Impulse and absolutely continuous ergodic control of one-dimensional Itô diffusions

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Summary. We consider a problem that combines impulse control with absolutely continuous control of the drift of a general one-dimensional Itô diffusion. The objective of the control problem is to minimise an ergodic or long-term average criterion that penalises both deviations of the state process from a given nominal point and the use of control effort. Our analysis completely characterises the optimal strategy.

Keywords: Itô diffusions, impulse control, absolutely continuous control, ergodic criterion

Mathematics Subject Classification (2000): 93E20, 49J40, 49N25

1 Introduction

We consider a stochastic system, the state of which is modelled by the controlled, one-dimensional Itô diffusion

$$dX_t = U_t\,dt + dZ_t + \sigma(X_t)\,dW_t, \quad X_0 = x \in \mathbb{R}, \tag{1}$$

where W is a standard, one-dimensional Brownian motion, U is a progressively measurable process such that

$$U_t \in [-b(X_t), b(X_t)], \quad \text{for all } t \geq 0, \tag{2}$$

and Z is a controlled, piece-wise constant process, the jumps of which occur at the times when control effort is exercised in an impulsive way to reposition the system's state by an amount equal to the associated jump sizes. The objective of the optimisation problem is to minimise the long-term average criterion

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$$\limsup_{T\to\infty} \frac{1}{T} E_x \Bigg[\int_0^T h(X_t)\,dt + \sum_{t\in[0,T]} \left(K^+ \Delta Z_t + c^+\right) \mathbf{1}_{\{\Delta Z_t > 0\}} \\ + \sum_{t\in[0,T]} \left(-K^- \Delta Z_t + c^-\right) \mathbf{1}_{\{\Delta Z_t < 0\}} \Bigg],$$

which is taken to be equal to ∞ if X explodes in finite time with positive probability, over all admissible choices of the controlled processes U and Z. Here, h is a given function that is strictly decreasing in $]-\infty, 0[$ and strictly increasing in $]0, \infty[$, and c^+, c^-, K^+, K^- are positive constants. This performance index penalises deviations of the state process X from the nominal operating point 0. While the index does not explicitly penalise the expenditure of control effort associated with an admissible choice of U, which is constrained by (2), it reflects a cost each time that control is exercised in an impulsive way. In particular the constants c^+ and K^+ (resp., c^- and K^-) provide a fixed and a proportional cost each time that the controller incurs a jump of the system's state in the positive (resp., negative) direction.

This problem provides one of the few non-trivial examples of optimal stochastic control models that admit a solution of an explicit analytic nature. The version of the problem that arises when the drift of (1) is not controllable has been solved by Jack and Zervos [5]. Both of these problems have been motivated by the research presented in Jeanblanc-Picqué [6], Mundaca and Øksendal [8], Cadenillas and Zapatero [1, 2], and Chiarolla and Haussmann [3] who consider the issue of controlling in an optimal way the stochastic dynamics of a foreign exchange (FX) or an inflation rate by means of a central bank intervention policy.

With regard to these references, we can see that the optimisation problem that we consider can be of use to a central bank in its task of controlling an FX rate as follows. The process X is used to model the stochastic dynamics of the logarithm of an FX rate relative to a given nominal point. The central bank wishes to keep the rate as close as possible to its given nominal point, which translates to 0 in the state space of X. To achieve this aim, the central bank uses the function h to penalise deviations of the rate from its nominal value. To control the rate, the central bank has two intervention policies at its disposal. The first one is through the continuous adjustment of its interest rate, the effect of which is modelled by the process U. The second policy is to purchase or sell large amounts of foreign capital at discrete times, the effect of which is incorporated into the model through the jumps of the process Z. In contrast to the above mentioned references where discounted criteria are considered, here, as well as in Jack and Zervos [5], we consider a long-term average criterion. Since an FX rate is not an asset and the function h does not represent a tangible cost, the choice of a discounting factor does not have a clear economic interpretation. This observation suggests that addressing this type of application using a long-term average criterion rather than a discounted one conforms better with the standard economics theory.

Our analysis is based on the explicit construction of an appropriate solution to the associated Hamilton-Jacobi-Bellman (HJB) equation. This construction relies on the use of the so-called "smooth-pasting condition" that was first observed to characterise a wide class of optimal stopping problems (e.g., see Shiryaev [9] and Krylov [7]). Also, part of it follows steps that parallel the ones used in the analysis of Harrison, Sellke and Taylor [4] who consider the impulse control of a Brownian motion with an expected, discounted criterion. With regard to the structure of the problem that we solve, it is worth noting that, even though the dynamics modelled by (1) allow for the possibility that the state process X explodes in finite time, our assumptions ensure that the optimal control strategy is a "stabilising" one.

2 The control problem

We consider a stochastic system, the state process X of which is driven by a Brownian motion W, a controlled process U that affects the system's dynamics in an absolutely continuous way and a controlled process Z that affects the system's dynamics impulsively. In particular, we assume that the system's state process satisfies the controlled SDE

$$dX_t = U_t \, dt + dZ_t + \sigma(X_t) \, dW_t, \quad X_0 = x \in \mathbb{R}, \tag{3}$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is a given function and W is a standard, one-dimensional Brownian motion. Here, U is a process such that, for some given function $b : \mathbb{R} \to [0, \infty[$,

$$U_t \in [-b(X_t), b(X_t)], \quad \text{for all } t \geq 0, \tag{4}$$

and Z is a piece-wise constant, càglàd process. The time evolution of both of these processes is determined by the system's controller. With reference to the current impulse control literature, it is worth observing that an admissible choice of a process Z can equivalently be described by the collection

$$\mathcal{Z} = (\tau_1, \tau_2, \ldots, \tau_n, \ldots; \Delta Z_{\tau_1}, \Delta Z_{\tau_2}, \ldots, \Delta Z_{\tau_n}, \ldots),$$

where $(\tau_n, \ n \geq 1)$ is the sequence of random times at which the jumps of Z occur and $(\Delta Z_{\tau_n}, \ n \geq 1)$ are the sizes of the corresponding jumps.

We adopt a weak formulation of the control problem that we study:

Definition 1. *Given an initial condition $x \in \mathbb{R}$, a control of a stochastic system governed by dynamics as in (3) is any nine-tuple*

$$\mathbb{C}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, P_x, W, U, Z, X, \tau),$$

where

$(\Omega, \mathcal{F}, \mathcal{F}_t, P_x)$ is a filtered probability space satisfying the usual conditions,
W is a standard, one-dimensional $(\mathcal{F}_t)$-Brownian motion,
U is an $(\mathcal{F}_t)$-progressively measurable process,
Z is a finite variation, piece-wise constant, càglàd, $(\mathcal{F}_t)$-adapted process with $Z_0 = 0$, and
X is a càglàd, $(\mathcal{F}_t)$-adapted process such that (3) and (4) are well defined and satisfied up to the explosion time τ.

We define $\mathcal{C}_x$ to be the family of all such controls $\mathbb{C}_x$.

With each control $\mathbb{C}_x \in \mathcal{C}_x$, we associate the performance criterion defined by

$$J(\mathbb{C}_x) := \limsup_{T\to\infty} \frac{1}{T} E_x \Bigg[\int_0^T h(X_t)\, dt + \sum_{t\in[0,T]} \left(K^+ \Delta Z_t + c^+ \right) \mathbf{1}_{\{\Delta Z_t > 0\}} + \sum_{t\in[0,T]} \left(-K^- \Delta Z_t + c^- \right) \mathbf{1}_{\{\Delta Z_t < 0\}} \Bigg], \quad \text{if } P_x\left(\tau = \infty\right) = 1, \tag{5}$$

where $\Delta Z_t := Z_{t+} - Z_t$, and by

$$J(\mathbb{C}_x) := \infty, \quad \text{if } P_x\left(\tau = \infty\right) < 1. \tag{6}$$

Here, $h : \mathbb{R} \to \mathbb{R}$ is a given function that models the running cost resulting from the system's operation and $K^+, c^+, K^-, c^- > 0$ are given constants penalising the use of impulsive control effort.

The objective of the control problem is to minimise the performance criterion defined by (5)–(6) over all controls $\mathbb{C}_x \in \mathcal{C}_x$. The following assumption on the problem's data is sufficient for our optimisation problem to be well posed.

Assumption 1 *The following conditions hold:*
(a) There exists $C_1 > 0$ such that

$$0 < \sigma^2(x) \le C_1(1 + |x|), \quad \textit{for all } x \in \mathbb{R}, \tag{7}$$

(b) For all $x \in \mathbb{R}$, there exists $\varepsilon > 0$ such that

$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + b(s)}{\sigma^2(s)}\, ds < \infty, \tag{8}$$

(c) h is continuous, strictly decreasing on $]-\infty, 0[$ and strictly increasing on $]0, \infty[$. Also, $h(0) = 0$, and there exists a constant $C_2 > 0$ such that

$$h(x) \ge C_2(|x| - 1), \quad \textit{for all } x \in \mathbb{R}. \tag{9}$$

(d) Given any constant $\gamma \in \mathbb{R}$,

$$\lim_{x \to \pm\infty} \frac{1}{\sigma^2(x)} \left[h(x) + b(x)\gamma\right] = \infty. \tag{10}$$

(e) There exist $a_- \leq a_+$ such that the function

$$h(\cdot) - b(\cdot)K^- \begin{cases} \textit{is strictly decreasing on }]-\infty, a_-[, \\ \textit{is strictly negative inside }]a_-, a_+[, \ \textit{ if } a_- < a_+, \\ \textit{is strictly increasing on }]a_+, \infty[. \end{cases} \tag{11}$$

(f) There exist $\alpha_- \leq \alpha_+$ such that the function

$$h(\cdot) - b(\cdot)K^+ \begin{cases} \textit{is strictly decreasing on }]-\infty, \alpha_-[, \\ \textit{is strictly negative inside }]\alpha_-, \alpha_+[, \ \textit{ if } \alpha_- < \alpha_+, \\ \textit{is strictly increasing on }]\alpha_+, \infty[. \end{cases} \tag{12}$$

(g) $K^+, c^+, K^-, c^- > 0$.

It is worth noting that the conditions in this assumption involve no convexity properties such as the ones often imposed in the stochastic control literature. Also, although they appear to be involved, they are quite general and easy to verify in practice.

Example 1. If we choose

$$b(x) = \beta|x| + \gamma, \quad \sigma(x) = \zeta \quad \text{and} \quad h(x) = \theta|x|^p,$$

for some constants $\beta, \gamma > 0$, $\zeta \neq 0$, $\theta > 0$ and $p > 1$, then Assumption 1 holds.

Remark 1. It is worth noting that we can easily dispense of the assumption that h is continuous. However, we decided against such a relaxation because it would complicate the presentation of part of our analysis.

3 The solution to the control problem

With regard to the general theory of stochastic control, the solution to the control problem formulated in the previous section can be obtained by finding a sufficiently, for an application of Itô's formula, smooth function w and a constant λ satisfying the HJB equation

$$\begin{aligned} \min\Big\{ &\frac{1}{2}\sigma^2(x)w''(x) - b(x)|w'(x)| + h(x) - \lambda, \\ &c^+ - w(x) + \inf_{z \geq 0} \left[w(x+z) + K^+ z\right], \\ &c^- - w(x) + \inf_{z \leq 0} \left[w(x+z) - K^- z\right] \Big\} = 0. \end{aligned} \tag{13}$$

If such a pair (w, λ) exists, then, subject to suitable technical conditions, we expect the following. Given any initial condition $x \in \mathbb{R}$,

$$\lambda = \inf_{\mathbb{C}_x \in \mathcal{C}_x} J(\mathbb{C}_x).$$

Note that this expression also reflects the fact that the optimal value of the performance criterion is independent of the system's initial condition. The set of all $x \in \mathbb{R}$ such that

$$c^- - w(x) + \inf_{z \le 0} \big[w(x+z) - K^- z \big] = 0 \tag{14}$$

is the part of the state space where the controller should act immediately with an impulse in the negative direction, while the set of all $x \in \mathbb{R}$ such that

$$c^+ - w(x) + \inf_{z \ge 0} \big[w(x+z) + K^+ z \big] = 0 \tag{15}$$

is the region of the state space where the controller should act with an impulse in the positive direction. The interior of the set of all $x \in \mathbb{R}$ such that

$$\frac{1}{2}\sigma^2(x) w''(x) - b(x)|w'(x)| + h(x) - \lambda = 0 \tag{16}$$

defines the part of the state space in which the controller should act only through the exercise of absolutely continuous control of the drift. Inside this region, it is optimal to choose

$$U_t = -\operatorname{sgn}(w'(X_t)) b(X_t). \tag{17}$$

It turns out that all of these statements, are indeed true.

Now, we conjecture that an optimal strategy is characterised by five points, $y_2 < y_1 < a < x_1 < x_2$, and takes the form that can be described as follows. If the state space process X assumes any value $x \ge x_2$, then impulsive control is exercised to "push" it instantaneously to the level x_1. Similarly, whenever the state process X assumes a value $x \le y_2$, impulsive control action is used to reposition it at y_1. As long as the state process is inside the interval $]y_2, x_2[$, the controller expends absolutely continuous control effort at the maximum rate, given by $b(X)$, to "push" the state process X towards a, which, in view of (17) is associated with (21) below. We therefore look for a solution (w, λ) to the HJB equation (13) such that

$$w(x) = w(x_1) + K^-(x - x_1) + c^-, \quad \text{for } x \ge x_2, \tag{18}$$

$$\frac{1}{2}\sigma^2(x) w''(x) - b(x)|w'(x)| + h(x) - \lambda = 0, \quad \text{for } x \in]y_2, x_2[, \tag{19}$$

$$w(x) = w(y_1) + K^+(y_1 - x) + c^+, \quad \text{for } x \le y_2, \tag{20}$$

$$w'(x) \begin{cases} < 0, & \text{for } x < a, \\ = 0, & \text{for } x = a, \\ > 0, & \text{for } x > a. \end{cases} \tag{21}$$

Assuming that this strategy is indeed optimal, we need a system of appropriate equations to determine the free-boundary points y_2, y_1, a, x_1, x_2 and the constant λ. To derive such equations, we argue as follows. With regard to the boundary points y_2 and x_2 that separate the three regions defined by (14)–(16) and the so-called "smooth-pasting condition", we impose

$$w'(y_2+) = -K^+ \quad \text{and} \quad w'(x_2-) = K^-. \tag{22}$$

Now, relative to impulses in the negative direction, let us consider the inequality

$$c^- - w(x) + \inf_{z \le 0} \big[w(x+z) - K^- z \big] \ge 0.$$

Assuming for the sake of the argument that we have somehow calculated w, this inequality implies

$$c^- - w(x_2) + w(x) - K^-(x - x_2) \ge 0, \quad \text{for all } x \le x_2.$$

With regard to (18) and the fact that x_2 is a constant, this observation implies that the function $x \mapsto w(x) - K^- x$ has a local minimum at $x = x_1$, which can be true only if

$$w'(x_1) = K^-. \tag{23}$$

Moreover, for $x = x_2$, (18) implies

$$\int_{x_1}^{x_2} w'(s)\, ds = K^-\, (x_2 - x_1) + c^-. \tag{24}$$

Similarly, a consideration of impulses in the positive direction leads to

$$w'(y_1) = -K^+ \quad \text{and} \quad \int_{y_2}^{y_1} w'(s)\, ds = -K^+\, (y_1 - y_2) - c^+. \tag{25}$$

Summarising the considerations above, a candidate for an optimal strategy is characterised by six parameters, namely $y_2 < y_1 < a < x_1 < x_2$ and λ, and a function w such that (18)–(25) are all true. Now, (19) and (21) can both be true only if w satisfies

$$\frac{1}{2}\sigma^2(x) w''(x) - \operatorname{sgn}(x-a) b(x) w'(x) + h(x) - \lambda = 0, \quad \text{for } x \in]y_2, x_2[,$$

which is the case if

$$w'(x) = g(x, \lambda, a), \quad \text{for all } x \in]y_2, x_2[, \tag{26}$$

where g is defined by

$$g(x, \lambda, a) := p_a'(x) \int_a^x [\lambda - h(s)]\; m_a(ds), \quad x \in]y_2, x_2[. \tag{27}$$

Here, p_a and m_a are defined by

$$p_a(x) := \begin{cases} \int_a^x \exp\left(2\int_a^s b(u)\sigma^{-2}(u)\,du\right) ds, & \text{if } x \geq a, \\ -\int_x^a \exp\left(2\int_s^a b(u)\sigma^{-2}(u)\,du\right) ds, & \text{if } x < a, \end{cases} \tag{28}$$

$$m_a(dx) := \frac{2}{p_a'(x)\sigma^2(x)}\,dx. \tag{29}$$

It follows that, to determine the six parameters $y_2 < y_1 < a < x_1 < x_2$ and λ, we have to solve the system of the following six algebraic, non-linear equations:

$$g(x_2, \lambda, a) = K^-, \qquad g(x_1, \lambda, a) = K^-, \tag{30}$$

$$g(y_2, \lambda, a) = -K^+, \qquad g(y_1, \lambda, a) = -K^+, \tag{31}$$

$$\int_{x_1}^{x_2} g(s, \lambda, a)\,ds = K^-\,(x_2 - x_1) + c^-, \tag{32}$$

$$\int_{y_2}^{y_1} g(s, \lambda, a)\,ds = -K^+\,(y_1 - y_2) - c^+. \tag{33}$$

where g is as in (27).

At this point, it is worth observing that p_a and m_a are the *scale function* and the *speed measure*, respectively, of the uncontrolled Itô-diffusion

$$dX_t = -\operatorname{sgn}(X_t - a)b(X_t)\,dt + \sigma(X_t)\,dW_t.$$

The following result asserts that a solution to the HJB equation (13) that conforms with all of the heuristic considerations above indeed exists. Its proof is developed in the Appendix.

Lemma 1. *Suppose that Assumption 1 holds. The system of equations (30)–(33), has a solution $(y_2, y_1, a, x_1, x_2, \lambda)$ such that $y_2 < y_1 < a < x_1 < x_2$, and, if w is the function defined by (18), (20) and (26), then $w \in W^{2,\infty}_{\text{loc}}(\mathbb{R})$, w satisfies (21), and the pair (w, λ) is a classical solution to the HJB equation (13).*

We can now establish our main result.

Theorem 1. *Consider the control problem formulated in Section 2, suppose that Assumption 1 holds and let (w, λ) be the solution to the HJB equation (13) provided by Lemma 1. Given any initial condition $x \in \mathbb{R}$,*

$$\lambda = \inf_{\mathbb{C}_x \in \mathcal{C}_x} J(\mathbb{C}_x), \tag{34}$$

and the strategy discussed above, which is constructed rigorously in the proof below, is optimal.

Proof. Throughout this proof, we fix the solution (w, λ) to the HJB equation (13) constructed in Lemma 1. We also fix an initial condition $x \in \mathbb{R}$.

Consider any admissible control $\mathbb{C}_x \in \mathcal{C}_x$ such that $J(\mathbb{C}_x) < \infty$. Using Itô's formula, we calculate

$$w(X_{T+}) = w(x) + \int_0^T \left[\frac{1}{2}\sigma^2(X_s)w''(X_s) + U_s w'(X_s)\right] ds + \int_0^T \sigma(X_s)w'(X_s)\, dW_s + \sum_{s\in[0,T]} \left[w(X_s + \Delta Z_s) - w(X_s)\right],$$

which implies

$$\begin{aligned} I_T(\mathbb{C}_x) := {} & \int_0^T h(X_s)\, ds + \sum_{s\in[0,T]} \left(K^+ \Delta Z_t + c^+\right) \mathbf{1}_{\{\Delta Z_t > 0\}} \\ & + \sum_{s\in[0,T]} \left(-K^- \Delta Z_t + c^-\right) \mathbf{1}_{\{\Delta Z_t < 0\}} \\ = {} & \lambda T + w(x) - w(X_{T+}) + \int_0^T \sigma(X_s)w'(X_s)\, dW_s \\ & + \int_0^T \left[\frac{1}{2}\sigma^2(X_s)w''(X_s) + U_s w'(X_s) + h(X_s) - \lambda\right] ds \\ & + \sum_{s\in[0,T]} \left[w(X_s + \Delta Z_s) - w(X_s) + K^+ \Delta Z_s + c^+\right] \mathbf{1}_{\{\Delta Z_s > 0\}} \\ & + \sum_{s\in[0,T]} \left[w(X_s + \Delta Z_s) - w(X_s) - K^- \Delta Z_s + c^-\right] \mathbf{1}_{\{\Delta Z_s < 0\}}. \end{aligned} \tag{35}$$

With reference to (4), we note that $U_t w'(X_t) \geq -b(X_t)|w'(X_t)|$. Combining this observation with the fact that (w, λ) satisfies the HJB equation (13), we calculate

$$I_T(\mathbb{C}_x) \geq \lambda T + w(x) - w(X_{T+}) + \int_0^T \sigma(X_s)w'(X_s)\, dW_s. \tag{36}$$

By construction, w is C^1, $w'(x) = K^-$, for all $x \geq x_2$, and $w'(x) = -K^+$, for all $x \leq y_2$. Therefore, there exists a constant $C_3 > 0$ such that

$$w(x) \leq C_3(1 + |x|) \quad \text{and} \quad |w'(x)| \leq C_3, \quad \text{for all } x \in \mathbb{R}. \tag{37}$$

For such a choice of C_3, (36) yields

$$I_T(\mathbb{C}_x) \geq \lambda T + w(x) - C_3 - C_3\, |X_{T+}| + \int_0^T \sigma(X_s)w'(X_s)\, dW_s. \tag{38}$$

Now, with respect to Assumption 1.(c),

$$\infty > J(\mathbb{C}_x) \geq -C_2 + C_2 \limsup_{T\to\infty} \frac{1}{T} E_x \left[\int_0^T |X_s| \, ds \right]. \tag{39}$$

These inequalities imply

$$E_x \left[\int_0^T |X_s| \, ds \right] < \infty, \text{ for all } T > 0, \tag{40}$$

$$\text{and} \quad \liminf_{T\to\infty} \frac{1}{T} E_x \left[|X_{T+}| \right] = 0. \tag{41}$$

To see (41), suppose that $\liminf_{T\to\infty} T^{-1} E_x \left[|X_{T+}| \right] > \varepsilon > 0$. This implies that there exists $T_1 \geq 0$ such that $E_x \left[|X_{s+}| \right] > \varepsilon s/2$, for all $s \geq T_1$. Since the sample paths of X have countable discontinuities, it follows that

$$\limsup_{T\to\infty} \frac{1}{T} E_x \left[\int_0^T |X_s| \, ds \right] \geq \limsup_{T\to\infty} \frac{1}{T} \int_{T_1}^T \frac{\varepsilon s}{2} \, ds = \infty,$$

which contradicts (39).

With regard to (7) in Assumption 1, the second inequality in (37), and (40), we calculate

$$E_x \left[\int_0^T \left[\sigma(X_s) w'(X_s) \right]^2 ds \right] \leq C_3^2 C_1 \left[T + E_x \left[\int_0^T |X_s| \, ds \right] \right] < \infty, \tag{42}$$

for all $T > 0$, which proves that the stochastic integral in (38) is a square integrable martingale and therefore has zero expectation. In view of this observation, we can take expectations in (38) and divide by T to obtain

$$\frac{1}{T} E_x \left[I_T(\mathbb{C}_x) \right] \geq \lambda + \frac{w(x)}{T} - \frac{C_3}{T} - \frac{C_3}{T} E_x \left[|X_{T+}| \right].$$

In view of (41) and the definition of $I_T(\mathbb{C}_x)$ in (35), we can pass to the limit $T \to \infty$ to obtain $J(\mathbb{C}_x) \geq \lambda$.

To prove the reverse inequality, suppose that we can find a control

$$\hat{\mathbb{C}}_x = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{P}_x, \hat{W}, \hat{U}, \hat{Z}, \hat{X}, \hat{\tau}) \in \mathcal{C}_x$$

such that

$$\hat{U}_t = -\operatorname{sgn}(\hat{X}_t - a) b(\hat{X}_t), \tag{43}$$

$$\hat{X}_{t+} \in [y_2, x_2], \tag{44}$$

$$\Delta \hat{Z}_t \mathbf{1}_{\{\Delta \hat{Z}_t > 0\}} = (y_1 - y_2) \mathbf{1}_{\{\hat{X}_t = y_2\}}, \tag{45}$$

$$\Delta \hat{Z}_t \mathbf{1}_{\{\Delta \hat{Z}_t < 0\}} = -(x_2 - x_1) \mathbf{1}_{\{\hat{X}_t = x_2\}}, \tag{46}$$

for all $t \geq 0$, $\hat{P}_x$-a.s.. Plainly, (44) implies that $\hat{X}$ is non-explosive, so that $\hat{\tau} = \infty$, $\hat{P}_x$-a.s.. Also, since w satisfies (21), $\hat{U}_t w'(\hat{X}_t) = -b(\hat{X}_t)|w'(\hat{X}_t)|$. In view of this observation and (18)–(20), we can see that, in this context, (35) implies

$$I_T(\hat{\mathbb{C}}_x) = \lambda T + w(x) - w(\hat{X}_{T+}) + \int_0^T \sigma(\hat{X}_s) w'(\hat{X}_s)\, d\hat{W}_s. \tag{47}$$

Now, (7) in Assumption 1, (37) and (44) imply

$$E_x\left[\int_0^T \left[\sigma(\hat{X}_s) w'(\hat{X}_s)\right]^2 ds\right] \leq C_3^2 C_1 \left(1 + |y_2| \vee |x_2|\right) T < \infty,$$

for all $T > 0$, which proves that the stochastic integral in (47) is a square integrable martingale, and

$$\lim_{T\to\infty} \frac{1}{T} E_x\left[\left|w(\hat{X}_{T+})\right|\right] \leq \lim_{T\to\infty} \frac{C_3\left(1 + |y_2| \vee |x_2|\right)}{T} = 0.$$

It follows that

$$\lim_{T\to\infty} \frac{1}{T} E_x\left[I_T(\hat{\mathbb{C}}_x)\right] = \lambda,$$

which proves that $J(\hat{\mathbb{C}}_x) = \lambda$, and establishes (34).

It remains to construct a control $\hat{\mathbb{C}}_x \in \mathcal{C}_x$ satisfying (43)–(46), which amounts to constructing a weak solution $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{P}_x, \hat{W}, \hat{Z}, \hat{X})$ to the SDE

$$d\hat{X}_t = -\operatorname{sgn}(\hat{X}_t - a) b(\hat{X}_t)\, dt + d\hat{Z}_t + \sigma(\hat{X}_t)\, d\hat{W}_t \tag{48}$$

that satisfies (44)–(46). To this end, we fix a filtered probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \bar{\mathcal{F}}_t, \hat{P}_x)$ satisfying the usual conditions and supporting a standard, one-dimensional Brownian motion $\bar{W}$. By appealing to a simple induction argument, we construct a càglàd, piece-wise constant process $\bar{Z}$ with $\bar{Z}_0 = 0$ such that, if

$$\bar{X}_t := p_a(x) + \bar{Z}_t + \bar{W}_t, \tag{49}$$

then

$$\bar{X}_{t+} \in \left[p_a(y_2), p_a(x_2)\right], \tag{50}$$

$$\Delta\bar{Z}_t \mathbf{1}_{\{\Delta\bar{Z}_t > 0\}} = \left(p_a(y_1) - p_a(y_2)\right) \mathbf{1}_{\{\bar{X}_t = p_a(y_2)\}}, \tag{51}$$

$$\Delta\bar{Z}_t \mathbf{1}_{\{\Delta\bar{Z}_t < 0\}} = -\left(p_a(x_2) - p_a(x_1)\right) \mathbf{1}_{\{\bar{X}_t = p_a(x_2)\}}, \tag{52}$$

for all $t \geq 0$, $\hat{P}_x$-a.s.. The function p_a appearing here is the solution to the ODE

$$\frac{1}{2}\sigma^2(x) p_a''(x) - \operatorname{sgn}(x - a) b(x) p_a'(x) = 0. \tag{53}$$

that is given by (28). In what follows, we denote by q_a the inverse function of p_a. For future reference, we note that q_a satisfies

$$q_a'\left(p_a(x)\right) = \frac{1}{p_a'(x)} \quad \text{and} \quad q_a''\left(p_a(x)\right) = -\frac{p_a''(x)}{\left[p_a'(x)\right]^3}. \tag{54}$$

Now, we consider the continuous, increasing process A defined by

$$A_t := \int_0^t \tilde{\sigma}^{-2}(\bar{X}_s)\, ds,$$

where

$$\tilde{\sigma}(x) := p_a'\big(q_a(x)\big)\sigma\big(q_a(x)\big), \quad x \in \mathbb{R}, \tag{55}$$

and we observe that $\lim_{t\to\infty} A_t = \infty$ thanks to (7) in Assumption 1 and (50). Also, we denote by C the inverse of A defined by

$$C_t := \inf\left\{s \geq 0 \mid \ A_s > t\right\},$$

and we note that $\lim_{t\to\infty} C_t = \infty$. Since C is continuous, if we define

$$\hat{\mathcal{F}}_t := \bar{\mathcal{F}}_{C_t}, \quad \tilde{X}_t := \bar{X}_{C_t}, \quad \tilde{Z}_t := \bar{Z}_{C_t} \quad \text{and} \quad M_t := \bar{W}_{C_t}, \tag{56}$$

then

$$\tilde{X}, \tilde{Z} \text{ are càglàd, } (\hat{\mathcal{F}}_t)\text{-adapted processes satisfying (50)–(52),} \tag{57}$$

and M is a continuous, $(\hat{\mathcal{F}}_t)$-local martingale. Furthermore, if we define

$$\hat{W}_t := \int_0^t \tilde{\sigma}^{-1}(\tilde{X}_s)\, dM_s,$$

then, in view of (49) and (56),

$$d\tilde{X}_t = d\tilde{Z}_t + \tilde{\sigma}(\tilde{X}_t)\, d\hat{W}_t, \quad \tilde{X}_0 = p_a(x).$$

To see that $\hat{W}$ is a standard $(\hat{\mathcal{F}}_t)$-Brownian motion, we first observe that

$$\langle M\rangle_t = C_t = \int_0^{C_t} \tilde{\sigma}^2(\bar{X}_s)\, dA_s = \int_0^t \tilde{\sigma}^2(\tilde{X}_s)\, ds,$$

the last equality following thanks to the time change formula and the fact that $A_{C_s} = s$. It follows that

$$\langle\hat{W}\rangle_t = \int_0^t \tilde{\sigma}^{-2}(\tilde{X}_s)\, d\langle M\rangle_s = t.$$

However, with reference to Lévy's characterisation theorem, this calculation and the fact that $\hat{W}$ is a continuous, $(\hat{\mathcal{F}}_t)$-local martingale imply that $\hat{W}$ is an $(\hat{\mathcal{F}}_t)$-Brownian motion.

Finally, we define

$$\hat{X}_t := q_a(\tilde{X}_t) \quad \text{and} \quad \hat{Z}_t := \mathbf{1}_{\{t>0\}} \sum_{s \in [0,t[} \Big[q_a(\tilde{X}_s + \Delta\tilde{Z}_s) - q_a(\tilde{X}_s) \Big]. \tag{58}$$

In view of (57), we can verify that these processes satisfy (44)–(46), while an application of Itô's formula yields

$$\hat{X}_t = x + \int_0^t \frac{1}{2} \tilde{\sigma}^2\big(p_a(\hat{X}_s)\big) q_a''\big(p_a(\hat{X}_s)\big)\, ds + \hat{Z}_t + \int_0^t \tilde{\sigma}\big(p_a(\hat{X}_s)\big) q_a'\big(p_a(\hat{X}_s)\big)\, d\hat{W}_s.$$

However, this SDE, (53), (54) and the identity

$$\tilde{\sigma}\big(p_a(x)\big) = p_a'(x)\sigma(x), \quad x \in \mathbb{R},$$

which follows from the definition of $\tilde{\sigma}$ in (55), imply that (48) is satisfied, and the construction is complete. □

Appendix: Proof of Lemma 1

Before addressing the proof of Lemma 1, we first establish some preliminary results. For easy future reference, we note the calculations

$$\frac{\partial g}{\partial x}(x, \lambda, a) = -\frac{2}{\sigma^2(x)} \big[h(x) - b(x)|g(x, \lambda, a)| - \lambda \big], \tag{59}$$

$$\frac{\partial g}{\partial \lambda}(x, \lambda, a) = \begin{cases} p_a'(x) m_a\left([a, x]\right) > 0, & \text{if } x > a, \\ -p_a'(x) m_a\left([x, a]\right) < 0, & \text{if } x < a, \end{cases} \tag{60}$$

which follow from the definition of g in (27). The development of our analysis requires the following definitions:

$$\lambda^*(a) := \inf \left\{ \lambda \in \mathbb{R} \mid \sup_{x \geq a} g(x, \lambda, a) = \infty \right\}, \quad \text{for } a \in \mathbb{R}, \tag{61}$$

$$^*\lambda(a) := \inf \left\{ \lambda \in \mathbb{R} \mid \inf_{x \leq a} g(x, \lambda, a) = -\infty \right\}, \quad \text{for } a \in \mathbb{R}, \tag{62}$$

with the usual convention that $\inf \emptyset = \infty$.

Lemma 2. *Fix any $a \in \mathbb{R}$ and suppose that Assumption 1 is true. If $\lambda^*(a)$ and $^*\lambda(a)$ are defined as in (61) and (62), respectively, then $\lambda^*(a), {}^*\lambda(a) \in]0, \infty]$, and*

$$\lim_{x \to \infty} g(x, \lambda, a) = \begin{cases} -\infty, & \text{if } \lambda < \lambda^*(a), \\ \infty, & \text{if } \lambda \in [\lambda^*(a), \infty] \cap \mathbb{R}, \end{cases} \tag{63}$$

$$\lim_{x \to -\infty} g(x, \lambda, a) = \begin{cases} \infty, & \text{if } \lambda < {}^*\lambda(a), \\ -\infty, & \text{if } \lambda \in [{}^*\lambda(a), \infty] \cap \mathbb{R}. \end{cases} \tag{64}$$

Proof. We first prove that, given any $\lambda,\ a \in \mathbb{R}$,

$$\begin{gathered}\text{the equation } g(x,\lambda,a)=0 \text{ has at most two solutions } x\in\,]a,\infty[,\\ \text{and at most two solutions } x\in\,]-\infty,a[.\end{gathered}\tag{65}$$

Fix any $\lambda, a \in \mathbb{R}$, and consider the solvability of $g(x,\lambda,a)=0$ for $x\in\,]a,\infty[$. Assumption 1.(c) implies that there exist at most two points $x>a$ such that $h(x)=\lambda$. Also, (59) implies that

$$\begin{gathered}\text{given any } x>a \text{ such that } g(x,\lambda,a)=0,\\ \frac{\partial g}{\partial x}(x,\lambda,a) = -\frac{2}{\sigma^2(x)}\left[h(x)-\lambda\right].\end{gathered}\tag{66}$$

Combining these observations with the boundary condition $g(a,\lambda,a)=0$, we can conclude that the number of solutions of $g(x,\lambda,a)=0$ inside $]a,\infty[$ is less than or equal to the number of solutions of $h(x)=\lambda$ inside $]a,\infty[$, which is at most two. Similarly, we show that the number of solutions of $g(x,\lambda,a)=0$ inside $]-\infty,a[$ is also less than or equal to two.

Now, we show that

$$\lim_{x\to\infty} g(x,\lambda,a),\ \lim_{x\to-\infty} g(x,\lambda,a) \in \{-\infty,\infty\}, \quad \text{for all } a,\lambda\in\mathbb{R}. \tag{67}$$

With reference to (65), the conclusion $\lim_{x\to\infty} g(x,\lambda,a)\in\{-\infty,\infty\}$ will follow if we show that either of

$$\liminf_{x\to\infty} g(x,\lambda,a)\in[0,\infty[, \quad \limsup_{x\to\infty} g(x,\lambda,a)\in\,]-\infty,0], \tag{68}$$

leads to a contradiction. Assuming that the first limit in (68) is true, we choose a sequence $x_n\to\infty$ such that

$$\lim_{n\to\infty} g(x_n,\lambda,a) = \liminf_{x\to\infty} g(x,\lambda,a) \quad \text{and} \quad \lim_{n\to\infty}\frac{\partial g}{\partial x}(x_n,\lambda,a)=0.$$

If we assume that the second limit in (68) is true, then we choose a sequence (x_n) in a similar fashion. In either case, we define $\gamma := \sup_{n\geq 1}|g(x_n,\lambda,a)|$. Observing that $\gamma\in\mathbb{R}$, and referring to (59) we calculate

$$\begin{aligned}0 &= \lim_{n\to\infty}\frac{-2}{\sigma^2(x_n)}\left[h(x_n)-b(x_n)g(x_n,\lambda,a)-\lambda\right]\\ &\leq \lim_{n\to\infty}\frac{-2}{\sigma^2(x_n)}\left[h(x_n)-b(x_n)\gamma-\lambda\right]\\ &= -\infty,\end{aligned}$$

the inequality following because $b\geq 0$, and the last equality following thanks to Assumption 1.(d). However this calculation provides the required contradiction. Likewise, we can show that $\lim_{x\to-\infty} g(x,\lambda,a)\in\{-\infty,\infty\}$.

We can now prove the claims made relative to $\lambda^*(a)$. With regard to the definition of g in (27), the positivity of h and a simple continuity argument, we can see that $\lambda^*(a) \in]0,\infty]$. Also, the fact that $g(x,\cdot,a)$ is strictly increasing, for all $x > a$, which follows from (60), implies

$$\sup_{x\geq a} g(x,\lambda,a) \begin{cases} < \infty, & \text{for all } \lambda < \lambda^*(a), \\ = \infty, & \text{for all } \lambda \in]\lambda^*(a),\infty] \cap \mathbb{R}. \end{cases}$$

To show that $\sup_{x\geq a} g(x,\lambda^*(a),a) = \infty$, and thus, in the light of (67), complete the proof of (63), we argue by contradiction. To this end, we assume that $\lambda^*(a) < \infty$ and

$$\lim_{x\to\infty} g(x,\lambda^*(a),a) = -\infty.$$

This limit and Assumption 1.(c) imply that there exists $\hat{x}(a) > a$ such that

$$g(x,\lambda^*(a),a) < 0 \quad \text{and} \quad h(x) - \lambda^*(a) > 0, \quad \text{for all } x \geq \hat{x}(a). \tag{69}$$

In view of the fact that $\lim_{x\to\infty} g(x,\lambda,a) = \infty$, for all $\lambda > \lambda^*(a)$, (66) and the second inequality in (69), we can appeal to a simple contradiction argument to see that

$$g(x,\lambda,a) > 0, \quad \text{for all } x \geq \hat{x}(a) \text{ and } \lambda > \lambda^*(a).$$

However, this and the first inequality in (69) imply

$$\lim_{\lambda\downarrow\lambda^*(a)} g(x,\lambda,a) \geq 0 > g(x,\lambda^*(a),a), \quad \text{for all } x \geq \hat{x}(a),$$

which contradicts the continuity of g.

Proving the statements relating to ${}^*\lambda(a)$ involves similar arguments. □

It is worth noting that the consideration of λ^* and ${}^*\lambda$ is not a redundant exercise. Indeed, we can easily construct examples in which $\lambda^*(0), {}^*\lambda(0) < \infty$. With reference to the structure of the system of equations (30)–(33), which involves the functions $g(\cdot,\cdot,\cdot)+K^+$ and $g(\cdot,\cdot,\cdot)-K^-$, we consider the following definitions:

$$\lambda_*(a) := \inf\left\{\lambda > 0 \mid \sup_{x\geq a} g(x,\lambda,a) \geq K^-\right\}, \tag{70}$$

$$_*\lambda(a) := \inf\left\{\lambda > 0 \mid \inf_{x\leq a} g(x,\lambda,a) \leq -K^+\right\}. \tag{71}$$

Lemma 3. *Given $a \in \mathbb{R}$, $\lambda^*(a) > \lambda_*(a) > 0$, and the equation $g(x,\lambda,a) = K^-$ defines uniquely two C^1 functions $x_1(\cdot,a), x_2(\cdot,a) :]\lambda_*(a),\lambda^*(a)[\to \mathbb{R}$ such that*

$$a < x_1(\lambda,a) < x_2(\lambda,a) \text{ and } a_+ < x_2(\lambda,a), \text{ for all } \lambda \in]\lambda_*(a),\lambda^*(a)[,$$

where a_+ is as in Assumption 1.(e). Furthermore, the following statements are true:

$$x_1(\cdot,a) \text{ (resp., } x_2(\cdot,a)) \text{ is strictly decreasing (resp., increasing),} \tag{72}$$

$$\lim_{\lambda\downarrow\lambda_*(a)} x_1(\lambda,a) = \lim_{\lambda\downarrow\lambda_*(a)} x_2(\lambda,a), \quad \lim_{\lambda\uparrow\lambda^*(a)} x_2(\lambda,a) = \infty, \tag{73}$$

$$h(x) - b(x)K^- - \lambda > 0, \quad \text{for all } x > x_2(\lambda,a). \tag{74}$$

Proof. Fix any $a \in \mathbb{R}$. In view of (27) and the positivity of h, we can see that $\lambda_*(a) > 0$. Also, the definitions of $\lambda_*(a)$, $\lambda^*(a)$ and the continuity of g imply trivially that $\lambda_*(a) < \lambda^*(a)$.

Now, observe that a simple inspection of (59) reveals that

$$\begin{gathered}\text{if } x > a \text{ satisfies } g(x,\lambda,a) = K^-, \text{ then} \\ \frac{\partial g}{\partial x}(x,\lambda,a) = -\frac{2}{\sigma^2(x)}\left[h(x) - b(x)K^- - \lambda\right].\end{gathered} \tag{75}$$

With regard to the definitions of $\lambda_*(a)$ and $\lambda^*(a)$, (63) in Lemma 2, the fact that $g(a,\lambda,a) = 0$, Assumption 1.(e) and the continuity of g, this observation implies the following:

(I) If $\lambda < \lambda_*(a)$, then the equation $g(x,\lambda,a) = K^-$ has no solutions $x \in]a,\infty[$.

(II) If $\lambda \in]\lambda_*(a),\lambda^*(a)[$, then the equation $g(x,\lambda,a) = K^-$ has one solution $x_1(\lambda,a) > a$ such that

$$h(x_1(\lambda,a)) - b(x_1(\lambda,a))K^- - \lambda < 0, \tag{76}$$

and one solution $x_2(\lambda,a) > x_1(\lambda,a)$ such that

$$h(x_2(\lambda,a)) - b(x_2(\lambda,a))K^- - \lambda > 0. \tag{77}$$

Moreover, (74) is true.

(III) If $\lambda \geq \lambda^*(a)$, then the equation $g(x,\lambda,a) = K^-$ has one solution $x_1(\lambda,a) > a$ such that

$$h(x_1(\lambda,a)) - b(x_1(\lambda,a))K^- - \lambda < 0. \tag{78}$$

Since $\lambda_*(a) > 0$, Assumption 1.(e) and (77) imply that the solution x_2 in (II) above satisfies $x_2(\lambda,a) > a_+$. Also, (I) and (II) and the continuity of g imply the first equality in (73), while (II), (III) and (72) imply the second equality in (73). To prove (72), we differentiate $g(x_j(\lambda,a),\lambda,a) = K^-$ with respect to λ to calculate

$$\frac{\partial x_j}{\partial\lambda}(\lambda,a) = \frac{\sigma^2(x_j(\lambda,a))\frac{\partial g}{\partial\lambda}(x_j(\lambda,a),\lambda,a)}{2\left[h(x_j(\lambda,a)) - b(x_j(\lambda,a))K^- - \lambda\right]},$$

for all $\lambda \in]\lambda_*(a),\lambda^*(a)[$, $j = 1,2$. However, this calculation, (60) and (76) (resp., (77)) imply that the function $x_1(\cdot,a)$ (resp., $x_2(\cdot,a)$) is strictly decreasing (resp., increasing), and the proof is complete. □

With regard to the problem's data symmetry, we can trivially modify the arguments of the preceding proof to establish the following result.

Lemma 4. *Given $a \in \mathbb{R}$, ${}^*\lambda(a) > {}_*\lambda(a) > 0$, and the equation $g(x, \lambda, a) = -K^+$ defines uniquely two C^1 functions $y_1(\cdot, a), y_2(\cdot, a) :]{}_*\lambda(a), {}^*\lambda(a)[\to \mathbb{R}$ such that*

$$y_2(\lambda, a) < y_1(\lambda, a) < a \text{ and } y_2(\lambda, a) < \alpha_-, \text{ for all } \lambda \in]{}_*\lambda(a), {}^*\lambda(a)[$$

where α_- is as in Assumption 1.(f). Furthermore,

$$y_2(\cdot, a) \text{ (resp., } y_1(\cdot, a)\text{) is strictly decreasing (resp., increasing),} \tag{79}$$

$$\lim_{\lambda \downarrow {}_*\lambda(a)} y_1(\lambda, a) = \lim_{\lambda \downarrow {}_*\lambda(a)} y_2(\lambda, a), \quad \lim_{\lambda \uparrow {}^*\lambda(a)} y_2(\lambda, a) = -\infty, \tag{80}$$

$$h(x) - b(x)K^+ - \lambda > 0, \quad \text{for all } x < y_2(\lambda, a). \tag{81}$$

Proof of Lemma 1. With reference to (32)–(33), we define the functions $Q^*(\cdot, a) :]\lambda_*(a), \lambda^*(a)[\to \mathbb{R}$ and ${}^*Q(\cdot, a) :]{}_*\lambda(a), {}^*\lambda(a)[\to \mathbb{R}$ by

$$Q^*(\lambda, a) = \int_{x_1(\lambda,a)}^{x_2(\lambda,a)} \left[g(s, \lambda, a) - K^-\right] ds - c^-, \tag{82}$$

$${}^*Q(\lambda, a) = \int_{y_2(\lambda,a)}^{y_1(\lambda,a)} \left[g(s, \lambda, a) + K^+\right] ds + c^+, \tag{83}$$

respectively, where x_1, x_2 are as in Lemma 3, and y_1, y_2 are as in Lemma 4. Given these definitions, we will establish the claim regarding the solvability of the system of equations (30)–(33) if we prove that

$$\begin{gathered}\text{there exist } \tilde{a} \in \mathbb{R} \text{ and } \tilde{\lambda} \in]\lambda_*(\tilde{a}), \lambda^*(\tilde{a})[\cap]{}_*\lambda(\tilde{a}), {}^*\lambda(\tilde{a})[\\ \text{such that } Q^*(\tilde{\lambda}, \tilde{a}) = {}^*Q(\tilde{\lambda}, \tilde{a}) = 0.\end{gathered} \tag{84}$$

Differentiating (82) with respect to λ, and using the fact that both of $g(x_1(\lambda, a), \lambda, a)$ and $g(x_2(\lambda, a), \lambda, a)$ are equal to the constant K^-, we calculate

$$\frac{\partial Q^*}{\partial \lambda}(\lambda, a) = \int_{x_1(\lambda,a)}^{x_2(\lambda,a)} \frac{\partial g}{\partial \lambda}(s, \lambda, a)\, ds > 0, \quad \text{for } \lambda \in]\lambda_*(a), \lambda^*(a)[, \tag{85}$$

the inequality following thanks to (60) and the fact that $a < x_1 < x_2$. Also, with regard to (60), (63) and (72)–(73) in Lemma 3, we can see that

$$\lim_{\lambda \downarrow \lambda_*(a)} Q^*(\lambda, a) = -c^- < 0 \quad \text{and} \quad \lim_{\lambda \uparrow \lambda^*(a)} Q^*(\lambda, a) = \infty. \tag{86}$$

However, (85) and (86) imply that there exists a unique point $\Lambda^*(a) \in]\lambda_*(a), \lambda^*(a)[$ such that $Q^*(\Lambda^*(a), a) = 0$. Similarly, we can show that given any $a \in \mathbb{R}$, there exists a unique point ${}^*\Lambda(a) \in]{}_*\lambda(a), {}^*\lambda(a)[$ such that ${}^*Q({}^*\Lambda(a), a) = 0$.

With regard to these calculations, (84) will follow if we prove that

$$\text{there exists } \tilde{a} \in \mathbb{R} \text{ such that } \Lambda^*(\tilde{a}) = {}^*\Lambda(\tilde{a}). \tag{87}$$

To this end, we differentiate $Q^*(\Lambda^*(a), a) = 0$ with respect to a to obtain

$$\frac{d}{da}\Lambda^*(a) = -\frac{\frac{\partial Q^*}{\partial a}(\Lambda^*(a), a)}{\frac{\partial Q^*}{\partial \lambda}(\Lambda^*(a), a)}. \tag{88}$$

Furthermore, we calculate

$$\frac{\partial p_a'}{\partial a}(x) = -\operatorname{sgn}(x-a)\frac{2b(a)}{\sigma^2(a)}p_a'(x), \quad \text{for } x \neq a,$$

which, in view of the definition of g in (27), implies

$$\frac{\partial g}{\partial a}(x, \lambda, a) = \frac{2\left[h(a) - \lambda\right]}{\sigma^2(a)}p_a'(x), \quad \text{for } x \neq a.$$

Using this calculation and the fact that $g(x, \lambda, a) = K^-$ for $x = x_1(\lambda, a)$ or $x = x_2(\lambda, a)$, we can see that

$$\frac{\partial Q^*}{\partial a}(\lambda, a) = \frac{2\left[h(a) - \lambda\right]}{\sigma^2(a)}\int_{x_1(\lambda,a)}^{x_2(\lambda,a)} p_a'(s)\, ds.$$

which, combined with (85) and (88), it implies

$$\frac{d}{da}\Lambda^*(a) > 0 \text{ for all } a \in \mathbb{R} \text{ such that } h(a) < \Lambda^*(a). \tag{89}$$

Using similar arguments, we can also show that

$$\frac{d}{da}{}^*\Lambda(a) < 0, \text{ for all } a \in \mathbb{R} \text{ such that } h(a) < {}^*\Lambda(a). \tag{90}$$

Now, if we assume that $h(a) < \Lambda^*(a)$, for all $a \in \mathbb{R}$, then (89) implies

$$h(a) < \Lambda^*(a) < \Lambda^*(0), \quad \text{for all } a < 0,$$

which contradicts Assumption 1.(c). With respect to the usual convention $\sup \emptyset = -\infty$, it follows that $A_- := \sup\{a \in \mathbb{R} \mid \Lambda^*(a) \leq h(a)\} > -\infty$. Moreover, since $\lambda_*(a) < \Lambda^*(a)$, and $h(a) < \lambda_*(a)$ for all $a > 0$ (see (27) and recall the definition of $\lambda_*(a)$ and Assumption 1.(c)), it follows that

$$A_- := \sup\{a \in \mathbb{R} \mid \Lambda^*(a) \leq h(a)\} \in]-\infty, 0[. \tag{91}$$

Using a similar reasoning, we can also show that

$$A_+ := \inf\{a \in \mathbb{R} \mid {}^*\Lambda(a) \leq h(a)\} \in]0, \infty[. \tag{92}$$

With regard to (89)–(92), it follows that

$$\text{the function } \Lambda^*(\cdot) - {}^*\Lambda(\cdot) \text{ is strictly increasing on the interval }]A_-, A_+[. \tag{93}$$

To proceed further, suppose that ${}^*\Lambda(A_+) \geq \Lambda^*(A_+)$, so that $h(A_+) \geq {}^*\Lambda(A_+) \geq \Lambda^*(A_+)$. Then, (27) and Assumption 1.(c) combined with the fact that $A_+ > 0$ imply

$$g(x, \Lambda^*(A_+), A_+) < 0, \quad \text{for all } x > A_+,$$

which contradicts the definition of Λ^*. However, this proves that

$$\Lambda^*(A_+) - {}^*\Lambda(A_+) > 0. \tag{94}$$

Similarly, we can prove that $\Lambda^*(A_-) - {}^*\Lambda(A_-) < 0$, which, combined with (93) and (94), implies (87), and, therefore, (84). Moreover, these arguments show that

$$h(\tilde{a}) < \tilde{\lambda}. \tag{95}$$

Now, with $\tilde{a}$, $\tilde{\lambda}$ being as in (84), we define

$$w'(x) := g(x, \tilde{\lambda}, \tilde{a}), \quad \text{for } x \in [y_2, x_2] \equiv [y_2(\tilde{\lambda}, \tilde{a}), x_2(\tilde{\lambda}, \tilde{a})]. \tag{96}$$

With regard to our construction thus far, this, (18) and (20) define a unique, modulo an additive constant, function $w \in W^{2,\infty}_{\text{loc}}(\mathbb{R})$ satisfying (18)–(20). With reference to (63) and (64) in Lemma 2 and (84), we can see that

$$\lim_{x \to -\infty} g(x, \tilde{\lambda}, \tilde{a}) = \infty \quad \text{and} \quad \lim_{x \to \infty} g(x, \tilde{\lambda}, \tilde{a}) = -\infty.$$

With regard to the definition of g in (27) and (95), we can combine these asymptotics with (65), the fact that $g(\tilde{a}, \tilde{\lambda}, \tilde{a}) = 0$ and the fact that

$$g\left(y_2(\tilde{\lambda}, \tilde{a}), \tilde{\lambda}, \tilde{a}\right) = -K^- < 0 < K^+ = g\left(x_2(\tilde{\lambda}, \tilde{a}), \tilde{\lambda}, \tilde{a}\right),$$

to conclude that w satisfies (21) as well.

To complete the proof, we still need to prove that the function w satisfies the HJB equation (13). With regard to its construction, this will follow if we show that

$$\frac{1}{2}\sigma^2(x)w''(x) - b(x)w'(x) + h(x) - \lambda \geq 0, \quad \text{for all } x > x_2, \tag{97}$$

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) + h(x) - \lambda \geq 0, \quad \text{for all } x < y_2, \tag{98}$$

$$w(x+z) - w(x) - K^- z + c^- \geq 0, \quad \text{for } z < 0,\ x \in \mathbb{R}, \tag{99}$$

$$w(x+z) - w(x) + K^+ z + c^+ \geq 0, \quad \text{for } z > 0,\ x \in \mathbb{R}. \tag{100}$$

In view of (96), inequalities (97) and (98) follow by a straightforward calculation that shows that they are implied by (74) and (81) respectively. Inequality (99) is equivalent to

$$-\int_{x+z}^{x} \left[w'(s) - K^-\right] ds + c^- \geq 0, \quad \text{for } z < 0,\ x \in \mathbb{R}. \tag{101}$$

With regard to (21), the inequalities

$$w'(x) \begin{cases} < K^-, & \text{for } x < x_1, \\ > K^-, & \text{for } x \in]x_1, x_2[, \\ = K^-, & \text{for } x > x_2, \end{cases}$$

and equation (82), it is straightforward to show that (101) is true. Finally, the proof of (100) is similar. □

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