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Sequential testing problems for some diffusion processes*

Pavel V. Gapeev

We study the Bayesian problem of sequential testing of two simple hypotheses about the local drift of an observed diffusion process. The optimal stopping time is found as the first time when the a posteriori probability process leaves the region defined by two stochastic boundaries depending on the observation process. It is shown that under some nontrivial relationships on the coefficients of the observed diffusion the problem admits a closed form solution. The method of proof is based on embedding the initial problem into a two-dimensional optimal stopping problem and solving the equivalent free-boundary problem by means of the smooth-fit conditions.

1. Introduction

The problem of sequential testing of two simple hypotheses about the local drift $\mu(x)$ of an observed diffusion process seeks to determine as soon as possible and with minimal error probabilities if the true drift coefficient is either $\mu_0(x)$ or $\mu_1(x)$. This problem admits two different formulations (see Wald [20]). In the Bayesian formulation it is assumed that the drift coefficient $\mu(x)$ has an a priori given distribution, and in the variational formulation no probabilitsic assumption is made about the unknown drift $\mu(x)$. In this paper we only study the Bayesian formulation.

By means of the Bayesian approach, Wald and Wolfowitz [21]-[22] proved the optimality of the sequential probability ratio test (SPRT) in the variational formulation of the problem for sequences of i.i.d. observations. Dvoretzky, Kiefer and Wolfowitz [2] pointed out that if the (continuous time) likelihood ratio process has stationary independent increments, then the SPRT remains optimal in the variational problem. Mikhalevich [12] and Shiryaev [18] (see also [19; Chapter IV]) obtained an explicit solution of the Bayesian problem for an observed Wiener process by reducing the initial optimal stopping problem to a free-boundary problem for an ordinary second order operator. A complete proof of the statement of [2] (under some mild assumptions) was given by Irle and Schmitz [7]. Peskir and Shiryaev [14] obtained an explicit

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solution of the Bayesian problem of testing hypotheses about the intensity of an observed Poisson process by solving a free-boundary problem for a differential-difference operator. Sequential testing problems for a compound Poisson process having exponentially distributed jumps were explicitly solved in [4]. Recently, Dayanik and Sezer [1] obtained a solution of the Bayesian sequential testing problem for a general compound Poisson process. A finite horizon version of the Wiener sequential testing problem was studied in [5]. The main purpose of this paper is to present a solution of the problem of testing hypotheses about the local drift of an observed diffusion process in the Bayesian formulation under some nontrivial relationships on coefficients of the observed diffusion. In this case the optimal Bayes stopping time is the first time when the a posteriori probability process leaves a region defined by two stochastic boundaries depending on the observation process.

In the present paper we make an embedding of the initial Bayesian problem into an extended optimal stopping problem for a two-dimensional (time-homogeneous strong) Markov diffusion process (consisting of the a posteriori probability process and the observation process). We show that the continuation region (for the a posteriori probability process) is determined by two stochastic boundaries depending on the observation process where the behavior of the boundaries is characterized by the signal/noise ratio. In order to find analytic expressions for the value function and the stopping boundaries under some special nontrivial relationships on coefficients of the observed diffusion, we formulate an equivalent free-boundary problem. By applying smooth-fit conditions we show that the free-boundary problem admits an explicit solution and the boundaries are uniquely determined from a coupled system of transcendental equations. Then we verify that the solution of the free-boundary problem turns out to be a solution of the initial extended optimal stopping problem. The main result of the paper is stated in Theorem 2.1.

2. Formulation and solution of the Bayesian problem

In the Bayesian formulation of the problem (see [19; Chapter IV, Section 2] for the case of Wiener process) it is assumed that we observe a trajectory of the diffusion process $X = (X_t)_{t \ge 0}$ with drift $\mu_0(x) + \theta(\mu_1(x) - \mu_0(x))$ where the random parameter θ may be 1 or 0 with probability π or $1-\pi$, respectively.

2.1. For a precise probabilistic formulation of the Bayesian problem it is convenient to assume that all our considerations take place on a probability space $(\Omega, \mathcal{F}, P_\pi)$ where the probability measure P_π has the following structure:

$$P_\pi = \pi P_1 + (1-\pi) P_0 \tag{2.1}$$

for any $\pi \in [0,1]$. Let θ be a random variable taking two values 1 and 0 with probabilities $P_\pi[\theta = 1] = \pi$ and $P_\pi[\theta = 0] = 1 - \pi$, and let $W = (W_t)_{t \ge 0}$ be a standard Wiener process started at zero under P_π. It is assumed that θ and W are independent.

It is further assumed that we observe a continuous process $X = (X_t)_{t \ge 0}$ with the (open) state space $E \subseteq \mathbb{R}$ and solving the stochastic differential equation:

$$dX_t = [\mu_0(X_t) + \theta(\mu_1(X_t) - \mu_0(X_t))]\, dt + \sigma(X_t)\, dW_t \quad (X_0 = x) \tag{2.2}$$

where the functions $\mu_i(x)$ and $\sigma(x)$ are Lipschitz continuous on E, that is, there exists a constant $C > 0$ such that:

$$[\mu_i(x) - \mu_i(x')]^2 + [\sigma(x) - \sigma(x')]^2 \le C[x - x']^2 \tag{2.3}$$

for all $x, x' \in E$ and $i = 0, 1$. Thus, from [11; Chapter IV, Theorem 4.6] it follows that under fixed $\theta = i$ equation (2.2) has a unique strong solution, and hence, $P_\pi[X \in \cdot \,|\, \theta = i] = P_i[X \in \cdot\,]$ is the distribution law of a homogeneous diffusion process (starting at some fixed point $x \in E$) with local drift $\mu_i(x)$ and diffusion coefficient $\sigma^2(x)$ for $i = 0, 1$. We will also assume that either $\mu_0(x) < \mu_1(x)$ or $\mu_0(x) > \mu_1(x)$ holds and $\sigma^2(x) > 0$ for all $x \in E$. Let π and $1 - \pi$ play the role of a priopi probabilities of the statistical hypotheses:

$$H_1 : \theta = 1 \qquad \text{and} \qquad H_0 : \theta = 0 \tag{2.4}$$

respectively.

Being based upon the continuous observation of X our task is to test sequentially the hypotheses H_1 and H_0 with a minimal loss. For this, we consider a sequential decision rule (τ, d) where τ is a stopping time of the observed process X (i.e., a stopping time with respect to the natural filtration $\mathcal{F}_t^X = \sigma\{X_s \mid 0 \le s \le t\}$ generated by the process X for $t \ge 0$), and d is an $\mathcal{F}_\tau^X$-measurable function taking on values 0 and 1. After stopping the observations at time τ, the terminal decision function indicates which hypothesis should be accepted according to the following rule: if $d = 1$ we accept H_1, and if $d = 0$ we accept H_0. The problem consists of computing the risk function:

$$V(\pi) = \inf_{(\tau,d)} E_\pi[\tau + aI(d = 0, \theta = 1) + bI(d = 1, \theta = 0)] \tag{2.5}$$

and finding the optimal decision rule (τ_*, d_*), called the π-Bayes decision rule, at which the infimum in (2.5) is attained. Here $E_\pi[\tau]$ is the average cost of the observations, and $aP_\pi[d = 0, \theta = 1] + bP_\pi[d = 1, \theta = 0]$ is the average loss due to a wrong terminal decision, where $a > 0$ and $b > 0$ are some given constants.

2.2. By means of standard arguments (see [19; pages 166-167]) one can reduce the Bayesian problem (2.5) to the optimal stopping problem:

$$V(\pi) = \inf_\tau E_\pi[\tau + g_{a,b}(\pi_\tau)] \tag{2.6}$$

for the a posteriori probability process $\pi_t = P_\pi[\theta = 1 \,|\, \mathcal{F}_t^X]$ for $t \ge 0$ with $P_\pi[\pi_0 = \pi] = 1$. Here $g_{a,b}(\pi) = a\pi \wedge b(1 - \pi)$ for $\pi \in [0, 1]$, and the optimal decision function is given by $d_* = 1$ if $\pi_{\tau_*} \ge c$, and $d_* = 0$ if $\pi_{\tau_*} < c$, where here and in the sequel we set $c = b/(a + b)$.

2.3. Since for $i = 0, 1$ condition (2.3) is assumed to be satisfied, by applying Girsanov's theorem for diffusion-type processes [11; Theorem 7.19] we get that the loglikelihood ratio process $Z = (Z_t)_{t \ge 0}$ defined as logarithm of the Radon-Nikodym derivative:

$$Z_t = \log \frac{d(P_1 | \mathcal{F}_t^X)}{d(P_0 | \mathcal{F}_t^X)} \tag{2.7}$$

(here $P_i | \mathcal{F}_t^X$ denotes the restriction of P_i to $\mathcal{F}_t^X$ for $i = 0, 1$) takes the form:

$$Z_t = \int_0^t \frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma^2(X_s)} \, dX_s - \frac{1}{2} \int_0^t \frac{\mu_1^2(X_s) - \mu_0^2(X_s)}{\sigma^2(X_s)} \, ds \tag{2.8}$$

for all $t \geq 0$. According to the arguments in [19; pages 180-181], the a posteriori probability process $(\pi_t)_{t\geq 0}$ can be expressed as:

$$\pi_t = \left(\frac{\pi}{1-\pi} e^{Z_t} \right) \Big/ \left(1 + \frac{\pi}{1-\pi} e^{Z_t} \right) \tag{2.9}$$

and, by virtue of Itô's formula (see, e.g., [11; Chapter IV, Theorem 4.4]), it solves the equation:

$$d\pi_t = \frac{\mu_1(X_t) - \mu_0(X_t)}{\sigma(X_t)} \, \pi_t(1-\pi_t) \, d\overline{W}_t \quad (\pi_0 = \pi) \tag{2.10}$$

where, by means of P. Lévy's theorem [17; Chapter IV, Theorem 3.6], the innovation process $\overline{W} = (\overline{W}_t)_{t\geq 0}$ defined by:

$$\overline{W}_t = \int_0^t \frac{dX_s}{\sigma(X_s)} - \int_0^t \left(\frac{\mu_0(X_s)}{\sigma(X_s)} + \pi_s \, \frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma(X_s)} \right) ds \tag{2.11}$$

is a standard Wiener process under the measure P_π with respect to the filtration $(\mathcal{F}_t^X)_{t\geq 0}$. Therefore, from (2.11) it follows that the process $X = (X_t)_{t\geq 0}$ admits the representation:

$$dX_t = [\mu_0(X_t) + \pi_t(\mu_1(X_t) - \mu_0(X_t))] \, dt + \sigma(X_t) \, d\overline{W}_t \quad (X_0 = x). \tag{2.12}$$

Let us suppose that the *signal/noise ratio* function $r(x)$ defined by:

$$r(x) = \frac{\mu_1(x) - \mu_0(x)}{\sigma(x)} \tag{2.13}$$

is also Lipschitz continuous, that is, there exists a constant $C' > 0$ such that condition:

$$[r(x) - r(x')]^2 \leq C'[x - x']^2 \tag{2.14}$$

holds for all $x, x' \in E$, and there are constants r_* and r^* such that the inequalities:

$$0 < r_* \leq r(x) \leq r^* < \infty \tag{2.15}$$

are satisfied for all $x \in E$. Hence, by means of Remark to [11; Chapter IV, Theorem 4.6] (see also [13; Theorem 5.2.1]), we conclude that the process $(\pi_t, X_t)_{t\geq 0}$ turns out to be a unique strong solution of the (two-dimensional) stochastic differential equation (2.10)+(2.12), and thus, by virtue of [13; Theorem 7.2.4], it is a (time-homogeneous strong) Markov process with respect to its natural filtration which obviously coincides with $(\mathcal{F}_t^X)_{t\geq 0}$. Therefore, the infimum in (2.6) is taken over all stopping times of $(\pi_t, X_t)_{t\geq 0}$ being a Markovian sufficient statistic in the problem (see [19; Chapter II, Section 15]).

2.4. For the problem (2.6) let us consider the following extended optimal stopping problem for the Markov process $(\pi_t, X_t)_{t\geq 0}$:

$$V(\pi, x) = \inf_\tau E_{\pi,x}[\tau + g_{a,b}(\pi_\tau)] \tag{2.16}$$

where $P_{\pi,x}$ is a measure of the diffusion process $(\pi_t, X_t)_{t\geq 0}$ starting at the point (π, x) and solving the (two-dimensional) equation (2.10)+(2.12), and the infimum in (2.16) is taken over all stopping times τ of the process $(\pi_t, X_t)_{t\geq 0}$ such that $E_{\pi,x}[\tau] < \infty$ for all $(\pi, x) \in [0,1] \times E$.

2.5. Let us now determine the structure of the optimal stopping time in the problem (2.16).

(i) First, by applying applying Itô-Tanaka-Meyer formula (see, e.g., [8; Chapter V, (5.52)] or [16; Chapter IV, Theorem 51]) to the function $g_{a,b}(\pi) = a\pi \wedge b(1-\pi)$, we get:

$$g_{a,b}(\pi_t) = g_{a,b}(\pi) + \int_0^t (g_{a,b})_\pi(\pi_s)\, ds + \frac{1}{2}\int_0^t \Delta_\pi (g_{a,b})_\pi(\pi_s)\, d\ell_s^c(\pi) + N_t^c \tag{2.17}$$

where $\int_0^t \Delta_\pi(g_{a,b})_\pi(\pi_s) d\ell_s^c(\pi) = (-b-a)\ell_t^c(\pi)$, the process $(\ell_t^c(\pi))_{t\ge 0}$ is the local time of $(\pi_t)_{t\ge0}$ at the point c given by:

$$\ell_t^c(\pi) = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^t I(c-\varepsilon < \pi_s < c+\varepsilon)\, r^2(X_s)\pi_s^2(1-\pi_s)^2\, ds \tag{2.18}$$

as a limit in probability, and for any $(\mathcal{F}_t^X)_{t\ge0}$-stopping time τ satisfying $E_{\pi,x}[\tau] < \infty$ the process $(N_{\tau\wedge t}^c, \mathcal{F}_t^X, P_{\pi,x})_{t\ge0}$ defined by $N_{\tau\wedge t}^c = \int_0^{\tau\wedge t} (g_{a,b})_\pi(\pi_s) I(\pi_s \neq c) r(X_s)\pi_s(1-\pi_s) d\overline{W}_s$ is a continuous (uniformly integrable) martingale under $P_{\pi,x}$.

Let us fix some (π, x) from the continuation region C and let $\tau_* = \tau_*(\pi,x)$ denote the optimal stopping time in the problem (2.16). By applying Doob's optional sampling theorem (see, e.g., [9; Chapter I, Theorem 1.39] or [17; Chapter II, Theorem 3.1]) and by using (2.17), it follows that:

$$V(\pi,x) = E_{\pi,x}[\tau_* + g_{a,b}(\pi_{\tau_*})] = g_{a,b}(\pi) + E_{\pi,x}\left[\tau_* - \frac{1}{2}(a+b)\ell_{\tau_*}^c(\pi)\right] \tag{2.19}$$

and hence, by virtue of general optimal stopping theory for Markov processes (see [19; Chapter III]), we have:

$$V(\pi,x) - g_{a,b}(\pi) = E_{\pi,x}\left[\tau_* - \frac{1}{2}(a+b)\ell_{\tau_*}^c(\pi)\right] < 0. \tag{2.20}$$

Then taking any π' such that $\pi < \pi' \le c$ or $c \le \pi' < \pi$ and using the explicit expression (2.9), from (2.17)-(2.18) we obtain:

$$V(\pi',x) - g_{a,b}(\pi') \le E_{\pi',x}\left[\tau_* - \frac{1}{2}(a+b)\ell_{\tau_*}^c(\pi')\right] \le E_{\pi,x}\left[\tau_* - \frac{1}{2}(a+b)\ell_{\tau_*}^c(\pi)\right] \tag{2.21}$$

and thus, by means of (2.20), we see that $(\pi', x) \in C$. Therefore, according to the general optimal stopping theory (see, e.g., [19] and [15]), these arguments (together with the easily proved concavity of the function $\pi \mapsto V(\pi,x)$ on $[0,1]$, see also [10] or [19; pages 168-169]) show that there exists a couple of functions $(g_0(x), g_1(x))$, $x \in E$, such that $0 \le g_0(x) \le c \le g_1(x) \le 1$, and the continuation region for the optimal stopping problem (2.16) is an open set of the form:

$$C = \{(\pi,x) \in [0,1]\times E \mid \pi \in \langle g_0(x), g_1(x)\rangle\} \tag{2.22}$$

and the stopping region is the closure of the set:

$$D = \{(\pi,x) \in [0,1]\times E \mid \pi \in [0, g_0(x)\rangle \cup \langle g_1(x), 1]\}. \tag{2.23}$$

(ii) Now for given $(\pi, x) \in C$ let us take $x' \in E$ such that $x' < x$ if $x < c$ or $x < x'$ if $x > c$. Then using the facts that $(\pi_t, X_t)_{t\geq 0}$ is a time-homogeneous Markov process and $\tau_* = \tau_*(\pi, x)$ does not depend on x', from (2.17)-(2.18) we obtain:

$$\begin{aligned} V(\pi, x') - g_{a,b}(\pi) &\leq E_{\pi,x'} \left[\tau_* - \frac{1}{2}(a+b)\ell^c_{\tau_*}(\pi) \right] \\ &\leq E_{\pi,x} \left[\tau_* - \frac{1}{2}(a+b)\ell^c_{\tau_*}(\pi) \right] = V(\pi, x) - g_{a,b}(\pi) \end{aligned} \tag{2.24}$$

and hence, by means of (2.20), we see that $(\pi, x') \in C$. Therefore, we may conclude that in (2.22)-(2.23) the boundary $x \mapsto g_0(x)$ is increasing (decreasing) and the boundary $x \mapsto g_1(x)$ is decreasing (increasing) on E when the function $r(x)$ is increasing (decreasing), respectively.

(iii) Next, let us observe that the value function $V(\pi, x)$ from (2.16) and the boundaries $(g_0(x), g_1(x))$ from (2.22)-(2.23) also depend on $r(x)$ defined in (2.13) and denote them here by $V_*(x, \pi)$ and $V^*(\pi, x)$ as well as (A_*, B_*) and (A^*, B^*) when $r(x) = r_*$ and $r(x) = r^*$ for all $x \in E$, respectively. Using the fact that $x \mapsto V(\pi, x)$ is an increasing (decreasing) function when $r(x)$ is increasing (decreasing) on E, and $V(\pi, x) = g_{a,b}(\pi)$ for all $\pi \in [0, g_0(x)] \cup [g_1(x), 1]$, we conclude that $0 < A^* \leq g_0(x) \leq A_* < c < B_* \leq g_1(x) \leq B^* < 1$ for all $x \in E$. Here we note that if $r_* = r^*$ then $A^* = g_0(x) = A_*$ and $B_* = g_1(x) = B^*$ for all $x \in E$, where $0 < A^* < A_* < c < B_* < B^* < 1$ are uniquely determined from the system (4.85) in [19; Chapter IV].

2.6. Summarizing the facts proved in Subsection 2.5 above we may conclude that the following optimal decision rule is optimal in the extended problem (2.16):

$$\tau_* = \inf\{t \geq 0 \mid \pi_t \notin \langle g_0(X_t), g_1(X_t) \rangle\} \tag{2.25}$$

$$d_* = \begin{cases} 1, & \text{if } \pi_{\tau_*} = g_1(X_{\tau_*}) \\ 0, & \text{if } \pi_{\tau_*} = g_0(X_{\tau_*}) \end{cases} \tag{2.26}$$

(whenever $E_{\pi,x}[\tau_*] < \infty$) where the two boundaries $(g_0(x), g_1(x))$, $x \in E$, satisfy the following properties:

$$g_0(x) : E \to [0, 1] \quad \text{is continuous and increasing (decreasing)} \tag{2.27}$$

$$g_1(x) : E \to [0, 1] \quad \text{is continuous and decreasing (increasing)} \tag{2.28}$$

$$A^* \leq g_0(x) \leq A_* < c < B_* \leq g_1(x) \leq B^* \quad \text{for all} \quad x \in E \tag{2.29}$$

whenever $r(x)$ is an increasing (decreasing) function on E, respectively. Here (A_*, B_*) and (A^*, B^*) satisfying $0 < A^* < A_* < c < B_* < B^* < 1$ are the optimal stopping points for the corresponding infinite horizon problem with $r(x) = r_*$ and $r(x) = r^*$ for all $x \in E$, respectively, uniquely determined from the system of transcendental equations (4.85) in [19; Chapter IV].

2.7. Let us further assume that the state space of the process $X = (X_t)_{t\geq 0}$ under both hypotheses (2.4) is $E = \langle -\zeta, \infty \rangle$ for some $\zeta \in \mathbb{R}$ fixed, and under conditions of Subsections 2.1 and 2.3 the relationship:

$$\mu_i(x) = \frac{\eta_i \sigma^2(x)}{x + \zeta} \tag{2.30}$$

holds for all $x \in E$ and some constants $\eta_i \in \mathbb{R}$, $i = 0, 1$, such that $\eta_0 \neq \eta_1$ and $\eta_0 + \eta_1 = 1$. Let us define the process $Y = (Y_t)_{t \geq 0}$ by:

$$Y_t = \log \frac{\pi_t}{1 - \pi_t} - \frac{1}{\eta} \log \frac{x + \zeta}{X_t + \zeta} \tag{2.31}$$

with $\eta = 1/(\eta_1 - \eta_0)$. From the structure of (2.31) it is easily seen that there is a one-to-one correspondence between the processes $(\pi_t, X_t)_{t \geq 0}$ and $(\pi_t, Y_t)_{t \geq 0}$, and thus, the latter is also a (time-homogeneous strong) Markov process with respect to its natural filtration, which coincides with $(\mathcal{F}_t^X)_{t \geq 0}$. By deriving the expression for X_t from (2.31) and by substituting it into (2.10), we obtain:

$$d\pi_t = \frac{\sigma \left((x + \zeta) e^{-\eta Y_t} [\pi_t/(1 - \pi_t)]^\eta - \zeta \right)}{\eta (x + \zeta) e^{-\eta Y_t} [\pi_t/(1 - \pi_t)]^\eta} \, \pi_t (1 - \pi_t) \, d\overline{W}_t \quad (\pi_0 = \pi). \tag{2.32}$$

By applying Itô's formula to the expression (2.31) and by using the representations (2.10) and (2.12) as well as the assumption (2.30) with $\eta_0 \neq \eta_1$ and $\eta_0 + \eta_1 = 1$, we get $dY_t = 0$ and thus:

$$Y_t = \log \frac{\pi}{1 - \pi} \tag{2.33}$$

for all $t \geq 0$.

2.8. By means of standard arguments it is shown that under the assumptions of Subsection 2.7 the optimal stopping problems (2.6) and (2.16) are equivalent to:

$$\widetilde{V}(\pi, y) = \inf_\tau E_\pi [\tau + g_{a,b}(\pi_\tau)] \tag{2.34}$$

where the infimum is taken over all stopping times τ of the process $(\pi_t, Y_t)_{t \geq 0}$ such that $E_\pi[\tau] < \infty$ for all $(\pi, y) \in [0, 1] \times \mathbb{R}$ and $y = \log[\pi/(1 - \pi)]$ for each $\pi \in \langle 0, 1 \rangle$ and $x \in E = \langle -\zeta, \infty \rangle$ fixed. It also follows that there exists a couple of functions $(h_0(y), h_1(y))$, $y \in \mathbb{R}$, such that the continuation region C from (2.22) is equivalent to:

$$\widetilde{C} = \{ (\pi, y) \in [0, 1] \times \mathbb{R} \mid \pi \in \langle h_0(y), h_1(y) \rangle \} \tag{2.35}$$

and the set D from (2.23) is equivalent to:

$$\widetilde{D} = \{ (\pi, y) \in [0, 1] \times \mathbb{R} \mid \pi \in [0, h_0(y) \rangle \cup \langle h_1(y), 1] \} \tag{2.36}$$

for each $y \in \mathbb{R}$ and $x \in E$ fixed.

2.9. If the assumption (2.30) with $\eta_0 \neq \eta_1$ and $\eta_0 + \eta_1 = 1$ holds, then by means of standard arguments it is shown that the infinitesimal operator $\widetilde{\mathbb{L}}$ of the process $(\pi_t, Y_t)_{t \geq 0}$ from (2.32)-(2.33) acts on a function $F \in C^{2,0}(\langle 0, 1 \rangle \times \mathbb{R})$ like:

$$(\widetilde{\mathbb{L}} F)(\pi, y) = \frac{r^2(x; \pi, y)}{2} \pi^2 (1 - \pi)^2 \frac{\partial^2 F}{\partial \pi^2}(\pi, y) \tag{2.37}$$

with

$$r(x; \pi, y) = \frac{\sigma \left((x + \zeta) e^{-\eta y} [\pi/(1 - \pi)]^\eta - \zeta \right)}{\eta (x + \zeta) e^{-\eta y} [\pi/(1 - \pi)]^\eta} \tag{2.38}$$

for all $(\pi, y) \in \langle 0,1\rangle \times \mathbb{R}$ and each $x \in E = \langle -\zeta, \infty\rangle$ fixed.

Now let us use the results of general theory of optimal stopping problems for continuous time Markov processes (see, e.g., [6], [19; Chapter III, Section 8] and [15]) to formulate the corresponding *free-boundary problem* for the unknown value function $(\pi, y) \mapsto \widetilde{V}(\pi, y)$ from (2.16) (with $g_{a,b}(\pi) = a\pi \wedge b(1-\pi)$) and the couple of boundaries $(h_0(y), h_1(y))$, $y \in \mathbb{R}$, from (2.35)-(2.36):

$$(\widetilde{\mathbb{L}}\widetilde{V})(\pi, y) = -1 \quad \text{for} \quad (\pi, y) \in \widetilde{C} \tag{2.39}$$

$$\widetilde{V}(\pi, y)\big|_{\pi=h_0(y)+} = ah_0(y), \quad \widetilde{V}(\pi, y)\big|_{\pi=h_1(y)-} = b(1-h_1(y)) \tag{2.40}$$

$$\frac{\partial \widetilde{V}}{\partial \pi}(\pi, y)\big|_{\pi=h_0(y)+} = a, \quad \frac{\partial \widetilde{V}}{\partial \pi}(\pi, y)\big|_{\pi=h_1(y)-} = -b \tag{2.41}$$

$$\widetilde{V}(\pi, y) = g_{a,b}(\pi) \quad \text{for} \quad (\pi, y) \in \widetilde{D} \tag{2.42}$$

$$\widetilde{V}(\pi, y) < g_{a,b}(\pi) \quad \text{for} \quad (\pi, y) \in \widetilde{C} \tag{2.43}$$

where $\widetilde{C}$ and $\widetilde{D}$ are given by (2.35) and (2.36), and the *instantaneous-stopping* conditions (2.40) and the *smooth-fit* conditions (2.41) are assumed to be satisfied for all $y \in \mathbb{R}$ and each $x \in E$ fixed.

Note that by Dynkin's superharmonic characterization of the value function (see [3] and [19]) it follows that $\widetilde{V}(\pi, y)$ from (2.34) is the largest function satisfying (2.39)-(2.40) and (2.42)-(2.43) for each $y \in \mathbb{R}$ and $x \in E$ fixed.

2.10. Integrating the equation (2.39) with some $h_1(y) \in \langle c, 1\rangle$ fixed for any given $y \in \mathbb{R}$ and using the boundary conditions (2.40)-(2.41), we obtain:

$$\widetilde{V}(\pi, y; h_1(y)) = b(1-h_1(y)) - \int_\pi^{h_1(y)} \int_u^{h_1(y)} \frac{2}{r^2(x; v, y) v^2 (1-v)^2}\, dv du \tag{2.44}$$

with $r(x; \pi, y)$ given by (2.38) for all $\pi \in \langle 0, h_1(y)]$ and each $x \in E = \langle -\zeta, \infty\rangle$ fixed.

From (2.44) it is easily seen that for any $y \in \mathbb{R}$ given and fixed the function $\pi \mapsto \widetilde{V}(\pi, y; h_1(y))$ is concave on $\langle 0,1\rangle$, and hence $\widetilde{V}(h_1'(y), y; h_1''(y)) < \widetilde{V}(h_1'(y), y; h_1'(y))$ for $0 < h_1'(y) < h_1''(y) < 1$. This means that for different $h_1'(y)$ and $h_1''(y)$ the curves $\pi \mapsto \widetilde{V}(\pi, y; h_1'(y))$ and $\pi \mapsto \widetilde{V}(\pi, y; h_1''(y))$ have no points of intersection on the whole interval $\pi \in \langle 0, h_1'(y)]$. From (2.44) it also follows that $\widetilde{V}(\pi, y; h_1(y)) \to -\infty$ as $\pi \downarrow 0$ for all $h_1(y) \in [c, 1\rangle$ and $\widetilde{V}(\pi, y; 1-) < 0$ for all $\pi \in \langle 0,1\rangle$ and $\widetilde{V}(1-, y; 1-) = 0$. In this case, for some $\widetilde{h}_1(y) \in \langle c, 1\rangle$ the curve $\pi \mapsto \widetilde{V}(\pi, y; \widetilde{h}_1(y))$ intersects the line $\pi \mapsto a\pi$ at some point $h_0(y) \in \langle 0, c\rangle$. Since for different $h_1'(y) \in \langle c, 1\rangle$ the curves $\pi \mapsto \widetilde{V}(\pi, y; h_1'(y))$ do not intersect each other on the intervals $\langle 0, h_1'(y)\rangle$, we may conclude that there exists a unique point $h_1(y)$ obtained by moving the point $h_1'(y)$ from $\widetilde{h}_1(y)$ and such that in some point $h_0(y) \in \langle 0, c\rangle$ the boundary conditions (2.40)-(2.41) hold. It thus follows that the boundaries $(h_0(y), h_1(y))$ are uniquely determined from the system:

$$b + a = \int_{h_0(y)}^{h_1(y)} \frac{2}{r^2(x; u, y) u^2 (1-u)^2}\, du \tag{2.45}$$

$$b(1-h_1(y)) = ah_0(y) - \int_{h_0(y)}^{h_1(y)} \int_u^{h_1(y)} \frac{2}{r^2(x; v, y) v^2 (1-v)^2}\, dv du \tag{2.46}$$

for each $y \in \mathbb{R}$ and $x \in E = \langle -\zeta, \infty \rangle$ fixed.

2.11. Making use of the facts proved above we are now ready to formulate the main result of the paper.

Theorem 2.1. *Suppose that conditions (2.3) and (2.14)-(2.15) hold for all $x \in E = \langle -\zeta, \infty \rangle$ and some $\zeta \in \mathbb{R}$ fixed, and assumption (2.30) is satisfied with $\eta_0 \neq \eta_1$ and $\eta_0 + \eta_1 = 1$. Then in the Bayesian problem (2.6)+(2.16)+(2.34) of testing two simple hypotheses (2.4) for the process (2.2) the value function has the expression:*

$$V(\pi) = V(\pi, x) = \widetilde{V}(\pi, y) = \begin{cases} \widetilde{V}(\pi, y; h_1(y)), & \text{if } \pi \in \langle h_0(y), h_1(y) \rangle \\ g_{a,b}(\pi), & \text{if } \pi \in [0, h_0(y)] \cup [h_1(y), 1] \end{cases} \tag{2.47}$$

and the optimal π-Bayes decision rule is explicitly given by:

$$\tau_* = \inf\{t \geq 0 \mid \pi_t \notin \langle h_0(y), h_1(y) \rangle\} \tag{2.48}$$

$$d_* = \begin{cases} 1, & \text{if } \pi_{\tau_*} = h_1(y) \\ 0, & \text{if } \pi_{\tau_*} = h_0(y) \end{cases} \tag{2.49}$$

where the two boundaries $(h_0(y), h_1(y))$ are characterized as a unique solution of the coupled system of equations (2.45)-(2.46) for $y = \log[\pi/(1-\pi)]$ and each $\pi \in \langle 0, 1 \rangle$ and $x \in E$ fixed.

Proof. It remains to show that the function (2.47) coincides with the value function (2.34) and that the stopping time τ_* from (2.48) with the boundaries $(h_0(y), h_1(y))$, $y \in \mathbb{R}$, specified above is optimal. Let us denote by $\widetilde{V}(\pi, y)$ the right-hand side of the expression (**??**). It follows by construction from the previous section that the function $\widetilde{V}(\pi, y)$ solves the system (2.39)-(2.42). Thus, applying Itô's formula to $\widetilde{V}(\pi_t, y)$, we obtain:

$$\widetilde{V}(\pi_t, y) = \widetilde{V}(\pi, y) + \int_0^t (\widetilde{\mathbb{L}}\widetilde{V})(\pi_s, y) I(\pi_s \neq h_0(y), \pi_s \neq h_1(y))\, ds + \widetilde{M}_t \tag{2.50}$$

where the process $(\widetilde{M}_t)_{t \geq 0}$ defined by:

$$\widetilde{M}_t = \int_0^t \frac{\partial \widetilde{V}}{\partial \pi}(\pi_s, y) I(\pi_s \neq h_0(y), \pi_s \neq h_1(y)) \, \frac{\mu_1(X_t) - \mu_0(X_t)}{\sigma(X_t)} \pi_s(1-\pi_s)\, d\overline{W}_s \tag{2.51}$$

is a continuous local martingale under P_π with respect to $(\mathcal{F}_t^X)_{t \geq 0}$.

By using the arguments above it can be verified that $(\widetilde{\mathbb{L}}\widetilde{V})(\pi, y) \geq -1$ for all $(\pi, y) \in \langle 0, 1 \rangle \times \mathbb{R}$ such that $\pi \neq h_0(y)$ and $\pi \neq h_1(y)$. Moreover, by means of standard arguments and using the construction of $\widetilde{V}(\pi, y)$ above it can be checked that the property (2.43) also holds that together with (2.39)-(2.40)+(2.42) yields $\widetilde{V}(\pi, y) \leq g_{a,b}(\pi)$ for all $(\pi, y) \in [0, 1] \times \mathbb{R}$. Observe that the time spent by the process π at the boundaries $(h_0(y), h_1(y))$, $y \in \mathbb{R}$, is of Lebesgue measure zero, that allows to extend $(\widetilde{\mathbb{L}}\widetilde{V})(\pi, y)$ arbitrarily to $\pi = h_0(y)$ and to $\pi = h_1(y)$ and thus to ignore the indicators in (2.50)-(2.51). Hence, from the expressions (2.50) and the structure of the stopping time in (2.48) it follows that the inequalities:

$$\tau + g_{a,b}(\pi_\tau) \geq \tau + \widetilde{V}(\pi_\tau, y) \geq \widetilde{V}(\pi, y) + \widetilde{M}_\tau \tag{2.52}$$

hold for any stopping times τ of the process $(\pi_t)_{t \ge 0}$ started at $\pi \in [0,1]$ and for each $y \in \mathbb{R}$.

Let $(\tau_n)_{n \in \mathbb{N}}$ be an arbitrary localizing sequences of stopping times for the processes $(\widetilde{M}_t)_{t \ge 0}$. Taking in (2.52) the expectation with respect to the measure P_π, by means of the optional sampling theorem (see, e.g., [9; Chapter I, Theorem 1.39] or [17; Chapter II, Theorem 3.1]), we get:

$$E_\pi \left[\tau \wedge \tau_n + g_{a,b}(\pi_{\tau \wedge \tau_n}) \right] \ge E_\pi \left[\tau \wedge \tau_n + \widetilde{V}(\pi_{\tau \wedge \tau_n}, y) \right] \ge \widetilde{V}(\pi, y) + E_\pi \left[\widetilde{M}_{\tau \wedge \tau_n} \right] = \widetilde{V}(\pi, y) \quad (2.53)$$

for all $(\pi, y) \in [0,1] \times \mathbb{R}$. Hence, letting n go to infinity and using Fatou's lemma, for any stopping times τ such that $E_\pi[\tau] < \infty$ we obtain that the inequalities:

$$E_\pi \left[\tau + g_{a,b}(\pi_\tau) \right] \ge E_\pi \left[\tau + \widetilde{V}(\pi_\tau, y) \right] \ge \widetilde{V}(\pi, y) \quad (2.54)$$

are satisfied for all $(\pi, y) \in [0,1] \times \mathbb{R}$.

By virtue of the fact that the function $\widetilde{V}(\pi, y)$ together with the boundaries $h_0(y)$ and $h_1(y)$ satisfy the system (2.39)-(2.43), by the structure of the stopping time τ_* in (2.48) and the expressions (2.50) it follows that the equalities:

$$\tau_* \wedge \tau_n + g_{a,b}(\pi_{\tau_* \wedge \tau_n}) = \tau_* \wedge \tau_n + \widetilde{V}(\pi_{\tau_* \wedge \tau_n}, y) = \widetilde{V}(\pi, y) + \widetilde{M}_{\tau_* \wedge \tau_n} \quad (2.55)$$

hold for all $(\pi, y) \in [0,1] \times \mathbb{R}$ and any localizing sequence $(\tau_n)_{n \in \mathbb{N}}$ of $(\widetilde{M}_t)_{t \ge 0}$. Note that, by means of standard arguments and by using the structure of the process (2.32) and of the stopping time (2.48), we have $E_\pi[\tau_*] < \infty$ for all $\pi \in [0,1]$. Hence, letting n go to infinity and using conditions (2.39)-(2.40), we can apply the Lebesgue bounded convergence theorem for (2.55) to obtain the equality:

$$E_\pi \left[\tau_* \wedge \tau_n + g_{a,b}(\pi_{\tau_* \wedge \tau_n}) \right] = \widetilde{V}(\pi, y) \quad (2.56)$$

for all $(\pi, y) \in [0,1] \times \mathbb{R}$, which together with (2.54) directly imply the desired assertion. □

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Pavel V. Gapeev
Weierstraß Institute
for Applied Analysis and Stochastics (WIAS)
Mohrenstr. 39, D-10117 Berlin, Germany
e-mail: gapeev@wias-berlin.de

(Russian Academy of Sciences
Institute of Control Sciences
Profsoyuznaya Str. 65
117997 Moscow, Russia)