WIAS Preprint No. 1181 (2006) (14 pp)

Nonadditive disorder problems for some diffusion processes*

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We study nonadditive Bayesian problems of detecting a change in drift of an observed diffusion process where the cost function of the detection delay has the same structure as in [27] and construct a finite-dimensional Markovian sufficient statistic for that case. We show that when the cost function is linear the optimal stopping time is found as the first time when the a posteriori probability process hits a stochastic boundary depending on the observation process. It is shown that under some nontrivial relationships on the coefficients of the observed diffusion the problem admits a closed form solution. The method of proof is based on embedding the initial problem into a two-dimensional optimal stopping problem and solving the equivalent free-boundary problem by means of the smooth-fit conditions.

1. Introduction

The problem of quickest disorder detection for an observed diffusion process seeks to determine a stopping time which is as close as possible to the (unknown) time of 'disorder' (change-point) when the drift coefficient $\mu(x)$ of the observed process changes from $\mu_0(x)$ to $\mu_1(x)$. This problem admits at least two different Bayesian formulations. In the 'free' formulation (below referred to as the 'Bayesian problem') one looks for a time of 'alarm' minimizing a linear combination of the probability of a 'false alarm' and the expectation of a 'delay' in detecting the time of disorder correctly with no constraint on the former. In the 'fixed false-alarm' formulation (sometimes referred to as the 'variational problem') one looks for a time of 'alarm' minimizing the same linear combination under the constraint that the probability of a 'false alarm' cannot exceed a given value. In this paper we only study the 'Bayesian problem' where we also use the customary assumption that the time of disorder is exponentially distributed.

Shiryaev [25]-[26] and [28]-[29] (see also [30; Chapter IV]) derived an explicit solution of the Bayesian and variational problem of detecting a change in drift of an observed Wiener process by reducing the initial optimal stopping problem to a free-boundary problem for an ordinary second order operator. Some particular cases of the Bayesian problem of detecting a change

* This research was supported by Deutsche Forschungsgemeinschaft through the SFB 649 Economic Risk.

Mathematics Subject Classification 2000: Primary 60G40, 62M20, 34K10. Secondary 62C10, 62L15, 60J60.

Key words and phrases: Quickest 'disorder' (change-point) detection, diffusion process, optimal stopping, sufficient statistic, free-boundary problem, smooth-fit and normal-entrance conditions, Itô's formula.

in the intensity of an observed Poisson process were considered by Gal'chuk and Rozovskii [10] and Davis [6]. Peskir and Shiryaev [20] obtained a complete solution of the latter problem by reducing the initial optimal stopping problem for a differential-difference operator. The Bayesian and variational disorder problems for a compound Poisson process with exponentially distributed jumps were explicitly solved in [11]. Recently, Dayanik and Sezer [7] obtained a solution of the Bayesian disorder problem for a general compound Poisson process. In all these problems the optimal stopping time was the first time when the a posteriori probability process hits a constant boundary. A finite horizon version of the Bayesian and variational Wiener disorder problem was solved in [12] by reducing a parabolic free-boundary problem to an equivalent a nonlinear integral equation for the curved optimal stopping boundary depending on time. The main purpose of this paper is to study the Bayesian problem of detecting a change in local drift of an observed diffusion process and present a closed form solution of the problem under some nontrivial relationships on coefficients of the observed diffusion. In this case the optimal stopping time is the first time when the a posteriori probability process hits a stochastic boundary depending on the observation process.

Shiryaev [27] studied the problem of finding finite-dimensional Markovian sufficient statistics in the disorder problem with nonadditive minimizing functionals. More recently, it was shown by Poor [22] for the case of observed sequences of random variables, by Beibel [4] for the case of observed Wiener process, and then by Bayraktar and Dayanik [1] for the case of observed Poisson process, that when the cost function of the detection delay has an exponential form the Markovian sufficient statistic turns out to be one-dimensional that essentially simplifies the solution of the related optimal stopping problem. Some other formulations of the Poisson disorder problem were considered by Bayraktar, Dayanik and Karatzas [2]. Another 'adaptive' formulation of the Poisson disorder problem where the arrival rate of the observed process changes to an unobservable value and the related problem of finding finite-dimensional Markovian sufficient statistics were studied by Bayraktar, Dayanik and Karatzas [3]. It was shown that when the new arrival rate has Bernoulli distribution the Markovian sufficient statistic turns out to be two-dimensional that makes possible to observe interesting analytic properties of the solution of the corresponding optimal stopping problem.

The paper is organized as follows. In Section 2, after formulating the Bayesian disorder problem for an observed diffusion processes we study the problem of finding Markovian sufficient statistics for the case where the minimizing nonadditive functional has the same form as in [27]. In Section 3, we consider the case where the cost function of the detection delay is linear and make an embedding of the initial Bayesian problem into an extended optimal stopping problem for a two-dimensional (time-homogeneous strong) Markov process (consisting of the a posteriori probability process and the observation process). We show that the continuation region (for the a posteriori probability process) is determined by a stochastic boundary depending on the observation process where the behavior of the boundary is characterized by the signal/noise ratio. In order to find analytic expressions for the value function and the stopping boundary under some special nontrivial relationships on coefficients of the observed diffusion, we formulate an equivalent free-boundary problem. By applying smooth-fit condition we show that the free-boundary problem admits an explicit solution and the boundary is uniquely determined from a transcendental equation. Then we verify that the solution of the free-boundary problem turns out to be a solution of the initial extended optimal stopping problem. The main results of the paper are stated in Theorems 2.1 and 3.1.

2. Formulation of the problem

In the Bayesian formulation of the problem (see [30; Chapter IV, Section 4] for the case of Wiener process) it is assumed that we observe a trajectory of the diffusion process $X = (X_t)_{t \ge 0}$ with drift coefficient changing from $\mu_0(x)$ to $\mu_1(x)$ at some random time θ taking the value 0 with probability π and being exponentially distributed with parameter $\lambda > 0$ under $\theta > 0$.

2.1. For a precise probabilistic formulation of the Bayesian problem it is convenient to assume that all our considerations take place on a probability space $(\Omega, \mathcal{F}, P_\pi)$ where the probability measure P_π has the following structure:

$$P_\pi = \pi P^0 + (1-\pi) \int_0^\infty \lambda e^{-\lambda s} P^s \, ds \tag{2.1}$$

for any $\pi \in [0,1]$ and the measures P^s for $s \in [0, \infty]$ are specified below. Let θ be a nonnegative random variable satisfying $P_\pi[\theta = 0] = \pi$ and $P_\pi[\theta > t \,|\, \theta > 0] = e^{-\lambda t}$ for all $t \ge 0$ and some $\lambda > 0$, and let $W = (W_t)_{t \ge 0}$ be a standard Wiener process started at zero under P_π. It is assumed that θ and W are independent.

It is further assumed that we observe a continuous process $X = (X_t)_{t \ge 0}$ with the (open) state space $E \subseteq \mathbb{R}$ and solving the stochastic differential equation:

$$dX_t = [\mu_0(X_t) + I(t \ge \theta)(\mu_1(X_t) - \mu_0(X_t))] \, dt + \sigma(X_t) \, dW_t \quad (X_0 = x) \tag{2.2}$$

where the functions $\mu_i(x)$ and $\sigma(x)$ are Lipschitz continuous on E, that is, there exists a constant $C > 0$ such that:

$$[\mu_i(x) - \mu_i(x')]^2 + [\sigma(x) - \sigma(x')]^2 \le C[x - x']^2 \tag{2.3}$$

for all $x, x' \in E$ and $i = 0, 1$. Thus, from [17; Chapter IV, Theorem 4.6] it follows that under fixed $\theta = s$ equation (2.2) has a unique strong solution, and hence, $P_\pi[X \in \cdot \,|\, \theta = s] = P^s[X \in \cdot\,]$ is the distribution law of a homogeneous diffusion process (starting at some fixed point $x \in E$) with diffusion coefficient $\sigma^2(x)$ and local drift changing from $\mu_0(x)$ to $\mu_1(x)$ at time $s \in [0, \infty]$. We will also assume that either $\mu_0(x) < \mu_1(x)$ or $\mu_0(x) > \mu_1(x)$ holds and $\sigma^2(x) > 0$ for all $x \in E$. It is assumed that the time of *'disorder'* is unknown (i.e., it cannot be observed directly).

Being based upon the continuous observation of X our task is to find among the stopping times τ of X (i.e., stopping times with respect to the natural filtration $\mathcal{F}_t^X = \sigma\{X_s \,|\, 0 \le s \le t\}$ generated by the process X for $t \ge 0$) an optimal stopping time (a time of 'alarm') being *'as close as possible'* to the unknown time θ. More precisely, the problem consists of computing the risk function:

$$V(\pi) = \inf_\tau \big(P_\pi[\tau < \theta] + E_\pi[f(\tau - \theta) I(\tau \ge \theta)] \big), \tag{2.4}$$

and finding the optimal stopping time τ_*, called the π-Bayes time, at which the infimum in (2.4) is attained. Here $P_\pi[\tau < \theta]$ is the probability of a *'false alarm'*, $E_\pi[f(\tau - \theta) I(\tau \ge \theta)]$ is the *'average cost of delay'* in detecting disorder correctly (i.e., when $\tau \ge \theta$), so that the cost function $f(t)$ of the detection delay satisfies $f(t) \ge 0$ for $t \ge 0$ and $f(t) = 0$ for $t \le 0$.

2.2. Following the schema of arguments in [27], one can easily show that the Bayesian problem (2.4) is reduced to the optimal stopping problem:

$$V(\pi) = \inf_\tau E_\pi \left[1 - \pi_{\tau,\tau} + f(\tau) \pi_{0,\tau} + \int_0^\tau f(\tau - u) \, d_u \pi_{u,\tau} \right] \tag{2.5}$$

for the conditional probability $\pi_{u,t} = P_\pi[\theta \le u \,|\, \mathcal{F}_t^X]$ for each $0 \le u \le t$ and all $t \ge 0$ with $\pi_{0,0} = \pi$ (P_π-a.s.).

It follows by definition of the measures P^s above that:

$$\frac{d(P^s|\mathcal{F}_t^X)}{d(P^\infty|\mathcal{F}_t^X)} = I(s \le t) + \frac{L_t}{L_s}\, I(s > t) \tag{2.6}$$

for each $s \in [0, \infty]$, where by applying Girsanov's theorem for diffusion-type processes [17; Chapter VII, Theorem 7.19], we have:

$$L_t = \exp\left(\int_0^t \frac{\mu_1(X_u) - \mu_0(X_u)}{\sigma^2(X_u)}\, dX_u - \frac{1}{2} \int_0^t \frac{\mu_1^2(X_u) - \mu_0^2(X_u)}{\sigma^2(X_u)}\, du \right) \tag{2.7}$$

for all $t \ge 0$. Then, by means of generalized Bayes' formula [17; Chapter VII, Lemma 7.4], for the conditional probability $\pi_{u,t} = P_\pi[\theta \le u \,|\, \mathcal{F}_t^X]$ we get:

$$\pi_{u,t} = \frac{\pi L_t + (1-\pi) \int_0^u (L_t/L_s) \lambda e^{-\lambda s}\, ds}{\pi L_t + (1-\pi) \int_0^t (L_t/L_s) \lambda e^{-\lambda s}\, ds + (1-\pi) e^{-\lambda t}} \tag{2.8}$$

for each $0 \le u \le t$ and all $t \ge 0$. Hence, after setting $\pi_t = \pi_{t,t}$, we obtain:

$$\frac{\pi_{u,t}}{1-\pi_t} = \frac{L_t}{e^{-\lambda t}} \left(\frac{\pi}{1-\pi} + \int_0^u \frac{\lambda e^{-\lambda s}}{L_s}\, ds \right) \tag{2.9}$$

for each $0 \le u \le t$ and all $t \ge 0$. Thus, by means of standard arguments it follows that the value function (2.5) takes the form:

$$V(\pi) = \inf_\tau E_\pi \left[(1-\pi_\tau) \left(1 + \frac{L_\tau}{e^{-\lambda \tau}} \left(f(\tau) \frac{\pi}{1-\pi} + \int_0^\tau f(\tau - u) \frac{\lambda e^{-\lambda u}}{L_u}\, du \right) \right) \right]. \tag{2.10}$$

2.3. Following the schema of arguments from [27], from now on we assume that the Laplace transform $\widehat{f}(z)$ of the function $f(t)$ is a rational function of the form:

$$\widehat{f}(z) = \int_0^\infty f(u) e^{-zu}\, du = \sum_{k=1}^n \sum_{l=1}^{m(k)} \frac{c_{kl}}{(z - \lambda_k)^l} \tag{2.11}$$

for all $z > 0$, $z \ne \lambda_1, \ldots, z \ne \lambda_n$, where c_{kl}, $l = 1, \ldots, m(k)$, $k = 1, \ldots, n$, are some real constants. Let us denote by e^{-Su} the shift operator acting on an arbitrary function $a(t)$ like $e^{-Su}\,[a(t)] = a(t-u)$ for all $t \ge 0$ and some $u \ge 0$ fixed. In this case, by virtue of the assumption (2.11) we may set:

$$\widehat{f}(S)\,[\,\cdot\,] = \int_0^\infty f(u) e^{-Su}\,[\,\cdot\,]\, du = \sum_{k=1}^n \sum_{l=1}^{m(k)} \frac{c_{kl}}{(S - \lambda_k)^l}\,[\,\cdot\,] \tag{2.12}$$

and thus, we have:

$$\widehat{f}(S) \left[\frac{\lambda e^{-\lambda t}}{L_t} \right] = \int_0^\infty f(u) \frac{\lambda e^{-\lambda(t-u)}}{L_{t-u}}\, du = \sum_{k=1}^n \sum_{l=1}^{m(k)} \frac{c_{kl}}{(S - \lambda_k)^l} \left[\frac{\lambda e^{-\lambda t}}{L_t} \right] \tag{2.13}$$

for all $t \geq 0$. By means of standard arguments from [8], we get:

$$\frac{1}{(S-\lambda_k)^l}\left[\frac{\lambda e^{-\lambda t}}{L_t}\right] = \psi_t^{kl}\frac{e^{-\lambda t}}{L_t} \equiv \int_0^\infty e^{\lambda_k u}\frac{u^{l-1}}{(l-1)!}\frac{\lambda e^{-\lambda(t-u)}}{L_{t-u}}\,du \tag{2.14}$$

for each $0 \leq u \leq t$ and all $t \geq 0$, where the process $(\psi_t^{kl})_{t\geq 0}$ is defined by:

$$\psi_t^{kl} = \frac{L_t}{e^{-\lambda t}}\int_0^t e^{\lambda_k(t-u)}\frac{(t-u)^{l-1}}{(l-1)!}\frac{\lambda e^{-\lambda u}}{L_u}\,du \tag{2.15}$$

for every $l = 1, \ldots, m(k)$ and $k = 1, \ldots, n$. By using the fact that:

$$\int_0^t f(t-u)\,\frac{\lambda e^{-\lambda u}}{L_u}\,du = \int_0^\infty f(u)\,\frac{\lambda e^{-\lambda(t-u)}}{L_{t-u}}\,du \tag{2.16}$$

for each $0 \leq u \leq t$ and all $t \geq 0$, we therefore conclude that the value function (2.10) admits the representation:

$$V(\pi) = \inf_\tau E_\pi\left[(1-\pi_\tau)\left(1 + \frac{L_\tau f(\tau)}{e^{-\lambda\tau}}\frac{\pi}{1-\pi} + \sum_{k=1}^{n}\sum_{l=1}^{m(k)} c_{kl}\psi_\tau^{kl}\right)\right]. \tag{2.17}$$

2.4. Let us now introduce the likelihood ratio process $(\varphi_t)_{t\geq 0}$ defined by $\varphi_t = \pi_t/(1-\pi_t)$, and thus, by virtue of (2.9), taking the expression:

$$\varphi_t = \frac{L_t}{e^{-\lambda t}}\left(\frac{\pi}{1-\pi} + \int_0^t \frac{\lambda e^{-\lambda s}}{L_s}\,ds\right) \tag{2.18}$$

for all $t \geq 0$. Then, by applying Itô's formula [17; Chapter IV, Theorem 4.4] to the expressions (2.18), (2.7) and (2.15), and by using the fact that:

$$\pi_t = \frac{\varphi_t}{1+\varphi_t} \tag{2.19}$$

we obtain the following representations:

$$d\pi_t = \lambda(1-\pi_t)\,dt + \frac{\mu_1(X_t)-\mu_0(X_t)}{\sigma(X_t)}\,\pi_t(1-\pi_t)\,d\overline{W}_t \quad (\pi_0 = \pi) \tag{2.20}$$

$$dL_t = \frac{\mu_1(X_t)-\mu_0(X_t)}{\sigma(X_t)}\,L_t\,d\overline{W}_t \quad (L_0 = 1) \tag{2.21}$$

$$d\psi_t^{k1} = \left[\lambda + (\lambda+\lambda_k)\,\psi_t^{k1}\right]\,dt + \frac{\mu_1(X_t)-\mu_0(X_t)}{\sigma(X_t)}\,\psi_t^{k1}\,d\overline{W}_t \quad (\psi_0^{k1} = 0) \tag{2.22}$$

$$d\psi_t^{kl} = \left[\psi_t^{k(l-1)} + (\lambda+\lambda_k)\,\psi_t^{kl}\right]\,dt + \frac{\mu_1(X_t)-\mu_0(X_t)}{\sigma(X_t)}\,\psi_t^{kl}\,d\overline{W}_t \quad (\psi_0^{kl} = 0) \tag{2.23}$$

for every $l = 2, \ldots, m(k)$ and $k = 1, \ldots, n$, where, by means of P. Lévy's theorem [24; Chapter IV, Theorem 3.6], the innovation process $\overline{W} = (\overline{W}_t)_{t\geq 0}$ defined by:

$$\overline{W}_t = \int_0^t \frac{dX_s}{\sigma(X_s)} - \int_0^t \left(\frac{\mu_0(X_s)}{\sigma(X_s)} + \pi_s\frac{\mu_1(X_s)-\mu_0(X_s)}{\sigma(X_s)}\right)ds \tag{2.24}$$

is a standard Wiener process under the measure P_π with respect to the filtration $(\mathcal{F}_t^X)_{t\ge 0}$. Therefore, from (2.24) it follows that the process $X=(X_t)_{t\ge 0}$ admits the representation:

$$dX_t = [\mu_0(X_t) + \pi_t(\mu_1(X_t) - \mu_0(X_t))]\, dt + \sigma(X_t)\, d\overline{W}_t \quad (X_0 = x). \tag{2.25}$$

Let us suppose that the *signal/noise ratio* function defined by:

$$r(x) = \frac{\mu_1(x) - \mu_0(x)}{\sigma(x)} \tag{2.26}$$

is also Lipschitz continuous, that is, there exists a constant $C' > 0$ such that condition:

$$[r(x) - r(x')]^2 \le C'[x - x']^2 \tag{2.27}$$

holds for all $x, x' \in E$, and there are constants r_* and r^* such that the inequalities:

$$0 < r_* \le r(x) \le r^* < \infty \tag{2.28}$$

are satisfied for all $x \in E$. Hence, by means of Remark to [17; Chapter IV, Theorem 4.6] (see also [18; Chapter V, Theorem 5.2.1]) we conclude that the process $(\pi_t, L_t, \psi_t, X_t)_{t\ge 0}$ where we denote $\psi_t = (\psi_t^{kl}, l = 1, \ldots, m(k), k = 1, \ldots, n)$ turns out to be a unique strong solution of the multi-dimensional stochastic differential equation (2.20)-(2.23)+(2.25), and thus, by virtue of [18; Chapter VII, Theorem 7.2.4], it is a (time-homogeneous strong) Markov process with respect to its natural filtration, which obviously coincides with $(\mathcal{F}_t^X)_{t\ge 0}$. Therefore, the infimum in (3.2) is taken over all stopping times of $(\pi_t, L_t, \psi_t, X_t)_{t\ge 0}$ playing the role of a Markovian sufficient statistic in the problem.

2.5. Summarizing the facts proved above we now formulate the following assertion.

Theorem 2.1. *Suppose that condition (2.3) holds and the Laplace transform $\widehat{f}(z)$ of the cost of delay $f(t)$ takes the form (2.11). Then in the Bayesian problem (2.4)-(2.5) the value function admits the representation (2.17), where the processes $(\pi_t)_{t\ge 0}$, $(L_t)_{t\ge 0}$ and $(\psi_t^{kl})_{t\ge 0}$, $l = 1, \ldots, m(k)$, $k = 1, \ldots, n$, defined in (2.19), (2.7) and (2.15) solve the stochastic differential equations (2.20)-(2.23). Moreover, if the conditions (2.27)-(2.28) are satisfied, then $(\pi_t, L_t, \psi_t, X_t)_{t\ge 0}$ turns out to be a Markovian sufficient statistic in the problem (2.4)-(2.5).*

Remark 2.2. The Laplace transform $\widehat{f}(z)$ has the form (2.11) if, for example, the function $f(t)$ satisfies a homogeneous ordinary differential equation of the order $q = \sum_{k=1}^n m(k)$: $f^{(q)}(t) + a_1 f^{(q-1)}(t) + \ldots + a_{q-1} f'(t) + a_q f(t) = 0$ with the initial conditions $f(0) = c_0, f'(0) = c_1, \ldots, f^{(q-1)}(0) = c_{q-1}$, where $a_1, \ldots, a_q$ and $c_0, c_1, \ldots, c_{q-1}$ are some real constants (see, e.g., [8]).

Example 2.3. Assume that $f(t) = ct$ for all $t \ge 0$ and some $c > 0$ given and fixed. Then it is easily shown (see, e.g., [30; Chapter IV]) that the value function (2.4) takes the form (3.1)-(3.2) below, so that $(\pi_t)_{t\ge 0}$ turns out to be a one-dimensional sufficient statistic.

Example 2.4. Assume that $f(t) = ct^\alpha$ for all $t \ge 0$ and some $\alpha > 0$ and $c > 0$ given and fixed. In this case, the Laplace transform takes the form $\widehat{f}(z) = c\Gamma(\alpha+1)/z^{\alpha+1}$ for $z > 0$, where Γ is the Euler Gamma function. If $\alpha \in \mathbb{N}$ then $\widehat{f}(z)$ is of the type (2.11), from where, by means of the arguments above we obtain that the value function (2.10) admits the representation:

$$L_\pi(\tau) = E_\pi \left[(1-\pi_\tau) \left(1 + \frac{L_\tau c \tau^\alpha}{e^{-\lambda\tau}} \frac{\pi}{1-\pi} + c\Gamma(\alpha+1) \sum_{l=1}^{\alpha+1} \psi_\tau^l \right) \right]. \tag{2.29}$$

Otherwise, if $\alpha > 0$ but $\alpha \notin \mathbb{N}$ then the function $\widehat{f}(z)$ cannot be expressed in the form (2.11) with finite number of summands.

3. The case of linear cost of delay

In this section we study the case considered in Example 2.3 above where the cost of delay is a linear function.

3.1. Assume that in (2.4) we have $f(t) = ct$ for all $t \geq 0$ and some $c > 0$ fixed. In this case the value function admits the representation:

$$V(\pi) = \inf_\tau \left(P_\pi[\tau < \theta] + c\, E_\pi[\tau - \theta]^+ \right) \tag{3.1}$$

where the infimum is taken over all stopping time of the process X. By means of standard arguments (see [30; pages 195-197]) one can easily show that the Bayesian problem (3.1) is reduced to the optimal stopping problem:

$$V(\pi) = \inf_\tau E_\pi \left[1 - \pi_\tau + c \int_0^\tau \pi_t \, dt \right] \tag{3.2}$$

for the a posteriori probability process $\pi_t = P_\pi[\theta \leq t \,|\, \mathcal{F}_t^X]$ for $t \geq 0$ with $\pi_0 = \pi$ (P_π-a.s.).

3.2. For the problem (3.2) let us consider the following extended optimal stopping problem for the Markov process $(\pi_t, X_t)_{t\geq 0}$:

$$V(\pi, x) = \inf_\tau E_{\pi,x} \left[1 - \pi_\tau + c \int_0^\tau \pi_t \, dt \right] \tag{3.3}$$

where $P_{\pi,x}$ is a measure of the diffusion process $(\pi_t, X_t)_{t\geq 0}$ starting at the point (π, x) and solving the (two-dimensional) equation (2.20)+(2.25), and the infimum in (3.3) is taken over all stopping times τ of the process $(\pi_t, X_t)_{t\geq 0}$ such that $E_{\pi,x}[\tau] < \infty$ for all $(\pi, x) \in [0,1] \times E$.

3.3. Let us now determine the structure of the optimal stopping time in the problem (3.3).

(i) First, by applying Itô's formula we get:

$$1 - \pi_t = 1 - \pi - \lambda \int_0^t (1 - \pi_s)\, ds + N_t \tag{3.4}$$

where for any $(\mathcal{F}_t^X)_{t\geq 0}$-stopping time τ satisfying $E_{\pi,x}[\tau] < \infty$ the process $(N_{\tau\wedge t})_{t\geq 0}$ defined by $N_{\tau\wedge t} = -\int_0^{\tau\wedge t} r(X_s)\pi_s(1-\pi_s) d\overline{W}_s$ is a continuous (uniformly integrable) martingale under

$P_{\pi,x}$. It follows from (3.4) using the optional sampling theorem (see, e.g., [24; Chapter II, Theorem 3.2]) that:

$$E_{\pi,x}\left[1-\pi_\sigma + c\int_0^\sigma \pi_u\,du\right] = 1-\pi + E_{\pi,x}\left[\int_0^\sigma (c\,\pi_t - \lambda(1-\pi_t))\,du\right] \tag{3.5}$$

for each $(\mathcal{F}_t^X)_{t\geq 0}$-stopping time σ. Choosing σ to be the exit time from a small ball, we see from (3.5) that it is never optimal to stop when $\pi_t < \lambda/(\lambda+c)$ for $t \geq 0$. In other words, this shows that all points (π,x) for $x\in E$ with $0\leq \pi < \lambda/(\lambda+c)$ belong to the continuation region:

$$C = \{(\pi,x)\in[0,1]\times E \mid V(\pi,x) < 1-\pi\}. \tag{3.6}$$

Since $\pi\mapsto V(\pi,x)$ with $x\in E$ given and fixed is concave on $[0,1]$ (this is easily deduced using the same arguments as in [30; pages 197-198]), it follows that there exists a function $g(x)$ satisfying $0<\lambda/(\lambda+c)\leq g(x)<1$ for all $t\geq 0$ such that the continuation region is an open set of the form:

$$C = \{(\pi,x)\in[0,1]\times E \mid \pi < g(x)\} \tag{3.7}$$

and the stopping region is the closure of the set:

$$D = \{(\pi,x)\in[0,1]\times E \mid \pi > g(x)\}. \tag{3.8}$$

(ii) Now for given $(\pi,x)\in C$ let us take $x'\in E$ such that $x'<x$ or $x>x'$. Then using the facts that $(\pi_t,X_t)_{t\geq 0}$ is a time-homogeneous Markov process and $\tau_*=\tau_*(\pi,x)$ does not depend on x', from (3.4) we obtain:

$$\begin{aligned} V(\pi,x') - (1-\pi) &\leq E_{\pi,x'}\left[\int_0^{\tau_*}(c\,\pi_t - \lambda(1-\pi_t))dt\right] \\ &\leq E_{\pi,x}\left[\int_0^{\tau_*}(c\,\pi_t - \lambda(1-\pi_t))dt\right] = V(\pi,x)-(1-\pi) \end{aligned} \tag{3.9}$$

and hence, by means of (3.6), we see that $(\pi,x)\in C$. Therefore, we may conclude that in (3.7)-(3.8) and the boundary $x\mapsto g(x)$ is decreasing (increasing) on E when the function $r(x)$ is increasing (decreasing), respectively.

(iii) Next, let us observe that the value function $V(\pi,x)$ from (3.3) and the boundary $g(x)$ from (3.7)-(3.8) also depend on $r(x)$ defined in (2.26) and denote them here by $V_*(\pi,x)$ and $V^*(\pi,x)$ as well as A_* and A^* when $r(x)=r_*$ and $r(x)=r^*$ for all $x\in E$, respectively. Using the fact that $x\mapsto V(\pi,x)$ is an increasing (decreasing) function when $r(x)$ is increasing (decreasing) on E, and $V(\pi,x)=1-\pi$ for all $g(x)\leq\pi\leq 1$, we conclude that $0<A_*\leq g(x)\leq A^*<1$ for all $x\in E$. Here we note that if $r_*=r^*$ then $A_*=g(x)=A^*$ for all $x\in E$, where $0<\lambda/(c+\lambda)<A_*<A^*<1$ are uniquely determined from the system (4.147) in [30; Chapter IV].

3.4. Summarizing the facts proved in Subsection 3.3 above we may conclude that the following optimal decision rule is optimal in the extended problem (3.3):

$$\tau_* = \inf\{t\geq 0 \mid \pi_t \geq g(X_t)\} \tag{3.10}$$

(whenever $E_{\pi,x}[\tau_*] < \infty$) where the boundary $g(x)$, $x \in E$, satisfies the following properties:

$$g(x) : E \to [0,1] \quad \text{is continuous and decreasing (increasing)} \tag{3.11}$$

$$A_* \le g(x) \le A^* \quad \text{for all} \quad x \in E \tag{3.12}$$

whenever $r(x)$ is an increasing (decreasing) function on E, respectively. Here A_* and A^* satisfying $0 < \lambda/(c+\lambda) < A_* < A^* < 1$ are the optimal stopping points for the corresponding infinite horizon problem with $r(x) = r_*$ and $r(x) = r^*$ for all $x \in E$, respectively, uniquely determined from the system of transcendental equations (4.147) in [30; Chapter IV].

3.5. Let us further assume that the state space of the process $X = (X_t)_{t \ge 0}$ under $\theta = s$ for all $s \in [0, \infty]$ is $E = \langle -\zeta, \infty \rangle$ for some $\zeta \in \mathbb{R}$ fixed, and under conditions of Subsections 2.1 and 2.4 as well as of the Example 2.3 the relationship:

$$\mu_i(x) = \frac{\eta_i \sigma^2(x)}{x + \zeta} \tag{3.13}$$

holds for all $x \in E$ and some constants $\eta_i \in \mathbb{R}$, $i = 0, 1$, such that $\eta_0 \ne \eta_1$ and $\eta_0 + \eta_1 = 1$. Let us define the process $Y = (Y_t)_{t \ge 0}$ by:

$$Y_t = \log \frac{\pi_t}{1 - \pi_t} - \frac{1}{\eta} \log \frac{x + \zeta}{X_t + \zeta} \tag{3.14}$$

with $\eta = 1/(\eta_1 - \eta_0)$. From the structure of (3.14) it is easily seen that there is a one-to-one correspondence between the processes $(\pi_t, X_t)_{t \ge 0}$ and $(\pi_t, Y_t)_{t \ge 0}$, and thus, the latter is also a (time-homogeneous strong) Markov process with respect to its natural filtration, which coincides with $(\mathcal{F}_t^X)_{t \ge 0}$. By deriving the expression for X_t from (3.14) and by substituting it into (2.20), we obtain:

$$d\pi_t = \lambda(1 - \pi_t)\, dt + \frac{\sigma\big((x+\zeta)e^{-\eta Y_t}[\pi_t/(1-\pi_t)]^\eta - \zeta\big)}{\eta(x+\zeta)e^{-\eta Y_t}[\pi_t/(1-\pi_t)]^\eta}\, \pi_t(1-\pi_t)\, d\overline{W}_t \quad (\pi_0 = \pi). \tag{3.15}$$

By applying Itô's formula to the expression (3.14) and by using the representations (2.20) and (2.25) as well as the assumption (3.13) with $\eta_0 \ne \eta_1$ and $\eta_0 + \eta_1 = 1$, we get $dY_t = 0$ and thus:

$$Y_t = \log \frac{\pi}{1 - \pi} \tag{3.16}$$

for all $t \ge 0$.

3.6. By means of standard arguments it is shown that under the assumptions of Subsection 3.5 the optimal stopping problems (3.2) and (3.3) are equivalent to:

$$\widetilde{V}(\pi, y) = \inf_\tau E_\pi \left[1 - \pi_\tau + c \int_0^\tau \pi_t\, dt \right] \tag{3.17}$$

where the infimum is taken over all stopping times τ of the process $(\pi_t, Y_t)_{t \ge 0}$ such that $E_\pi[\tau] < \infty$ for all $(\pi, y) \in [0,1] \times \mathbb{R}$ and $y = \log[\pi/(1-\pi)]$ for each $\pi \in \langle 0, 1 \rangle$ and $x \in E = \langle -\zeta, \infty \rangle$ fixed. It also follows that there exists a function $h(y)$, $y \in \mathbb{R}$, such that the continuation region C from (3.7) is equivalent to:

$$\widetilde{C} = \{(\pi, y) \in [0,1] \times \mathbb{R} \mid \pi < h(y)\} \tag{3.18}$$

and the set D from (3.8) is equivalent to:

$$\widetilde{D} = \{(\pi, y) \in [0,1] \times \mathbb{R} \mid \pi > h(y)\} \tag{3.19}$$

for each $y \in \mathbb{R}$ and $x \in E$ fixed.

3.7. If the assumption (3.13) with $\eta_0 \neq \eta_1$ and $\eta_0 + \eta_1 = 1$ holds, then by means of standard arguments it is shown that the infinitesimal operator $\widetilde{\mathbb{L}}$ of the process $(\pi_t, Y_t)_{t\geq 0}$ from (3.15)-(3.16) acts on a function $F \in C^{2,0}(\langle 0,1\rangle \times \mathbb{R})$ like:

$$(\widetilde{\mathbb{L}}F)(\pi, y) = \left(\lambda(1-\pi)\frac{\partial F}{\partial \pi} + \frac{r^2(x;\pi,y)}{2}\pi^2(1-\pi)^2\frac{\partial^2 F}{\partial \pi^2}\right)(\pi, y) \tag{3.20}$$

with

$$r(x;\pi,y) = \frac{\sigma\left((x+\zeta)e^{-\eta y}[\pi/(1-\pi)]^{\eta} - \zeta\right)}{\eta(x+\zeta)e^{-\eta y}[\pi/(1-\pi)]^{\eta}} \tag{3.21}$$

for all $(\pi, y) \in \langle 0,1\rangle \times \mathbb{R}$ and each $x \in E = \langle -\zeta, \infty\rangle$ fixed.

Now let us use the results of general theory of optimal stopping problems for continuous time Markov processes (see, e.g., [13], [30; Chapter III, Section 8] and [21]) to formulate the corresponding *free-boundary problem* for the unknown value function $(\pi, y) \mapsto \widetilde{V}(\pi, y)$ from (3.3) and the boundary $h(y)$, $y \in \mathbb{R}$, from (3.18)-(3.19):

$$(\widetilde{\mathbb{L}}\widetilde{V})(\pi, y) = -c\,\pi \quad \text{for} \quad (\pi, y) \in \widetilde{C} \tag{3.22}$$

$$\widetilde{V}(\pi, y)\big|_{\pi=h(x)-} = 1 - h(x) \tag{3.23}$$

$$\frac{\partial \widetilde{V}}{\partial \pi}(\pi, y)\big|_{\pi=h(x)-} = -1 \tag{3.24}$$

$$\frac{\partial \widetilde{V}}{\partial \pi}(\pi, y)\big|_{\pi=0+} = 0 \tag{3.25}$$

$$\widetilde{V}(\pi, y) = 1 - \pi \quad \text{for} \quad (\pi, y) \in \widetilde{D} \tag{3.26}$$

$$\widetilde{V}(\pi, y) < 1 - \pi \quad \text{for} \quad (\pi, y) \in \widetilde{C} \tag{3.27}$$

where $\widetilde{C}$ and $\widetilde{D}$ are given by (3.18) and (3.19), and the *instantaneous-stopping* condition (3.23) and the *smooth-fit* condition (3.24) as well as the *normal-entrance* condition (3.25) are assumed to be satisfied for all $y \in \mathbb{R}$ and each $x \in E$ fixed.

Note that by Dynkin's superharmonic characterization of the value function (see [9] and [30]) it follows that $\widetilde{V}(\pi, y)$ from (3.17) is the largest function satisfying (3.22)-(3.23) and (3.26)-(3.27) for each $y \in \mathbb{R}$ and $x \in E$ fixed.

3.8. By integrating the equation (3.22) and by using the boundary conditions (3.23)-(3.25), we obtain:

$$\widetilde{V}(\pi, y; h(y)) = 1 - h(y) - \int_{\pi}^{h(y)} \int_0^z \exp\left(-\int_u^{h(y)} \frac{2\lambda}{r^2(x;v,y)v^2(1-v)}\,dv\right) \frac{2c}{r^2(x;u,y)u(1-u)^2}\,du dz \tag{3.28}$$

with $r(x;\pi,y)$ given by (3.21) for all $\pi \in \langle 0, h(y)]$ and each $y \in \mathbb{R}$ and $x \in E = \langle -\zeta, \infty \rangle$ fixed. By means of the arguments similar to [30; pages 203-204] it follows that the boundary $h(y)$ is uniquely determined from the equation:

$$\int_0^{h(y)} \exp\left(-\int_u^{h(y)} \frac{2\lambda}{r^2(x;v,y)v^2(1-v)}\, dv \right) \frac{2c}{r^2(x;u,y)u(1-u)^2}\, du dz = 1 \tag{3.29}$$

for each $y \in \mathbb{R}$ and $x \in E = \langle -\zeta, \infty \rangle$ fixed.

3.9. Making use of the facts proved above we are now ready to formulate the main result of the paper.

Theorem 3.1. *Suppose that conditions (2.3) and (2.27)-(2.28) hold for all* $x \in E = \langle -\zeta, \infty \rangle$ *and some* $\zeta \in \mathbb{R}$ *fixed, and assumption (3.13) is satisfied with* $\eta_0 \neq \eta_1$ *and* $\eta_0 + \eta_1 = 1$. *Then in the Bayesian problem (3.2)+(3.3)+(3.17) of quickest disorder detection for the process (2.2) the value function has the expression:*

$$V(\pi) = V(\pi,x) = \widetilde{V}(\pi,y) = \begin{cases} \widetilde{V}(\pi,y;h(y)), & \text{if } \pi \in [0,h(y)\rangle \\ 1-\pi, & \text{if } \pi \in [h(y),1] \end{cases} \tag{3.30}$$

and the optimal π*-Bayes stopping time is explicitly given by:*

$$\tau_* = \inf\{t \geq 0 \mid \pi_t \geq h(y)\} \tag{3.31}$$

where the boundary $h(y)$ *is characterized as a unique solution of the equation (3.29) for* $y = \log[\pi/(1-\pi)]$ *and each* $\pi \in \langle 0,1 \rangle$ *and* $x \in E$ *fixed.*

Proof. It remains to show that the function (3.30) coincides with the value function (3.17) and that the stopping time τ_* from (3.31) with the boundary $h(y)$, $y \in \mathbb{R}$, specified above is optimal. Let us denote by $\widetilde{V}(\pi,y)$ the right-hand side of the expression (3.30). It follows by construction from the previous section that the function $\widetilde{V}(\pi,y)$ solves the system (3.22)-(3.26). Thus, applying Itô's formula to $\widetilde{V}(\pi_t,y)$, we obtain:

$$\widetilde{V}(\pi_t,y) = \widetilde{V}(\pi,y) + \int_0^t (\mathbb{L}\widetilde{V})(\pi_s,y) I(\pi_s \neq h(y))\, ds + \widetilde{M}_t \tag{3.32}$$

where the process $(\widetilde{M}_t)_{t\geq 0}$ defined by:

$$\widetilde{M}_t = \int_0^t \frac{\partial \widetilde{V}}{\partial \pi}(\pi_s,y) I(\pi_s \neq h(y))\, \frac{\mu_1(X_t)-\mu_0(X_t)}{\sigma(X_t)} \pi_s(1-\pi_s)\, d\overline{W}_s \tag{3.33}$$

is a continuous local martingale under P_π with respect to $(\mathcal{F}_t^X)_{t\geq 0}$.

By using the arguments above it can be verified that $(\mathbb{L}\widetilde{V})(\pi,y) \geq -c\pi$ for all $(\pi,y) \in \langle 0,1 \rangle \times \mathbb{R}$ such that $\pi \neq h(y)$. Moreover, by means of standard arguments and using the construction of $\widetilde{V}(\pi,y)$ above it can be checked that the property (3.27) also holds that together with (3.22)-(3.23)+(3.26) yields $\widetilde{V}(\pi,y) \leq 1-\pi$ for all $(\pi,y) \in [0,1] \times \mathbb{R}$. Observe that the time spent by the process π at the boundary $h(y)$, $y \in \mathbb{R}$, is of Lebesgue measure zero, that allows to extend $(\widetilde{\mathbb{L}}\widetilde{V})(\pi, y)$ arbitrarily to $\pi = h(y)$ and thus to ignore the indicators in (3.32)-(3.33). Hence, from the expressions (3.32) and the structure of the stopping time in (3.31) it follows that the inequalities:

$$1 - \pi_\tau + c \int_0^\tau \pi_s \, ds \geq \widetilde{V}(\pi_\tau, y) + c \int_0^\tau \pi_s \, ds \geq \widetilde{V}(\pi, y) + \widetilde{M}_\tau \tag{3.34}$$

hold for any stopping times τ of the process $(\pi_t)_{t \geq 0}$ started at $\pi \in [0, 1]$ and for each $y \in \mathbb{R}$.

Let $(\tau_n)_{n \in \mathbb{N}}$ be an arbitrary localizing sequences of stopping times for the processes $(\widetilde{M}_t)_{t \geq 0}$. Taking in (3.34) the expectation with respect to the measure P_π, by means of the optional sampling theorem (see, e.g., [15; Chapter I, Theorem 1.39] or [24; Chapter II, Theorem 3.1]), we get:

$$\begin{aligned} E_\pi \left[1 - \pi_{\tau \wedge \tau_n} + c \int_0^{\tau \wedge \tau_n} \pi_s \, ds \right] &\geq E_\pi \left[\widetilde{V}(\pi_{\tau \wedge \tau_n}, y) + c \int_0^{\tau \wedge \tau_n} \pi_s \, ds \right] \\ &\geq \widetilde{V}(\pi, y) + E_\pi \big[\widetilde{M}_{\tau \wedge \tau_n} \big] = \widetilde{V}(\pi, y) \end{aligned} \tag{3.35}$$

for all $(\pi, y) \in [0, 1] \times \mathbb{R}$. Hence, letting n go to infinity and using Fatou's lemma, for any stopping times τ such that $E_\pi[\tau] < \infty$ we obtain that the inequalities:

$$E_\pi \left[1 - \pi_\tau + c \int_0^\tau \pi_s \, ds \right] \geq E_\pi \left[\widetilde{V}(\pi_\tau, y) + c \int_0^\tau \pi_s \, ds \right] \geq \widetilde{V}(\pi, y) \tag{3.36}$$

are satisfied for all $(\pi, y) \in [0, 1] \times \mathbb{R}$.

By virtue of the fact that the function $\widetilde{V}(\pi, y)$ together with the boundary $h(y)$ satisfy the system (3.22)-(3.27), by the structure of the stopping time τ_* in (3.31) and the expressions (3.32) it follows that the equalities:

$$1 - \pi_{\tau_* \wedge \tau_n} + c \int_0^{\tau_* \wedge \tau_n} \pi_s \, ds = \widetilde{V}(\pi_{\tau_* \wedge \tau_n}, y) + c \int_0^{\tau_* \wedge \tau_n} \pi_s \, ds = \widetilde{V}(\pi, y) + \widetilde{M}_{\tau_* \wedge \tau_n} \tag{3.37}$$

hold for all $(\pi, y) \in [0, 1] \times \mathbb{R}$ and any localizing sequence $(\tau_n)_{n \in \mathbb{N}}$ of $(\widetilde{M}_t)_{t \geq 0}$. Note that, by means of standard arguments and by using the structure of the process (3.15) and of the stopping time (3.31), we have $E_\pi[\tau_*] < \infty$ for all $\pi \in [0, 1]$. Hence, letting n go to infinity and using conditions (3.22)-(3.23), we can apply the Lebesgue bounded convergence theorem for (3.37) to obtain the equality:

$$E_\pi \left[1 - \pi_{\tau_* \wedge \tau_n} + c \int_0^{\tau_* \wedge \tau_n} \pi_s \, ds \right] = \widetilde{V}(\pi, y) \tag{3.38}$$

for all $(\pi, y) \in [0, 1] \times \mathbb{R}$, which together with (3.36) directly imply the desired assertion. □

Acknowledgments. A part of these results was presented at the Conference 'Kolmogorov and Contemporary Mathematics' held at Moscow State University in June 2003. The author thanks Hans Rudolf Lerche and Goran Peskir for their interest. The author is grateful Alexander A. Gushchin for many helpful discussions concerning a general formulation of the disorder problem.

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