A MODEL FOR THE LONG-TERM OPTIMAL CAPACITY LEVEL OF AN INVESTMENT PROJECT

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Abstract. We consider an investment project that produces a single commodity. The project's operation yields payoff at a rate that depends on the project's installed capacity level and on an underlying economic indicator such as the output commodity's price or demand, which we model by an ergodic, one-dimensional Itô diffusion. The project's capacity level can be increased dynamically over time. The objective is to determine a capacity expansion strategy that maximises the ergodic or long-term average payoff resulting from the project's management. We prove that it is optimal to increase the project's capacity level to a certain value and then take no further actions. The optimal capacity level depends on both the long-term average and the volatility of the underlying diffusion.

1. Introduction

We consider an investment project, the capacity of which can be expanded irreversibly over time. The project yields payoff at a rate that depends on the installed capacity level and on the value of an underlying state process that we model with a recurrent, ergodic Itô diffusion. This state process can represent an economic indicator that evolves randomly over time such as the demand for or the discounted price of the project's unique output commodity. The objective is to determine the capacity expansion strategy that maximises the long-term average payoff resulting from the project's operation in a pathwise as well as in an expected sense. It is worth noting that establishing the optimality of a given capacity expansion strategy in a mathematically rigorous way is surprisingly involved, even though the strategy is straightforward to come up with.

Under suitable general assumptions, we prove that it is optimal to increase the project's capacity to a given level, which is the unique solution of a given algebraic equation, and then take no further actions. Thus, we show that the *dynamical* optimisation problem that we consider is equivalent to a *static* one that involves only the underlying state process' stationary distribution and the running payoff function. This is one of the main contributions of the paper because it shows that apparently "simple" static models such as the ones encountered in undergraduate microeconomics textbooks can be identified with models involving non-trivial stochastic dynamics. As a result, we can expect that this paper will motivate research aiming at identifying static microeconomics models with appropriate dynamical ones. The output of such research is important because (a) it shows that the "simple" microeconomics models considered are not that simple after all, and (b) it

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can provide a methodological way for estimating the various parameters associated with a static model. As a matter of fact, the example that we consider in Section 4 reveals that the volatility of the underlying state process is as important as the long-term mean of the time series.

Capacity expansion models have attracted considerable interest in the literature and can be traced beck to Manne [9]; see Van Mieghem [13] for a recent survey. A number of other related models have been studied by Abel and Eberly [1], Davis, Dempster, Sethi and Vermes [5] (see also Davis [4]), Kobila [7], Øksendal [11], Wang [14], Bank [2], Chiarolla and Haussmann [3], Merhi and Zervos [10], Guo and Tomecek [6] and in references therein. This research output has considered the optimisation of expected discounted performance indices, which quantify the expected present value of the cash flow associated with each decision strategy. With regard to applications, such optimisation objectives require the modelling of stochastic dynamics for the underlying state (e.g., price) process, which involves the use of historical data taking into account future expectations, as well as the discounting factor. In fact, determining appropriate discounting for the payoff flow resulting from an investment project, which factors in expectations about future economic growth, has been a controversial issue in economics. Considering a long-term average criterion involves *only* the modelling of the underlying state (e.g., discounted price) process, which involves the use of historical data moderated by future expectations. In view of these considerations, we can conclude that long-term average criteria have an advantage relative to expected discounted ones at least as long as investment is not motivated by speculation and applications in sustainable development are concerned. In connection to the type of applications that we consider here, a further major advantage of the long-term average criterion approach arises from the fact that it leads to results of an explicit analytic nature that require minimal computational effort for a most wide class of stochastic dynamics for the underlying state process.

2. Problem formulation

We consider an investment project that operates within a random economic environment, the state of which is modelled by the one-dimensional Itô diffusion

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = x > 0, \tag{2.1}$$

where $b, \sigma : (0, \infty) \to \mathbb{R}$ are given functions, and W is a one-dimensional Brownian motion. In practice, we can think of such an investment project as a unit that can produce a single commodity. In this context, the state process X can be used to model an economic indicator such as the commodity's demand or the commodity's price.

With reference to the general theory of one-dimensional diffusions (e.g., see Section 5.5 in Karatzas and Shreve [8]), we impose the following standard assumption that is sufficient for (2.1) to define a diffusion that is unique in the sense of probability law.

Assumption 2.1. The deterministic functions $b, \sigma : (0, \infty) \to \mathbb{R}$ satisfy the following conditions:

$$\sigma^2(x) > 0, \quad \text{for all } x \in (0, \infty), \tag{2.2}$$

$$\text{for all } x \in (0,\infty), \text{ there exists } \varepsilon > 0 \text{ such that } \int_{x-\varepsilon}^{x+\varepsilon} \frac{1+|b(s)|}{\sigma^2(s)}\, ds < \infty. \tag{2.3}$$

This assumption also ensures that the *scale function* p and the *speed measure* $\widetilde{m}$ given by

$$p(1) = 0 \quad \text{and} \quad p'(x) = \exp\left(-2\int_1^x \frac{b(s)}{\sigma^2(s)}\, ds\right), \quad \text{for } x \in (0,\infty), \tag{2.4}$$

and

$$\widetilde{m}(dx) = \frac{2}{\sigma^2(x)p'(x)}\, dx, \tag{2.5}$$

respectively, are well defined. We denote by m the normalised speed measure, given by

$$m(dx) = \frac{1}{\widetilde{m}((0,\infty))}\widetilde{m}(dx).$$

We also assume that the solution to (2.1) is *non-explosive* and *recurrent*. With regard to Proposition 5.5.22 in Karatzas and Shreve [8], we therefore impose the following assumption.

Assumption 2.2. The scale function p defined by (2.4) satisfies $\lim_{x\downarrow 0} p(x) = -\infty$ and $\lim_{x\to\infty} p(x) = \infty$.

We assume that the investment project's capacity can be increased to any positive level dynamically over time. We denote by Y_t the project's capacity level at time t, and we assume that Y is a càglàd, increasing process. The constant $y = Y_0$ is the project's initial capacity level, while $Y_{t+} - y$ is the total additional capacity installed following the project's management decisions by time t.

We adopt a weak formulation of the capacity expansion problem that we study.

Definition 2.1. Given an initial condition $(x,y) \in (0,\infty)\times[0,\infty)$, a *capacity expansion strategy* is any seven-tuple $\mathbb{S}_{x,y} = \big(\Omega, \mathcal{F}, \mathcal{F}_t, P_{x,y}, W, X, Y\big)$ such that:

(i) $(\Omega, \mathcal{F}, \mathcal{F}_t, P_{x,y})$ is a filtered probability space satisfying the usual conditions,
(ii) W is a standard, one-dimensional $(\mathcal{F}_t)$ - Brownian motion,
(iii) X is a continuous, $(\mathcal{F}_t)$-adapted process satisfying (2.1), and
(iv) Y is a càglàd, increasing process such that $Y_0 = y$.

We denote by $\mathcal{A}_{x,y}$ the set of all such capacity expansion strategies, by $\mathcal{A}^F_{x,y} \subseteq \mathcal{A}_{x,y}$ the family of all capacity expansion strategies such that $Y_\infty = \lim_{t\to\infty} Y_t < \infty$, P-a.s., and by $\mathcal{A}^I_{x,y} \subseteq \mathcal{A}^F_{x,y}$ the class of all capacity expansion strategies such that $\mathbb{E}_{x,y}\big[Y_\infty\big] < \infty$.

With each capacity expansion strategy $\mathbb{S}_{x,y} \in \mathcal{A}_{x,y}$, we associate the *pathwise* performance criterion

$$J_P\big(\mathbb{S}_{x,y}\big) = \limsup_{T\to\infty} \frac{1}{T}\left[\int_0^T h\big(X_t, Y_t\big)\, dt - K\big(Y_{T+} - y\big)\right], \tag{2.6}$$

as well as the *expected* performance index

$$J_E\big(\mathbb{S}_{x,y}\big) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_{x,y}\left[\int_0^T h\big(X_t, Y_t\big)\, dt - K\big(Y_{T+} - y\big)\right]. \tag{2.7}$$

Here, the function $h : (0,\infty) \times [0,\infty) \to \mathbb{R}$ provides the running payoff resulting from the project's operation, while the constant $K > 0$ is associated with modelling the cost incurred by the project's capacity expansion.

The objective is to maximise J_P and J_E over $\mathcal{A}_{x,y}$. To achieve this aim, we impose the following assumption

Assumption 2.3. There exist a measurable function $k : (0,\infty) \to \mathbb{R}$, a continuous function $g : [0,\infty) \to [0,\infty)$ and constants $C_1, C_2 > 0$ such that

$$\int_0^\infty k(x)\, m(dx) < \infty, \quad \lim_{y\downarrow 0} g(y) = 0, \tag{2.8}$$

$$-C_1(1+y) \le h(x,y) \le k(x) - C_2(1+y), \quad \text{for all } (x,y) \in (0,\infty)\times[0,\infty), \tag{2.9}$$

$$\big|\, h(x,y) - h(x,y')\,\big| \le k(x)\, g(y-y'), \quad \text{for all } x \in (0,\infty) \text{ and } y, y' \in [0,\infty). \tag{2.10}$$

Note that (2.8) and (2.9) imply

$$\int_0^\infty \big|\, h(x,y)\,\big|\; m(dx) < \infty, \text{ for all } y \ge 0.$$

It follows that, since X is a recurrent, ergodic diffusion,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^\infty h\big(X_t, y\big)\, dt = \int_0^\infty h(x,y)\, m(dx), \text{ for all } y \ge 0, \tag{2.11}$$

(see Theorem V.53.1 in Rogers and Williams [12]), and

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}_{x,y}\left[\int_0^\infty h\big(X_t, y\big)\, dt\right] = \int_0^\infty h(x,y)\, m(dx), \text{ for all } y \ge 0, \tag{2.12}$$

(see Theorem V.54.5 in Rogers and Williams [12]).

Remark 2.1. For future reference, note that given a weak solution $(\Omega, \mathcal{F}, \mathcal{F}_t, P_x, W, X)$ to (2.1), if Z is a measurable mapping from Ω into $\mathbb{R}$, then (2.11) implies

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T h\big(X_t, Z\big)\, dt = \int_0^\infty h\big(x, Z\big)\, m(dx), \quad P_x\text{-a.s.}. \tag{2.13}$$

3. The solution to the optimisation problem

To solve the optimisation problems formulated in the previous section, we first show that maximising J_P and J_E over $\mathcal{A}_{x,y}$ is equivalent to maximising J_P and J_E over the smaller sets $\mathcal{A}^F_{x,y}$ and $\mathcal{A}^I_{x,y}$, respectively.

Lemma 3.1. *Given any* $(x,y) \in (0,\infty)\times[0,\infty)$,

$$\sup_{\mathbb{S}_{x,y}\in\mathcal{A}_{x,y}} J_P\big(\mathbb{S}_{x,y}\big) < \infty, \tag{3.1}$$

$$J_P(\mathbb{S}_{x,y})\mathbf{1}_{\{Y_\infty=\infty\}} = -\infty\mathbf{1}_{\{Y_\infty=\infty\}}, \quad \textit{for all } \mathbb{S}_{x,y} \in \mathcal{A}_{x,y} \tag{3.2}$$

and

$$\sup_{\mathbb{S}_{x,y}\in\mathcal{A}_{x,y}} J_E(\mathbb{S}_{x,y}) = \sup_{\mathbb{S}_{x,y}\in\mathcal{A}^I_{x,y}} \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_{x,y}\left[\int_0^T h(X_t, Y_t)\, dt\right] \in \mathbb{R}. \tag{3.3}$$

Proof. In view of (2.8), (2.11), and (2.12) with k in place of $h(\cdot, y)$,

$$\sup_{\mathbb{S}_{x,y}\in\mathcal{A}_{x,y}} J_E(\mathbb{S}_{x,y}) \le \int_0^\infty k(x)\, m(dx) < \infty.$$

Also, by considering the strategy that involves no capacity increases at any time, we can see that, in the presence of assumption (2.9),

$$\sup_{\mathbb{S}_{x,y}\in\mathcal{A}_{x,y}} J_E(\mathbb{S}_{x,y}) \ge -C_1(1+y) > -\infty.$$

Now, fix any $\mathbb{S}_{x,y} \in \mathcal{A}_{x,y} \setminus \mathcal{A}^I_{x,y}$. Assumption (2.9) implies that, given any constant time $T_1 > 0$,

$$\begin{aligned} J_E(\mathbb{S}_{x,y}) &\le \int_0^\infty k(x)\, m(dx) - C_2\Big(1 + \liminf_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}_{x,y}\big[Y_t\big]\, dt\Big) \\ &\le \int_0^\infty k(x)\, m(dx) - C_2\Big(1 + \mathbb{E}_{x,y}\big[Y_{T_1}\big]\Big). \end{aligned}$$

Since T_1 is arbitrary, it follows that

$$\begin{aligned} J_E(\mathbb{S}_{x,y}) &\le \int_0^\infty k(x)\, m(dx) - C_2\Big(1 + \lim_{T_1\to\infty} \mathbb{E}_{x,y}\big[Y_{T_1}\big]\Big) \\ &= \int_0^\infty k(x)\, m(dx) - C_2(1 + \mathbb{E}_{x,y}\big[Y_\infty\big]\Big) \\ &= -\infty, \end{aligned}$$

the first equality following thanks to the monotone convergence theorem. However, this calculation establishes the identity in (3.3).

Using similar arguments, we can prove (3.1) and (3.2). □

The next result shows that, to maximise J_P, we only need to consider the "eventual" capacity level.

Lemma 3.2. *Given any* $\mathbb{S}_{x,y} \in \mathcal{A}_{x,y}$, $\lim_{T\to\infty} T^{-1}\int_0^T h(X_t, Y_t)\, dt$ *exists and*

$$\begin{aligned} J_P(\mathbb{S}_{x,y}) &= \mathbf{1}_{\{Y_\infty<\infty\}} \lim_{T\to\infty} \frac{1}{T}\int_0^T h(X_t, Y_t)\, dt - \infty\mathbf{1}_{\{Y_\infty=\infty\}} \\ &= \mathbf{1}_{\{Y_\infty<\infty\}} \int_0^\infty h(x, Y_\infty)\, m(dx) - \infty\mathbf{1}_{\{Y_\infty=\infty\}}. \end{aligned} \tag{3.4}$$

Proof. With reference to (3.2) in Lemma 3.1, we shall prove this result if we establish (3.4) on the event $\{Y_\infty < \infty\}$. To simplify the notation, we therefore fix $\mathbb{S}_{x,y} \in \mathcal{A}^F_{x,y}$ in what follows, without loss of generality. We define the function

$$G(y) = g(y) \int_0^\infty k(x)\, m(dx), \tag{3.5}$$

where k and g are as in Assumption 2.3, and we consider any $\varepsilon > 0$. With reference to the properties of g, choose any $\delta > 0$ such that $G(y) \leq \varepsilon$ for all $y \in [0, \delta]$. Since $Y_\infty < \infty$, $P_{x,y}$-a.s., there exists a finite *random* time T_δ such that $Y_\infty - Y_t \leq \delta$, for all $t \geq T_\delta$, $P_{x,y}$-a.s.. In view of (2.10), we calculate

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T}\left[\int_0^T \Big[h\big(X_t, Y_\infty\big) - h\big(X_t, Y_t\big)\Big]\, dt\right] \\
&\qquad = \limsup_{T\to\infty} \frac{1}{T}\left[\int_{T_\delta}^T \Big[h\big(X_t, Y_\infty\big) - h\big(X_t, Y_t\big)\Big]\, dt\right] \\
&\qquad \leq \limsup_{T\to\infty} \frac{1}{T} \int_{T_\delta}^T \big|\, h\big(X_t, Y_\infty\big) - h\big(X_t, Y_t\big)\,\big|\, dt \\
&\qquad \leq g(\delta)\, \limsup_{T\to\infty} \frac{1}{T}\left[\int_{T_\delta}^T k(X_t)\, dt\right] \\
&\qquad \leq G(\delta) \leq \varepsilon.
\end{aligned}$$

With reference to (2.13), it follows that

$$\begin{aligned}
\int_0^\infty h\big(x, Y_\infty\big)\, m(dx) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T h\big(X_t, Y_\infty\big)\, dt \\
&\leq \liminf_{T\to\infty} \frac{1}{T} \int_0^T h\big(X_t, Y_t\big)\, dt + \varepsilon.
\end{aligned} \tag{3.6}$$

Similarly, we can see that

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \Big[h\big(X_t, Y_t\big) - h\big(X_t, Y_\infty\big)\Big]\, dt \leq \varepsilon,$$

which, combined with (2.13), implies

$$\begin{aligned}
\limsup_{T\to\infty} \frac{1}{T} \int_0^T h\big(X_t, Y_t\big)\, dt - \varepsilon &\leq \lim_{T\to\infty} \frac{1}{T} \int_0^T h\big(X_t, Y_\infty\big)\, dt \\
&= \int_0^\infty h\big(x, Y_\infty\big)\, m(dx).
\end{aligned} \tag{3.7}$$

However, since $\varepsilon > 0$ is arbitrary, (3.6) and (3.7) establish the claims made. □

The following result, which parallels the preceding one, is concerned with maximising J_E.

Lemma 3.3. *Given any* $\mathbb{S}_{x,y} \in \mathcal{A}^I_{x,y}$,

$$J_E\big(\mathbb{S}_{x,y}\big) = \mathbb{E}_{x,y}\left[\int_0^\infty h\big(x, Y_\infty\big)\, m(dx)\right].$$

Proof. Let any $\mathbb{S}_{x,y} \in \mathcal{A}^I_{x,y}$. Observing that

$$\int_0^T k\big(X_t\big) dt - \int_0^T h\big(X_t, Y_t\big) dt \geq 0,$$

which follows from the upper bound in (2.9), we can appeal to (2.8) and Fatou's lemma to see that

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T}\mathbb{E}_{x,y}\left[\int_0^T k\big(X_t\big)\, dt\right] - \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_{x,y}\left[\int_0^T h\big(X_t, Y_t\big)\, dt\right] \\
&\quad \geq \mathbb{E}_{x,y}\left[\liminf_{T\to\infty}\left(\frac{1}{T}\int_0^T k\big(X_t\big)\, dt - \frac{1}{T}\int_0^T h\big(X_t, Y_t\big)\, dt\right)\right] \\
&\quad = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}_{x,y}\left[\int_0^T k\big(X_t\big)\, dt\right] - \mathbb{E}_{x,y}\left[\limsup_{T\to\infty} \frac{1}{T}\int_0^T h\big(X_t, Y_t\big)\, dt\right],
\end{aligned}$$

the equality following thanks to (2.11) and (2.12). However, this observation and Lemma 3.2 imply

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_{x,y}\left[\int_0^T h\big(X_t, Y_t\big)\, dt\right] \leq \mathbb{E}_{x,y}\left[\int_0^\infty h\big(x, Y_\infty\big)\, m(dx)\right]. \tag{3.8}$$

To proceed further, we note that

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}_{x,y}\left[\int_0^T Y_t\, dt\right] = \mathbb{E}_{x,y}\big[Y_\infty\big] < \infty. \tag{3.9}$$

To see this claim, we first note that $\mathbb{E}_{x,y}\big[Y_t\big] \uparrow \mathbb{E}_{x,y}\big[Y_\infty\big]$ thanks to the fact that Y is an increasing process and the monotone convergence theorem. In view of this observation, given any $\varepsilon > 0$, there exists $T_1 > 0$ such that $\mathbb{E}_{x,y}\big[Y_t\big] \geq \mathbb{E}_{x,y}\big[Y_\infty\big] - \varepsilon$, for all $t \geq T_1$. It follows that

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}_{x,y}\left[\int_0^T Y_t\, dt\right] \geq \mathbb{E}_{x,y}\big[Y_\infty\big] - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves that $\lim_{T\to\infty} \frac{1}{T}\mathbb{E}_{x,y}\left[\int_0^T Y_t\, dt\right] \geq \mathbb{E}_{x,y}\left[Y_\infty\right]$. The reverse inequality is obvious because $\mathbb{E}_{x,y}\left[Y_t\right] \leq \mathbb{E}_{x,y}\left[Y_\infty\right]$, for all $t \geq 0$. Using similar arguments, we can see that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T Y_t\, dt = Y_\infty. \tag{3.10}$$

Now, in view of (3.9), (3.10), the lower bound in (2.9) in Assumption 2.3 and Fatou's lemma, we calculate

$$
\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_{x,y}\left[\int_0^T h\big(X_t, Y_t\big)\, dt\right] + C_1\big(1+\mathbb{E}_{x,y}\big[Y_\infty\big]\big) \\
&\quad = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_{x,y}\left[\int_0^T \big[h\big(X_t, Y_t\big) + C_1\big(1+Y_t\big)\big]\, dt\right] \\
&\quad \geq \mathbb{E}_{x,y}\left[\liminf_{T\to\infty} \frac{1}{T}\left(\int_0^T \big[h\big(X_t, Y_t\big) + C_1\big(1+Y_t\big)\big]\, dt\right)\right] \\
&\quad = \mathbb{E}_{x,y}\left[\liminf_{T\to\infty} \frac{1}{T}\int_0^T h\big(X_t, Y_t\big)\, dt + C_1\big(1+Y_\infty\big)\right] \\
&\quad = \mathbb{E}_{x,y}\left[\int_0^\infty h\big(x, Y_\infty\big)\, m(dx)\right] + C_1\big(1+\mathbb{E}_{x,y}\big[Y_\infty\big]\big),
\end{aligned}
$$

the last equality following thanks to Lemma 3.2. However, this inequality and (3.8) imply the claim made. □

Given the results that we have established thus far, it is straightforward to see that the following theorem provides the solution to the optimisation problems that we consider.

Theorem 3.4. *Consider the optimisation problems formulated in Section 2, fix any initial condition $(x, y) \in (0,\infty) \times [0,\infty)$, and define*

$$
\bar{y} = \arg\max_{z\in[y,\infty)} \int_0^\infty h(x, z)\, m(dx).
$$

In the presence of Assumption 2.3,

$$
\sup_{\mathbb{S}_{x,y}\in\mathcal{A}_{x,y}} J_P\big(\mathbb{S}_{x,y}\big) = \sup_{\mathbb{S}_{x,y}\in\mathcal{A}_{x,y}} J_E\big(\mathbb{S}_{x,y}\big) = \int_0^\infty h(x, \bar{y})\, m(dx),
$$

In either case, immediately increasing the project's capacity level to $\bar{y}$ and then take no further action provides an optimal capacity expansion strategy.

4. A special case

We now consider the special case that arises when the state process X is modelled by the SDE

$$
dX_t = \kappa(\theta - X_t)\, dt + \sigma\sqrt{X_t}\, dW_t, \tag{4.1}
$$

where $\kappa, \theta, \sigma > 0$ are constants satisfying $2\kappa\theta > \sigma^2$. This diffusion is identical to the short rate process in the Cox-Ingersoll-Ross interest rate model, and is widely adopted as a model for commodity prices. Verifying that this diffusion satisfies Assumption 2.1

and Assumption 2.2 is a standard exercise. Also, it is straightforward to verify that the normalised speed measure of X is given by

$$m(dx) = \Gamma^{-1}\left(\frac{2\kappa\theta}{\sigma^2}\right) x^{\frac{2\kappa\theta}{\sigma^2}-1} \exp\left(\frac{2\kappa}{\sigma^2}\left[\theta \ln\left(\frac{2\kappa}{\sigma^2}\right) - x\right]\right) dx,$$

where Γ is the gamma function and $\Gamma^{-1}(\cdot) = 1/\Gamma(\cdot)$.

We also assume that the running payoff function h is given by

$$h(x, y) = x^\alpha y^\beta - c\, y,$$

where $\alpha, \beta \in (0, 1)$ and $c \in (0, \infty)$ are constants, which is a choice satisfying Assumption 2.3. The term $x^\alpha y^\beta$ here identifies with the so-called Cobb-Douglas production function, while the term cy provides a measure for the cost of capital utilisation.

With reference to Theorem 3.4, the project's optimal capacity level $\bar{y}$ is the maximum of the project's initial capacity y and the solution to the algebraic equation

$$\left(\int_0^\infty x^\alpha\, m(dx)\right) y^\beta - cy = 0.$$

In the light of the calculation

$$\int_0^\infty x^\alpha\, m(dx) = \Gamma\left(\frac{2\kappa\theta}{\sigma^2} + \alpha\right)\Gamma^{-1}\left(\frac{2\kappa\theta}{\sigma^2}\right)\left(\frac{\sigma^2}{2\kappa}\right)^\alpha,$$

it follows that the investment project's optimal capacity level is given by

$$\bar{y} = \max\left\{y,\ \left[\frac{\beta\Gamma\big(z\theta + \alpha\big)}{c\Gamma\big(z\theta\big) z^\alpha}\right]^{\frac{1}{1-\beta}}\right\},$$

where $z = 2\kappa/\sigma^2$.

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