d}s, \tag{A.1}$$

where α *and* β *are some positive constants and where the supremum runs over the nonempty set of all weak solutions of* (2.2), *each of which is given by a pair* $(\tilde{Y}, \tilde{B})$ *defined on some filtered probability space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ *satisfying the usual conditions.*

Proof. We follow the proof of Lemma 2.4. Assume w.l.o.g. that $S_0 = 1$ and define the probability measure $\mathrm{d}\hat{\mathbb{P}} = S_T\mathrm{d}\mathbb{P}$. From Girsanov's theorem we obtain that

$$\begin{pmatrix} W^{(1)} \\ W^{(2)} \end{pmatrix} = \begin{pmatrix} B - \rho\int_0^{\cdot\wedge T} \sqrt{v_s}\mathrm{d}s \\ \tilde{W} - \sqrt{1-\rho^2}\int_0^{\cdot\wedge T} \sqrt{v_s}\mathrm{d}s \end{pmatrix}$$

is a Brownian motion under $\hat{\mathbb{P}}$. Lemma 2.1 yields

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} S_t\bigg] \leq \frac{e}{e-1}\bigg(1+\mathbb{E}_{\hat{\mathbb{P}}}\bigg[\int_0^T \sqrt{v_s}\mathrm{d}\big(\rho W_s^{(1)}+\sqrt{1-\rho^2}W_s^{(2)}\big)+\frac{1}{2}\int_0^T v_s\mathrm{d}s\bigg]\bigg).$$

By the Burkholder-Davis-Gundy inequality we then have some $\alpha, \beta > 0$ such that

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} S_t\bigg] \leq \alpha + \beta\int_0^T \mathbb{E}_{\hat{\mathbb{P}}}\left[v_s\right]\mathrm{d}s.$$

Note that

$$\begin{aligned} Y_t &= y_0 + \int_0^t K(t,s)b(s,Y_s)\mathrm{d}s + \int_0^t K(t,s)\sigma(s,Y_s)\mathrm{d}B_s \\ &= y_0 + \int_0^t K(t,s)\left(b(s,Y_s)+\rho\sqrt{f(s,Y_s)}\sigma(s,Y_s)\right)\mathrm{d}s + \int_0^t K(t,s)\sigma(s,Y_s)\mathrm{d}W_s^{(1)}, \end{aligned}$$

for $t \in [0,T]$. Denoting our original model probability space as $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ the last equation implies that $(Y, W^{(1)})$ defined on the space $(\Omega, \mathcal{F}, \mathbb{F}, \hat{\mathbb{P}})$ is a weak solution to (2.2). Hence, there exists a weak solution and obviously we get an upper estimate, by using the supremum over all weak solutions. This concludes the proof. □

Lastly, we want to illustrate Lemma A.3 by an application to affine Volterra processes. We keep the presentation brief, as the integrability of the supremum in Theorem A.6 below follows easily by an application of Doob's L^p-inequality; see the remarks after Theorem 2.6. Recall the general stochastic Volterra equation

$$Y_t = Y_0 + \int_0^t K_{\alpha,\eta}(t,s)\mu(Y_s)\mathrm{d}s + \int_0^t K_{\alpha,\eta}(t,s)\sigma(Y_s)\mathrm{d}W_s, \quad t \geq 0, \tag{A.2}$$

where the kernel $K_{\alpha,\eta}$ is defined as in (3.1) with $\eta > 0$ and $\alpha \in (1/2, 1)$ and W is a Brownian motion. Following [2], we define affine Volterra processes as follows.

Definition A.4 (Affine Volterra process)**.** *Let a and b be affine functions given by*

$$a(x) = a^0 + a^1 y;$$

with scalars $b^0, b^1 \in \mathbb{R}$ and $a^0, a^1 \geq 0$. An affine Volterra process is a continuous weak solution of the stochastic Volterra equation (A.2) *with $\mu(y) = b(y)$ and $\sigma(y) = \sqrt{a(y)}$ and starting value $Y_0 \geq 0$.*

The following theorem, see [2, Lemma 3.1 and Theorem 6.1], allows us to show existence and uniqueness of affine Volterra processes and find uniform bounds on the moments of solutions to (A.2).

Theorem A.5. *The following holds.*

(i) Let Y be a continuous weak solution to (A.2)*, where σ and μ are continuous on $\mathbb{R}$ and satisfy the linear growth condition*

$$|b(y)| \vee |\sigma(y)| \leq c_{\mathrm{LG}}(1 + |y|), \quad y \in \mathbb{R},$$

for some constant c_{LG}. Then for any $T > 0$ and $p \geq 1$, we have

$$\sup_{t \in [0,T]} \mathbb{E}\left[|Y_t|^p\right] \leq c,$$

where the constant c only depends on T, p, $K_{\alpha,\eta}|_{[0,T]}$, Y_0 and c_{LG}.

(ii) Let μ and σ be given as in Definition A.4 with $a^0 = 0$ and $b^0 \geq 0$. Then (A.2) *admits a unique, continuous, and positive weak solution for every initial condition $Y_0 \geq 0$. In particular, the corresponding affine Volterra process exists and is unique in law.*

Proof. Part (i) immediately follows from [2, Lemma 3.1] and part (ii) follows from [2, Theorem 6.1] since the kernel $K_{\alpha,\eta}$ fulfills the necessary assumptions by [2, Examples 2.3 (ii) and 3.7]. □

Applying Theorem A.5 (ii), we can choose a positive affine Volterra process with $a^0 = 0$ and $b^0 \geq 0$. Using the link function $f(y) = \max\{0, y\}$ allows us to consider the associated stochastic volatility model (S, v) as given in Definition A.2. Noting that the stock price process S is a martingale by [2, Theorem 7.1(iii)], we can readily apply Lemma A.3. The corresponding equation for $\tilde{Y}$, see (2.2), is given by

$$\tilde{Y}_t = Y_0 + \int_0^t K(t,s)\left(b^0 + b^1\tilde{Y}_s + \rho\sqrt{a^1\tilde{Y}_s\tilde{Y}_s^+}\right)\mathrm{d}s + \int_0^t K(t,s)\sqrt{a^1\tilde{Y}_s}\mathrm{d}B_s.$$

Since the coefficients of the above equation for $\tilde{Y}$ grow at most linearly, we can apply Theorem A.5 (i) and hence find a uniform upper bound for the supremum in (A.1). We therefore have the following theorem.

Theorem A.6. *Let Y be an affine Volterra process according to Definition A.4 with $a^0 = 0$, $b^0 \geq 0$ and initial condition $Y_0 \geq 0$. Consider the associated stochastic volatility (S, v) model for the link function $f(y) =$ $\max\{0, y\}$. Then it holds*

$$\mathbb{E}\left[\sup_{t \in [0,T]} S_t\right] < \infty.$$

Remark A.7 (Application to rough Heston model)**.** *In particular, the well-known rough Heston model, see [12], is covered by the above theorem, since its volatility process is an affine Volterra process as described in Definition A.4 with* $a^0 = 0$ *and* $b^0 \geq 0$*.*

B Strong existence and uniqueness for stochastic Volterra integral equations

We present an existence and uniqueness result for certain stochastic Volterra equations that in particular applies to equation (2.2) in the situation of the rough Bergomi model. To this end, we first define Volterra kernels of continuous type, see for example [18].

Definition B.1 (Volterra kernel of continuous type)**.** *Recall* $\Delta = \{(t,s) \in [0,\infty)^2 : t \geq s\}$*. Then a measurable function* $\kappa : \Delta \to [0,\infty)$ *is a Volterra kernel of continuous type*[1] *if for each* $t \in [0,\infty)$ *we have* $\kappa(t,.) \in L^1([0,\infty))$ *and the map* $[0,\infty) \ni t \mapsto \kappa(t,.) \in L^1([0,\infty))$ *is continuous.*

To apply [33] in the proof of the upcoming existence and uniqueness result for stochastic Volterra equations, we establish two facts for Volterra kernels of continuous type. The first part, regarding the continuity of the map $t \mapsto \int_0^t \kappa(t,s)g(s)\mathrm{d}s$ for measurable and bounded g, is a special case of [18, Theorem 9.5.3].

Lemma B.2. *Let* κ *be a Volterra kernel of continuous type. Then for each bounded measurable function* $g : \mathbb{R} \to \mathbb{R}$*, the map*

$$t \mapsto \int_0^t \kappa(t,s)g(s)\mathrm{d}s$$

is continuous. Further, for each $T > 0$ *we have*

$$\limsup_{\varepsilon \downarrow 0} \Big\| \int_{\cdot}^{\cdot+\varepsilon} \kappa(\cdot+\varepsilon, s)\mathrm{d}s \Big\|_{L^\infty([0,T])} = 0.$$

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ be bounded and measurable and let $h \geq 0$. Fix $t, u \geq 0$ and first assume $u > t$. Then

$$\begin{aligned} &\Big| \int_0^u \kappa(u+h,s)g(s)\mathrm{d}s - \int_0^t \kappa(t+h,s)g(s)\mathrm{d}s \Big| \\ &\qquad\qquad \leq \|g\|_\infty \Big(\int_t^u \kappa(u+h,s)\mathrm{d}s + \int_0^t |\kappa(u+h,s) - \kappa(t+h,s)|\mathrm{d}s \Big). \end{aligned}$$

[1] By abuse of notation, for $t \in [0,\infty)$ we extend $\kappa(t,.)$ from $[0,t]$ to $[0,\infty)$ and set $\kappa(t,s) = 0$ for $s > t$.

For $u \downarrow t$, the first integral on the right hand side vanishes via an application of dominated convergence and the second due to the assumed L^1-continuity. The case $u < t$ is similar. Hence, taking $h = 0$, the first claim is shown.

For the second claim, fix $\varepsilon > 0$ and note that

$$\int_t^{t+\varepsilon} \kappa(t+\varepsilon, s)\mathrm{d}s = \int_0^{t+\varepsilon} \kappa(t+\varepsilon, s)\mathrm{d}s - \int_0^t \kappa(t+\varepsilon, s)\mathrm{d}s$$

must be a continuous function of t since it is the difference of two continuous functions by the first part of the proof by taking $g = 1$ and $h = 0$ and $h = \varepsilon$, respectively. Hence, the supremum over the compact interval $[0, T]$ is attained by some maximizing $\hat{t}_\varepsilon \in [0, T]$. Hence, we have

$$\Big\| \int_\cdot^{\cdot+\varepsilon} \kappa(\cdot + \varepsilon, s)\mathrm{d}s \Big\|_{L^\infty([0,T])} = \int_{\hat{t}_\varepsilon}^{\hat{t}_\varepsilon+\varepsilon} \kappa(\hat{t}_\varepsilon + \varepsilon, s)\mathrm{d}s.$$

Since the continuous maps

$$t \mapsto \int_0^t \kappa(t, s)\mathrm{d}s \in \mathbb{R} \quad \text{and} \quad t \mapsto \kappa(t, \cdot) \in L^1([0, \infty))$$

are uniformly continuous on the compact set $[0, T+1]$, there exists for every $\eta > 0$ some $\delta > 0$ such that $|u - v| < \delta$ implies

$$\Big| \int_0^u \kappa(u, s)\mathrm{d}s - \int_0^v \kappa(v, s)\mathrm{d}s \Big| < \eta \quad \text{and} \quad \int_0^\infty \Big|\kappa(u, s) - \kappa(v, s)\Big|\mathrm{d}s < \eta.$$

Let $\eta > 0$ be arbitrary, choose δ accordingly, and fix an arbitrary $\varepsilon < \delta$. Then we have

$$\begin{aligned}
&\Big| \int_{\hat{t}_\varepsilon}^{\hat{t}_\varepsilon+\varepsilon} \kappa(\hat{t}_\varepsilon + \varepsilon, s)\mathrm{d}s \Big| \\
&\qquad = \Big| \int_0^{\hat{t}_\varepsilon+\varepsilon} \kappa(\hat{t}_\varepsilon + \varepsilon, s)\mathrm{d}s - \int_0^{\hat{t}_\varepsilon} \kappa(\hat{t}_\varepsilon + \varepsilon, s)\mathrm{d}s \Big| \\
&\qquad \leq \Big| \int_0^{\hat{t}_\varepsilon+\varepsilon} \kappa(\hat{t}_\varepsilon + \varepsilon, s)\mathrm{d}s - \int_0^{\hat{t}_\varepsilon} \kappa(\hat{t}_\varepsilon, s)\mathrm{d}s \Big| \\
&\qquad\qquad + \Big| \int_0^{\hat{t}_\varepsilon} \kappa(\hat{t}_\varepsilon, s)\mathrm{d}s - \int_0^{\hat{t}_\varepsilon} \kappa(\hat{t}_\varepsilon + \varepsilon, s)\mathrm{d}s \Big| \\
&\qquad < \eta + \int_0^\infty \Big|\kappa(\hat{t}_\varepsilon + \varepsilon, s) - \kappa(\hat{t}_\varepsilon, s)\Big|\mathrm{d}s < 2\eta.
\end{aligned}$$

Since η was arbitrary, this completes the proof. □

Theorem B.3. *Let κ be a Volterra kernel of continuous type and let Z be a continuous and adapted real-valued process starting in zero. Further, let $g : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function, non-decreasing in the second argument, and such that for every $n \in \mathbb{N}$ and any $T > 0$ there is some constant $L(n,T) > 0$ such that*

$$\sup_{t\in[0,T]} |g(t,x) - g(t,y)| \leq L(n,T)|x-y|$$

for all $x, y \in \mathbb{R}$ with $|x| \leq n$ and $|y| \leq n$. Then the stochastic Volterra integral equation

$$X_t = Z_t - \int_0^t \kappa(t,s)g(s,X_s)\mathrm{d}s, \quad t \geq 0,$$

has a unique solution X, which is adapted and continuous.

Proof. To truncate the function g for each $n \in \mathbb{N}$ we define

$$g_n(s,y) = \begin{cases} g(s\wedge n, y), & \text{for} \quad |y| \leq n, \\ g\left(s\wedge n, \frac{y}{|y|}n\right), & \text{for} \quad |y| > n, \end{cases}$$

for $s \geq 0$. The resulting functions g_n are continuous and bounded by the continuity of g. Further, we have for each $t \geq 0$ and all $x, y \in \mathbb{R}$

$$|g_n(t,x) - g_n(t,y)| \leq L(n,n)|x-y|.$$

Additionally define the stopping times $\sigma_n = \inf\{t \geq 0 : |Z_t| \geq n\}$. By Lemma B.2 we can apply [33, Theorem 3.1], which yields that there exist unique adapted processes X^n such that for almost all $t \geq 0$ we have

$$X_t^n = Z_t^{\sigma_n} - \int_0^t \kappa(t,s)g_n(s,X_s^n)\mathrm{d}s.$$

By the boundedness of g_n, the right-hand side of the above equation is continuous according to Lemma B.2. Hence, we may assume that X^n is continuous and satisfies the above equation for all $t \geq 0$ and up to indistinguishability. To remove the dependence on n, first define the stopping times $\tau_n = \inf\{t \geq 0 : |X_t^n| \geq n\} \wedge n \wedge \sigma_n$, for each $n \in \mathbb{N}_0$ and note that $\tau_0 = 0$. We first establish two facts:

(1) For $n \geq 1$ and each $t \in [0, \tau_{n-1}]$ we have $X_t^{n-1} = X_t^n$. To see this, note that for $s \leq \tau_{n-1}$ we have $|X_s^{n-1}| \leq n-1 < n$ and hence for $t \in [0, \tau_{n-1}]$

we have

$$\begin{aligned} X_t^{n-1} &= Z_t^{\sigma_{n-1}} - \int_0^t \kappa(t,s) g_{n-1}(s, X_s^{n-1}) \mathrm{d}s \\ &= Z_t^{\sigma_n} - \int_0^t \kappa(t,s) g_n(s, X_s^{n-1}) \mathrm{d}s. \end{aligned}$$

Uniqueness and continuity therefore yield $X_{n-1} = X_n$ on $[0, \tau_{n-1}]$.

(2) We have $\tau_{n-1} \leq \tau_n$ for each $n \geq 1$, which follows immediately from (1): On $[0, \tau_{n-1}]$ the process X^n agrees with X^{n-1}, which implies $|X_t^n| \leq n-1$ on $[0, \tau_{n-1}]$. Hence, for $t \leq \tau_{n-1}$ we have $|X_t^n| < n$.

Now, we define for each $t \geq 0$

$$X_t = \sum_{n=1}^{\infty} X_t^n 1_{[\tau_{n-1}, \tau_n)}(t),$$

which defines an adapted process because X^n is adapted and $\{\tau_{n-1} \leq t < \tau_n\} \in \mathcal{F}_t$ for each $n \in \mathbb{N}$. Clearly, X is continuous on $[0, \lim_{n\uparrow\infty} \tau_n)$ by (1) and (2) and jumps to zero afterwards.

To see that X is a solution of the Volterra equation on $[0, \lim_{n\uparrow\infty} \tau_n)$, we note for $t \in [\tau_{n-1}, \tau_n)$, using $Z_t^{\sigma_n} = Z_t$,

$$\begin{aligned} X_t = X_t^n &= Z_t - \int_0^t \kappa(t,s) g_n(s, X_s^n) \mathrm{d}s \\ &= Z_t - \sum_{l=1}^n \int_{\tau_{l-1}\wedge t}^{\tau_l \wedge t} \kappa(t,s) g_n(s, X_s^n) \mathrm{d}s \\ &= Z_t - \sum_{l=1}^n \int_{\tau_{l-1}\wedge t}^{\tau_l \wedge t} \kappa(t,s) g(s, X_s^l) \mathrm{d}s = Z_t - \int_0^t \kappa(t,s) g(s, X_s) \mathrm{d}s. \end{aligned}$$

Lastly, we have by the properties of the function g that any solution of our Volterra equation must satisfy

$$Z_t - \int_0^t \kappa(t,s) g(s, Z_s) \mathrm{d}s \leq X_t \leq Z_t.$$

Hence, we have continuous a-priori lower and upper bounds for the process X, which imply that it cannot explode in finite time, yielding $\lim_{n\uparrow\infty} \tau_n = \infty$ and hence global existence of the unique and continuous solution process X. This concludes the proof. □

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