Negative Call Prices*

Johannes Ruf†
Oxford-Man Institute of Quantitative Finance and Department of Mathematics
University of Oxford

Monday 17th December, 2012

Abstract

We show that the existence of an equivalent local martingale measure for asset prices does not prevent negative prices for European calls written on positive stock prices. In particular, we illustrate that many standard no-arbitrage arguments implicitly rely on conditions stronger than the No Free Lunch With Vanishing Risk (NFLVR) assumption. The discrepancy between replicating prices and market prices for a contingent claim may be observed in a model satisfying NFLVR since certain trading strategies of buying one portfolio and selling another one are often excluded by standard admissibility constraints.
Keywords: Pricing; Hedging; Arbitrage; Admissibility; NFLVR; Contingent Claim; Strict local martingale
JEL classification code: G13

1 Introduction

In the following, we illustrate how contingent claim prices can become negative, although the terminal payoff associated to the contingent claim is nonnegative. The economy that we consider as an example satisfies the assumption of *No Free Lunch With Vanishing Risk (NFLVR)*: In particular, there exists no (admissible) trading strategy that starts with zero initial wealth, has a wealth process uniformly bounded from below, and leads to a terminal wealth that is always nonnegative and strictly positive with positive probability. The NFLVR assumption in conjunction with local boundedness of asset prices is equivalent to the existence of an *equivalent local martingale measure (ELMM)*, to wit, a probability measure that is equivalent to the original one, and under which all asset price processes in the economy are local martingales. For a precise statement of NFLVR and the proof of this equivalence, we refer the reader to Delbaen and Schachermayer (1994, 1998, 2006).

The following discussion involves market prices; these usually do not have to agree with replicating prices. This is illustrated by the existence of *bubbles*, which are asset price processes whose current market price is higher than the costs for replicating them at some given time in the future; see Jarrow et al. (2007), Ruf (2012), and the discussion below. In the economic literature, a bubble is sometimes interpreted as an asset that is overpriced, but nevertheless bought by agents since they believe that the asset can be sold in the future at an even higher price before the "bubble bursts." Our example below can be interpreted similarly: It discusses an asset that is underpriced but nevertheless sold at the current price, which is lower than its

*I am grateful to Travis Fisher, Mike Hogan, Ioannis Karatzas, Arseniy Kukanov, Radka Pickova, Philip Protter, Sergio Pulido, Murad Taqqu, and Mike Tehranchi for fruitful discussions on the subject matter of this paper. I thank an anonymous referee for her or his helpful comments. This work was partially supported by the National Science Foundation DMS Grant 09-05754.
†E-mail: johannes.ruf@oxford-man.ox.ac.uk

intrinsic value, since the price might decrease even further in the future. To the best of our knowledge, models for an economic *depression* have not been discussed in the framework of arbitrage-free pricing.

It is not the purpose of this paper to make a case for the existence of negative call prices. On the contrary, we are convinced that negative call prices or, more generally, negative prices for contingent claims with positive terminal payoffs, should be excluded in an economy where agents prefer more to less.[1] However, it is our aim to convey that many no-arbitrage arguments, such as the one showing the equality of American and European call prices for stocks that do not pay dividends, implicitly rely on stronger assumptions than just the existence of an ELMM.

Admissibility constraints

Describing the class of *admissible* trading strategies, defined as trading strategies that an agent is allowed to follow, is essential for any formulation of a *Fundamental Theorem of Asset Pricing* (FTAP) when asset price processes are exogenously given. An FTAP is usually formulated as the equivalence of the lack of an arbitrage opportunity and the existence of a certain probability measure, the so-called risk-neutral measure, under which asset prices have certain dynamics. Towards this end, a precise definition of an arbitrage opportunity needs to be given. Indeed, in any non-trivial infinite-horizon discrete-time or continuous-time model, such as the Black-Scholes model, notorious *doubling strategies* exist, which lead to an arbitrage opportunity if they are not prohibited; see Section 6 of Harrison and Kreps (1979).

In order to avoid the trivial statement that an arbitrage opportunity exists in any continuous-time model, certain restrictions on the class of admissible trading strategies from which arbitrage opportunities may be selected have to be enforced. It is clear that the larger the class of admissible trading strategies is chosen, the stronger are the assumptions on the risk-neutral measure in order to have equivalence in the FTAP. In all cases we are aware of, admissible trading strategies are defined as the ones which lead to wealth processes that are somehow bounded from below.

The classical approach, as suggested by Harrison and Kreps (1979) and Delbaen and Schachermayer (1994), is to require the wealth process to be uniformly bounded from below by a (negative) constant. This can be motivated from an economic perspective as a margin requirement: As soon as an agent's ("she") wealth reaches some specified negative wealth, her broker forces her to cancel her position and prevents her from further trading. Under this admissibility constraint, no arbitrage (in the sense of NFLVR) corresponds to the existence of an equivalent probability measure, under which all asset price processes follow local martingale dynamics given they are locally bounded.

Yan (1998) suggests to use a larger class of admissible trading strategies, namely the ones whose associated wealth process is bounded from below by a (negative) constant times the market portfolio. In particular, as the asset prices increase, the wealth process of an admissible trading strategy is allowed to become more and more negative. As observed before, the extension of the class of admissible trading strategies implies a stronger no-arbitrage condition and thus leads to a risk-neutral measure satisfying a stronger condition; here, one under which all asset price processes follow true martingale dynamics.

The advantage of Yan's admissibility constraint is that it is independent of the choice of numéraire. Moreover, it excludes many pathologies such as the one studied here. However, its strong no-arbitrage assumptions exclude the possibility to model several important phenomena, such as bubbles as strict local martingales, relative arbitrage opportunities as in Fernholz and Karatzas (2009), or quadratic normal volatility models, which provide certain symmetry properties under a change of numéraire, as studied in Carr et al.

[1] Negative asset prices can, however, be observed in the market, for instance in the wind energy market. These negative prices occur primarily due to storage costs; see for the example the Bloomberg article *Windmill Boom Cuts Electricity Prices in Europe* by J. van Loon from April 23, 2010, retrieved from `http://www.bloomberg.com/news/2010-04-22/windmill-boom-curbs-electric-power-prices.html`. In this paper, we assume a frictionless market, in particular, an agent does not incur costs from holding an asset.

(2012a,b).

Given an economy, under which asset price processes follow local martingale dynamics, for instance, it is interesting to extend the class of admissible trading strategies without introducing arbitrage. This was for example done in Proposition 4.1 of Heston et al. (2007) and more generally, in Strasser (2003), where a criterion is given on trading strategies, such that the corresponding wealth processes are supermartingales.

After this discussion, the subtle reason for the existence of arbitrage-free models with counter-intuitive price processes is clear. A price might seem to imply an arbitrage opportunity but agents in the economy are not permitted to profit from this ostensible arbitrage, due to admissibility constraints in their set of trading strategies. More precisely, standard no-arbitrage arguments often imply the construction of a trading strategy consisting of selling one asset (for example, an European call) and buying another asset (for example, an American call). It is implicitly utilized that such a trading strategy is admissible. Thus, this argument resembles more an assumption on the admissibility of a trading strategy than a clean no-arbitrage argument.

A related way to think about the existence of prices that seem to contradict simple no-arbitrage arguments is to study strict local martingales; that is, local martingales that are not martingales. Any local martingale that is bounded from below by a constant is a supermartingale by Fatou's lemma; thus, any local martingale that is bounded from above by a constant is a submartingale. Therefore, if an asset price is modeled as a strict local martingale that is bounded from above, then the trading strategy of holding that asset for a fixed time is inadmissible, since its corresponding wealth process is not a supermartingale, but a submartingale. In the example below, we will make use of this insight.

Indeed, the existence of *bubbles*, modeled as positive strict local martingales, in models satisfying NFLVR is justified in the literature by the observation that selling such assets might represent an inadmissible trading strategy; see Cox and Hobson (2005), Heston et al. (2007), and Jarrow et al. (2007). Their argument can be marginally generalized by not restraining oneself to trading strategies that lead to a wealth process bounded from below, but by using the larger class of trading strategies discussed in Strasser (2003).

2 Example

In the following, we provide an example for an economy that satisfies NFLVR but allows for a negative call price. To this end, we fix a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$, where the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ is generated by a Brownian motion $B(\cdot)$. We model an asset with initial price $S_1(0) = 1$ and price dynamics given as a geometric Brownian motion; that is,

$$\mathrm{d}S_1(t) = S_1(t)\mathrm{d}B(t)$$

for all $t \geq 0$.

We now consider an European at-the-money call with maturity 1 written on $S_1(\cdot)$; to wit, we study an asset that at time 1 pays precisely $D = (S_1(1) - 1)^+$, where x^+ denotes the maximum of x and zero. Black and Scholes (1973) and Merton (1973) derive the replicating price $C(\cdot)$ of this call, assuming zero interest rate, as

$$\begin{aligned} C(t) &= \mathbb{E}[D|\mathcal{F}(t)] \\ &= S_1(t)\Phi\left(\frac{1}{\sqrt{1-t}}\left(\log(S_1(t)) + \frac{1-t}{2}\right)\right) - \Phi\left(\frac{1}{\sqrt{1-t}}\left(\log(S_1(t)) - \frac{1-t}{2}\right)\right) \\ &\geq 0 \end{aligned}$$

for all $t \in [0,1]$, where Φ denotes the standard normal cumulative distribution function.

We now set

$$M(t) = \int_0^t \mathbf{1}_{\{\varrho > s\}} \frac{1}{\sqrt{1-s}} \mathrm{d}B(s)$$

for all $t \in [0,1]$, where

$$\varrho := \inf \left\{ t \in [0,1] : \int_0^t \frac{1}{\sqrt{1-s}} \mathrm{d}B(s) = C(0) + 1 \right\}. \tag{1}$$

Then, we have $\varrho < 1$, which yields $M(1) = C(0) + 1$. This holds since the integral appearing in (1) is a continuous local martingale with quadratic variation $-\log(1-t)$, which tends to infinity as t tends to one. Thus, it can be represented as time-changed Brownian motion, which almost surely hits $C(0) + 1$.

We introduce a second asset with a price process $S_2(\cdot)$ specified as

$$S_2(t) = C(t \wedge 1) + M(t \wedge 1) - C(0) - 1 \tag{2}$$

for all $t \geq 0$, where $x \wedge y$ denotes the minimum of x and y. We observe that $S_2(0) = -1$ and that $S_2(\cdot)$ is a local martingale that is neither bounded from above nor from below by a constant, but is a submartingale, as it is the sum of a martingale and a submartingale. Furthermore, and most importantly, $S_2(1) = C(1) = D$.

We now consider an economy consisting of a money market account paying zero interest rate and two assets with price processes given by $S_1(\cdot)$ and $S_2(\cdot)$, as specified above. We observe that this economy satisfies NFLVR, since both $S_1(\cdot)$ and $S_2(\cdot)$ are local martingales under the probability measure $\mathbb{P}$. Moreover, the second asset can be considered the price of a call written on the first asset with exercise price 1, since its terminal payoff is exactly $D = (S_1(1) - 1)^+$. In accordance to standard theory, we take exactly these trading strategies that lead to wealth processes bounded from below by a constant as the class of admissible trading strategies.

Any agent in this economy can replicate this call written on $S_1(1)$ for the price of $C(0) > 0 > -1 = S_2(0)$. However, despite the existence of a market price for a call in the market, no arbitrage opportunity exists in this economy since the agent is not allowed to build a position that includes buying the call for a fixed time with price process $S_2(\cdot)$. To see this, consider the wealth process of a trading strategy that sells a portfolio that replicates the call with the dynamic Black-Scholes-Merton trading strategy, buys the second asset with price $S_2(0)$, and puts the profits of building this position in the money market account. The corresponding wealth process $W(\cdot)$ is thus the sum of three positions: a long position in $S_2(\cdot)$, a short position in the replicating portfolio, and a holding in the money market. In other words, the wealth $W(t)$ at time $t \in [0,1]$ is exactly

$$W(t) = S_2(t) - C(t) + (C(0) - S_2(0)) = M(t).$$

In particular, the initial wealth is zero, to wit, $W(0) = 0$, and the terminal wealth is strictly positive, to wit, $W(1) = C(0) + 1 > 0$. However, $W(\cdot)$ is not bounded from below as it is a time-changed (stopped) Brownian motion. Thus, this trading strategy is not an arbitrage strategy since it is not admissible. Even if one extends the class of admissible trading strategies in the sense of Strasser (2003), this trading strategy is still not admissible as $W(\cdot)$ is not a supermartingale.

Observe that we did not use any specific properties of call prices. Indeed, in (1) and (2), we can replace $C(\cdot)$ by any nonnegative martingale $\widetilde{C}(\cdot)$ representing the minimal replicating cost of a contingent claim with payoff $\widetilde{C}(1)$ at time 1. In this modified market, $S_2(\cdot)$ can then be interpreted as the price process of a contingent claim that pays $\widetilde{C}(1)$ at time 1; observe that the initial price is again $S_2(0) = -1$.

Admittedly, this example is quite pathological: It corresponds to an economy in which it is inadmissible to hold the second asset for a fixed deterministic time, although it is clearly admissible to hold the asset until a certain stopping time. However, this example also emphasizes that such pathological price processes as negative European call prices are not excluded by the NFLVR assumption. Thus, any no-arbitrage argument based on constructing a trading strategy must ensure that this trading strategy is admissible. To illustrate, the standard argument that a European call price for a strike K is bounded from below by $S_1(0) - K$ is often formulated as follows: Assume that the call price is smaller than $S_1(0) - K$. Then, consider the following trading strategy: Buy the call, sell the stock, borrow K dollars and put the leftover money in the bank account. At maturity, this trading strategy has corresponding wealth of at least the positive amount of money in the bank account and thus seems to imply the existence of an arbitrage opportunity. However, in our example above, this trading strategy would already be inadmissible.

3 Concluding remarks

It is important to emphasize that the discussion so far only involved European-style contingent claims. For example, American calls being in-the-money cannot have negative prices under the NFLVR assumption. An agent could buy such an American-style contingent claim and immediately exercise it, collecting at least the contingent claim's negative price. Bayraktar et al. (2012) observe that put-call parity holds, as long as the European call price is exchanged by the corresponding American call price; however, they also (explicitly) assume that both the European put price and the American call price are the corresponding replicating prices. In the same manner as above, it is easy to construct an arbitrage-free economy where put-call parity does not hold, even after replacing the European by the American call.

The discussion in Madan and Yor (2006) is related to different arbitrage arguments that can be made with respect to American and European-style contingent claims; they discuss, in the context of bubbles, robustness of trading strategies with respect to "random early liquidations." For example, in the economy above, consider the two trading strategies of selling the call with corresponding wealth process $W_1(t) = S_2(0) - S_2(t) = -1 - S_2(t)$ and of selling the Black-Scholes-Merton replicating portfolio with corresponding wealth process $W_2(t) = C(0) - C(t)$ for all $t \in [0, 1]$. Observe that $W_1(0) = 0 = W_2(0)$ and $W_1(T) < W_2(T)$. Both trading strategies are admissible under the weak admissibility constraints of Strasser (2003). The second trading strategy seems better than the first one as it leads to a higher terminal wealth. However, if an agent has to cover a short position in the call $S_2(\cdot)$ and bears the risk that her counterparty might liquidate this short position at some time $t \in (0, 1)$, she cannot follow the trading strategy of replicating the call's terminal payoff since the wealth process $W_2(t)$ can be (unboundedly far) below $W_1(t)$.

Similarly, Cox and Hobson (2005) discuss collateral requirements for European-style contingent claims. If an agent followed the Black-Scholes-Merton trading strategy to obtain a terminal wealth of $C(0) - C(1)$, her wealth process might not satisfy such a collateral requirement, which is basically an American-style feature and forces one's wealth process to stay above a certain barrier that, in this case, depends on the call price $S_2(\cdot)$.

We have illustrated that simple no-arbitrage arguments rely on more assumptions than only on the existence of an ELMM. Even if an agent observed negative European call prices quoted according to the example above, she could not achieve a nonnegative and with positive probability positive wealth at a later time by starting from zero initial wealth and following an admissible trading strategy. From the purely economic perspective of equilibrium pricing, the above example is of little insight. No agent in that economy is allowed to hold the call for a fixed time. However, standard no-arbitrage proofs do not include this point in their argument.

What assumptions do simple no-arbitrage arguments, relying on selling and buying certain assets, implicitly make? This question can be addressed in several ways. A technical assumption could be to consider only assets whose price processes are true martingales under a fixed ELMM. Then, both buying and selling these assets (and a combination of buying and selling) yield admissible trading strategies.

An assumption in more economic terms is the *no-dominance* principle, as suggested by Merton (1973), which is a slightly stronger assumption than NFLVR. The no-dominance principle basically states that if trading strategy A leads to a wealth greater than or equal to the wealth of trading strategy B, then the initial cost of trading according to A should be greater than or equal to the initial cost of trading according to

B. For instance, if no dominance holds, then European call prices on a nonnegative stock price have to be nonnegative. To see this, compare the trading strategy of holding the call to the trading strategy of doing nothing at all, costing zero and leading to a terminal wealth of zero, which is less than or equal to the terminal wealth corresponding to holding the call. Thus, the no-dominance principle yields that any call price has to be nonnegative. For details on the no-dominance principle and for a study how far this additional assumption can take us, see Jarrow et al. (2010).

References

Bayraktar, E., Kardaras, C., and Xing, H. (2012). Strict local martingale deflators and pricing American call-type options. *Finance and Stochastics*, 16(2):275–291.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654.

Carr, P., Fisher, T., and Ruf, J. (2012a). On the hedging of options on exploding exchange rates. Preprint, arXiv:1202.6188.

Carr, P., Fisher, T., and Ruf, J. (2012b). Why are quadratic normal volatility models analytically tractable? Preprint, arXiv:1202.6187.

Cox, A. and Hobson, D. (2005). Local martingales, bubbles and option prices. *Finance and Stochastics*, 9(4):477–492.

Delbaen, F. and Schachermayer, W. (1994). A general version of the Fundamental Theorem of Asset Pricing. *Mathematische Annalen*, 300(3):463–520.

Delbaen, F. and Schachermayer, W. (1998). The Fundamental Theorem of Asset Pricing for unbounded stochastic processes. *Mathematische Annalen*, 312(2):215–250.

Delbaen, F. and Schachermayer, W. (2006). *The Mathematics of Arbitrage*. Springer.

Fernholz, E. R. and Karatzas, I. (2009). Stochastic Portfolio Theory: a survey. In Bensoussan, A., editor, *Handbook of Numerical Analysis*, volume Mathematical Modeling and Numerical Methods in Finance. Elsevier.

Harrison, J. M. and Kreps, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20(3):381–408.

Heston, S., Loewenstein, M., and Willard, G. (2007). Options and bubbles. *Review of Financial Studies*, 20(2):359–390.

Jarrow, R., Protter, P., and Shimbo, K. (2007). Asset price bubbles in complete markets. In Fu, M. C., Jarrow, R. A., Yen, J.-Y. J., and Elliott, R. J., editors, *Advances in Mathematical Finance*, volume in honor of Dilip Madan, pages 97–121. Birkhäuser.

Jarrow, R., Protter, P., and Shimbo, K. (2010). Asset price bubbles in incomplete markets. *Mathematical Finance*, 20(2):145–185.

Madan, D. and Yor, M. (2006). Itô's integrated formula for strict local martingales. In *Séminaire de Probabilités, XXXIX*, pages 157–170. Springer.

Merton, R. C. (1973). Theory of rational option pricing. *Bell Journal of Economics*, 4(1):141–183.

Ruf, J. (2012). Hedging under arbitrage. *Mathematical Finance*, forthcoming.

Strasser, E. (2003). Necessary and sufficient conditions for the supermartingale property of a stochastic integral with respect to a local martingale. In *Séminaire de Probabilités, XXXVII*, pages 385–393. Springer.

Yan, J.-A. (1998). A new look at the Fundamental Theorem of Asset Pricing. *Journal of the Korean Mathematical Society*, 35(3):659–673.