The Linear Complementarity Problem

Richard W. Cottle
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The Linear Complementarity Problem

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The Linear Complementarity Problem

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To Our Families

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PREFACE TO THE CLASSICS EDITION

The first edition of this book was published by Academic Press in 1992 as a volume in the series Computer Science and Scientific Computing edited by Werner Rheinboldt. As the most up-to-date and comprehensive publication on the Linear Complementarity Problem (LCP), the book was a relatively instant success. It shared the 1994 Frederick W. Lanchester Prize from INFORMS as one of the two best contributions to operations research and the management sciences written in the English language during the preceding three years. In the intervening years, *The Linear Complementarity Problem* has become the standard reference on the subject. Despite its popularity, the book went out of print and out of stock around 2005. Since then, the supply of used copies offered for sale has dwindled to nearly zero at rare-book prices. This is attributable to the substantial growth of interest in the LCP coming from diverse fields such as pure and applied mathematics, operations research, computer science, game theory, economics, finance, and engineering. An important development that has made this growth possible is the availability of several robust complementarity solvers that allow realistic instances of the LCP to be solved efficiently; these solvers can be found on the Website http://neos.mcs.anl.gov/neos/solvers/index.html.

The present SIAM Classics edition is meant to address the grave imbalance between supply and demand for *The Linear Complementarity Problem*. In preparing this edition, we have resisted the temptation to enlarge an already sizable volume by adding new material. We have, however, corrected a large number of typographical errors, modified the wording of some opaque or faulty passages, and brought the entries in the Bibliography of the first edition up to date. We warmly thank all those who were kind enough to report ways in which the original edition would benefit from such improvements.

In addition, we are grateful to Sara Murphy, Developmental and Acquisitions Editor of SIAM, and to the Editorial Board of the SIAM Classics in Applied Mathematics for offering us the opportunity to bring forth this revised edition. The statements of gratitude to our families expressed in the first edition still stand, but now warrant the inclusion of Linda Stone.

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PREFACE

The linear complementarity problem (LCP) refers to an inequality system with a rich mathematical theory, a variety of algorithms, and a wide range of applications in applied science and technology. Although diverse instances of the linear complementarity problem can be traced to publications as far back as 1940, concentrated study of the LCP began in the mid 1960's. As with many subjects, its literature is to be found primarily in scientific journals, Ph.D. theses, and the like. We estimate that today there are nearly one thousand publications dealing with the LCP, and the number is growing. Only a handful of the existing publications are monographs, none of which are comprehensive treatments devoted to this subject alone. We believe there is a demand for such a book—one that will serve the needs of students as well as researchers. This is what we have endeavored to provide in writing *The Linear Complementarity Problem.*

The LCP is normally thought to belong to the realm of mathematical programming. For instance, its AMS (American Mathematical Society) subject classification is 90C33, which also includes nonlinear complementarity. The classification 90xxx refers to economics, operations research, programming, and games; the further classification 90Cxx is specifically for mathematical programming. This means that the subject is normally identified with (finite-dimensional) optimization and (physical or economic) equilibrium problems. As a result of this broad range of associations, the literature of the linear complementarity problem has benefitted from contributions made by operations researchers, mathematicians, computer scientists, economists, and engineers of many kinds (chemical, civil, electrical, industrial, and mechanical).

One particularly important and well known context in which linear complementarity problems are found is the first-order optimality conditions of quadratic programming. Indeed, in its infancy, the LCP was closely linked to the study of linear and quadratic programs. A new dimension was brought to the LCP when Lemke and Howson published their renowned algorithm for solving the bimatrix game problem. These two subjects played a major role in the early development of the linear complementarity problem. As the field of mathematical programming matured, and the need for solving complex equilibrium problems intensified, the fundamental importance of the LCP became increasingly apparent, and its scope expanded significantly. Today, many new research topics have come into being as a consequence of this expansion; needless to say, several classical questions remain an integral part in the overall study of the LCP. In this book, we have attempted to include every major aspect of the LCP; we have striven to be up to date, covering all topics of traditional and current importance, presenting them in a style consistent with a contemporary point of view, and providing the most comprehensive available list of references.

Besides its own diverse applications, the linear complementarity problem contains two basic features that are central to the study of general mathematical and equilibrium programming problems. One is the concept of *complementarity*. This is a prevalent property of nonlinear programs; in the context of an equilibrium problem (such as the bimatrix game), this property is typically equivalent to the essential equilibrium conditions. The other feature is the property of *linearity*; this is the building block for the treatment of all smooth nonlinear problems. Together, linearity and complementarity provide the fundamental elements needed for the analysis and understanding of the very complex nature of problems within mathematical and equilibrium programming.

Preview

All seven chapters of this volume are divided into sections. No chapter has fewer than seven or more than thirteen sections. Generally speaking, the chapters are rather long. Indeed, each chapter can be thought of as a *part* with its sections playing the role of *chapters.* We have avoided the latter terminology because it tends to obscure the interconnectedness of the subject matter presented here.

The last two sections of every chapter in this book are titled "Exercises" and "Notes and References," respectively. By design, this organization makes it possible to use *The Linear Complementarity Problem* as a textbook and as a reference work. We have written this monograph for readers with some background in linear algebra, linear programming, and real analysis. In academic terms, this essentially means graduate student status. Apart from these prerequisites, the book is practically self-contained. Just as a precaution, however, we have included an extensive discussion of background material (see Chapter 2) where many key definitions and results are given, and most of the proofs are omitted. Elsewhere, we have supplied a proof for each stated result unless it is obvious or given as an exercise.

In keeping with the textbook spirit of this volume, we have attempted to minimize the use of footnotes and literature citations in the main body of the work. That sort of scholarship is concentrated in the notes and references at the end of each chapter where we provide some history of the subjects treated, attributions for the results that are stated, and pointers to the relevant literature. The bibliographic details for all the references cited in *The Linear Complementarity Problem* are to be found in the comprehensive Bibliography at the back of the book. For the reader's convenience, we have included a Glossary of Notation and what we hope is a useful Index.

Admittedly, this volume is exceedingly lengthy as a textbook for a one-term course. Hence, the instructor is advised to be selective in deciding the topics to include. Some topics can be assigned as additional reading, while others can be skipped. This is the practice we have followed in teaching courses based on the book at Stanford and Johns Hopkins.

The opening chapter sets forth a precise statement of the linear complementarity problem and then offers a selection of settings in which such problems arise. These "source problems" should be of interest to readers representing various disciplines; further source problems (applications of the LCP) are mentioned in later chapters. Chapter 1 also includes a number of other topics, such as equivalent formulations and generalizations of the LCP. Among the latter is the nonlinear complementarity problem which likewise has an extensive theory and numerous applications.

Having already described the function of Chapter 2, we move on to the largely theoretical Chapter 3 which is concerned with questions on the existence and multiplicity of solutions to linear complementarity problems. Here we emphasize the *leitmotiv* of matrix classes in the study of the LCP. This important theme runs through Chapter 3 and all those which follow. We presume that one of these classes, the positive semi-definite matrices, will already be familiar to many readers, and that most of the other classes will not. Several of these matrix classes are of interest because they characterize certain properties of the linear complementarity problem. They are the answers to questions of the form "what is the class of all real square matrices such that every linear complementarity problem formulated with such a matrix has the property ... ?" For pedagogical purposes, the LCP is a splendid context in which to illustrate concepts of linear algebra and matrix theory. The matrix classes that abound in LCP theory also provide research opportunities for the mathematical and computational investigation of their characteristic properties.

In the literature on the LCP, one finds diverse terminology and notation for the matrix classes treated in Chapter 3. We hope that our systematic treatment of these classes and consistent use of notation will bring about their standardization.

Algorithms for the LCP provide another way of studying the existence of solutions, the so-called "constructive approach." Ideally, an LCP algorithm will either produce a solution to a given problem or else determine that no solution exists. This is called *processing* the problem. Beyond this, the ability to compute solutions for large classes of problems adds to the utility of the LCP as a model for real world problems.

There are two main families of algorithms for the linear complementarity problem: pivoting methods (direct methods) and iterative methods (indirect methods). We devote one chapter to each of these. Chapter 4 covers the better-known pivoting algorithms (notably principal pivoting methods and Lemke's method) for solving linear complementarity problems of various kinds; we also present their parametric versions. All these algorithms are specialized exchange methods, not unlike the familiar simplex method of linear programming. Under suitable conditions, these pivoting methods are finite whereas the iterative methods are convergent in the limit. Algorithms of the latter sort (e.g., matrix splitting methods, a damped Newton method, and interior-point methods) are treated in Chapter 5.

Chapter 6 offers a more geometric view of the linear complementarity problem. Much of it rests on the concept of complementary cones and the formulation of the LCP in terms of a particular piecewise linear function, both of which are introduced in Chapter 1. In addition, Chapter 6 features the application of degree theory to the study of the aforementioned piecewise linear function and, especially, its local behavior. Ideas from degree theory are also brought to bear on the parametric interpretation of Lemke's algorithm, which can be viewed as a homotopy method.

The seventh and concluding chapter focuses on sensitivity and stability analysis, the study of how small changes in the data affect various aspects of the problem. Under this heading, we investigate issues such as the local uniqueness of solutions, the local solvability of the perturbed problems, continuity properties of the solutions to the latter problems, and also the applications of these sensitivity results to the convergence analysis of algorithms. Several of the topics treated in this chapter are examples of the recent research items alluded to above.

Some comments are needed regarding the presentation of the algorithms in Chapters 4 and 5. Among several styles, we have chosen one that suits our taste and tried to use it consistently. This style is neither the terse "pidgeon algol" (or "pseudo code") nor the detailed listing of FORTRAN instructions. It probably lies somewhere in between. The reader is cautioned not to regard our algorithm statements as ready-to-use implementations, for their purpose is to identify the algorithmic tasks rather than the computationally sophisticated ways of performing them. With regard to the pivotal methods, we have included a section in Chapter 4 called "Computational Considerations" in which we briefly discuss a practical alternative to pivoting in schemas (tableaux). This approach, which is the counterpart of the revised simplex method of linear programming, paves the way for the entrance of matrix factorizations into the picture. In terms of the iterative methods, several of them (such as the point SOR method) are not difficult to implement, while others (such as the family of interior-point methods) are not as easy. A word of caution should perhaps be offered to the potential users of any of these algorithms. In general, special care is needed when dealing with practical problems. The effective management of the data, the efficient way of performing the computations, the precaution toward numerical round-off errors, the termination criteria, the problem characteristics, and the actual computational environment are all important considerations for a successful implementation of an algorithm.

The exercises in the book also deserve a few words. While most of them are not particularly difficult, several of them are rather challenging and may require some careful analysis. A fair number of the exercises are variations of known results in the literature; these are intended to be expansions of the results in the text.

An apology

Though long, this book gives short shrift (or no shrift at all) to several contemporary topics. For example, we pay almost no attention to the area of parallel methods for solving the LCP, though several papers of this kind are included in the Bibliography; and we do not cover enumerative methods and global optimization approaches, although we do cite a few publications in the Notes and References of Chapter 4 where such approaches can be seen. Another omission is a summary of existing computer codes and computational experience with them. Here, we regret to say, there is much work to be done to assemble and organize such information, not just by ourselves, but by the interested scientific community.

In Chapter 4, some discussion is devoted to the worst case behavior of certain pivoting methods, and two special polynomially bounded pivoting algorithms are presented in Sections 7 and 8. These two subjects are actually a small subset of the many issues concerning the computational complexity of the LCP and the design of polynomial algorithms for certain problems with special properties. Omitted in our treatment is the family of *ellipsoid methods* for a positive semi-definite LCP. The interior-point methods have been analyzed in great detail in the literature, and their polynomial complexity has been established for an LCP with a positive semi-definite matrix. Our presentation in Section 9 of Chapter 5 is greatly simplified and, in a way, has done an injustice to the burgeoning literature on this topic. Research in this area is still quite vigorous and, partly because of this, it is difficult to assemble the most current developments. We can only hope that by providing many references in the Bibliography, we have partially atoned for our insensitivity to this important class of methods.

Acknowledgments

Many years have passed since the writing of this book began, and in some respects (such as those just suggested) it is not finished yet. Be that as it may, we have managed to accumulate a debt of gratitude to a list of individuals who, at one time or another, have played a significant role in the completion of this work. Some have shared their technical expertise on the subject matter, some have provided the environment in which parts of the research and writing took place, some have assisted with the intricacies of computer-based typesetting, and some have criticized portions of the manuscript. We especially thank George Dantzig for siring linear programming; his continuing contributions to mathematical programming in general have inspired generations of students and researchers, including ourselves. The seminal work of Carlton Lemke did much to stimulate the growth of linear complementarity. We have benefitted from his research and his collegiality. In addition, we gratefully acknowledge the diverse contributions of Steve Brady, Rainer Burkard, Curtis Eaves, Ulrich Eckhardt, Gene Golub, George Herrmann, Bernhard Korte, Jerry Lieberman, Olvi Mangasarian, Ernst Mayr, Steve Robinson, Michael Saunders, Siegfried Schaible, Herb Shulman, John Stone, Albert Tucker, Pete Veinott, Franz Weinberg, Wes Winkler, Philip Wolfe, Margaret Wright, a host of graduate students at Stanford and Johns Hopkins, and colleagues there as well as at AT&T Bell Laboratories. To all these people, we express our sincere thanks and absolve them of guilt for any shortcomings of this book.

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GLOSSARY OF NOTATION

Spaces

R^n	real n-dimensional space
R	the real line
$R^{n\times m}$	the space of $n \times m$ real matrices
R^n_+	the nonnegative orthant of R^n
R^n_{++}	the positive orthant of R^n

Vectors

z^{T}	the transpose of a vector z
$\{z^\nu\}$	a sequence of vectors $z^1, z^2, z^3, \ldots$
e_m	an m-dimensional vector of all ones
	(m is sometimes omitted)
$\|\cdot\|$	a norm on R^n
$x^{\mathrm{T}}y$	the standard inner product of vectors in R^n
$x \geq y$	the (usual) partial ordering: $x_i \geq y_i$, $i = 1, \ldots n$
$x > y$	the strict ordering: $x_i > y_i$, $i = 1, \ldots, n$
$x \succ y$	x is lexicographically greater than y
$x \succeq y$	x is lexicographically greater than or equal to y
$\min(x, y)$	the vector whose i-th component is $\min(x_i, y_i)$
$\max(x, y)$	the vector whose i-th component is $\max(x_i, y_i)$
$x * y$	$(x_i y_i)$, the Hadamard product of x and y
$\|z\|$	the vector whose i-th component is $\|z_i\|$
z^+	$\max(0, z)$, the nonnegative part of a vector z
z^-	$\max(0, -z)$, the nonpositive part of a vector z
$z \simeq$	the sign pattern of the components of z is

Matrices

$A = (a_{ij})$	a matrix with entries a_{ij}
$\det A$	the determinant of a matrix A
A^{-1}	the inverse of a matrix A
$\|\|A\|\|$	a norm of a matrix A
$\rho(A)$	the spectral radius of a matrix A
A^{T}	the transpose of a matrix A
I	the identity matrix
$A_{\alpha\beta}$	$(a_{ij})_{i\in\alpha,\, j\in\beta}$, a submatrix of a matrix A
$A_{\alpha\bullet}$	$(a_{ij})_{i\in\alpha,\, \text{all } j}$, the rows of A indexed by α
$A_{\bullet\beta}$	$(a_{ij})_{\text{all } i,\, j\in\beta}$, the columns of A indexed by β
$\operatorname{diag}(a_1, \ldots, a_n)$	the diagonal matrix with elements $a_1, \ldots, a_n$
$A^{\mathrm{T}}_{\alpha\alpha}$	$(A^{\mathrm{T}})_{\alpha\alpha} = (A_{\alpha\alpha})^{\mathrm{T}}$, the transpose of a principal submatrix of A
$A^{-1}_{\alpha\alpha}$	$(A_{\alpha\alpha})^{-1}$, the inverse of a principal submatrix of A
$A^{-\mathrm{T}}$	$(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}$, the inverse transpose of a matrix A
$(A/A_{\alpha\alpha})$	$A_{\bar\alpha\bar\alpha} - A_{\bar\alpha\alpha}A^{-1}_{\alpha\alpha}A_{\alpha\bar\alpha}$, the Schur complement of $A_{\alpha\alpha}$ in A
$A \simeq$	the sign pattern of the entries of A is
$A \leq B$	$a_{ij} \leq b_{ij}$ for all i and j
$A < B$	$a_{ij} < b_{ij}$ for all i and j
$\wp_\alpha(M)$	the principal transform of M relative to $M_{\alpha\alpha}$

Index sets

$\bar\alpha$	the complement of an index set α
$\bar\imath$	the complement of the index set $\{i\}$
$\operatorname{supp} z$	$\{i : z_i \neq 0\}$, the support of a vector z
$\alpha(z)$	$\{i : z_i > (q + Mz)_i\}$
$\beta(z)$	$\{i : z_i = (q + Mz)_i\}$
$\gamma(z)$	$\{i : z_i < (q + Mz)_i\}$

Signs

$\oplus$	nonnegative
$\ominus$	nonpositive

Sets

$\in$	element membership
$\notin$	not an element of
$\emptyset$	the empty set
$\subseteq$	set inclusion
$\subset$	proper set inclusion
$\cup, \cap, \times$	union, intersection, Cartesian product
$S_1 \setminus S_2$	the difference of sets S_1 and S_2
$S_1 \triangle S_2$	$(S_1 \setminus S_2) \cup (S_2 \setminus S_1) = (S_1 \cup S_2) \setminus (S_1 \cap S_2)$, the symmetric difference of S_1 and S_2
$S_1 + S_2$	the (vector) sum of S_1 and S_2
$\lvert S \rvert$	the cardinality of a finite set S
S^c	the complement of a set S
$\dim S$	the dimension of a set S
$\operatorname{bd} S$	the (topological) boundary of a set S
$\operatorname{cl} S$	the (topological) closure of a set S
$\operatorname{int} S$	the topological interior of a set S
$\operatorname{ri} S$	the relative interior of a set S
$\operatorname{rb} S$	the relative boundary of a set S
S^*	the dual cone of S
0^+S	the set of recession directions of S
$\operatorname{affn} S$	the affine hull of S
$\operatorname{conv} S$	the convex hull of S
$\operatorname{pos} A$	the cone generated by the matrix A
$[A]$	see **6.9.1**
$\mathcal{G}(f)$	the graph of the function f
S^{n-1}	the unit sphere in R^n
$\ell[x, y]$	the closed line segment between x and y
$\ell(x, y)$	the open line segment between x and y
$B(x, \delta)$	an (open) neighborhood of x with radius δ
$N(x)$	an (open) neighborhood of x
$\arg\min_x f(x)$	the set of x attaining the minimum of $f(x)$
$\arg\max_x f(x)$	the set of x attaining the maximum of $f(x)$
$[a, b]$	a closed interval in R
(a, b)	an open interval in R
$\mathcal{S}$	the unit sphere
$\mathcal{B}$	the closed unit ball

Functions

$f : \mathcal{D} \to \mathcal{R}$	a mapping with domain $\mathcal{D}$ and range $\mathcal{R}$
∇f	$(\partial f_i/\partial x_j)$, the $m \times n$ Jacobian of a mapping $f : R^n \to R^m$ $(m \geq 2)$
$\nabla_\beta f_\alpha$	$(\partial f_i/\partial x_j)_{i\in\alpha}^{j\in\beta}$, a submatrix of ∇f
$\nabla\theta$	$(\partial\theta/\partial x_j)$, the gradient of a function $\theta : R^n \to R$
$f'(\cdot,\cdot)$	directional derivative of the mapping f
f^{-1}	the inverse of f
$o(t)$	any function such that $\lim_{t\to 0} \frac{o(t)}{t} = 0$
$O(n)$	any function such that $\sup_n \frac{\lvert O(n)\rvert}{n} < \infty$
$\Pi_K(x)$	the projection of x on the set K
$\inf f(x)$	the infimum of the function f
$\sup f(x)$	the supremum of the function f
$\mathrm{sgn}(x)$	the "sign" of $x \in R$, 0, +1, or −1

LCP symbols

(q, M)	the LCP with data q and M
(q, d, M)	the LCP $(q + dz_0, M)$, see (4.4.6)
$f_M(x)$	$x^+ - Mx^-$
$H_{q,M}(x)$	$\min(x, q + Mx)$
$\mathrm{FEA}(q, M)$	the feasible region of (q, M)
$\mathrm{SOL}(q, M)$	the solution set of (q, M)
$r(x, q, M)$	a residue function for (q, M)
$\deg M$	the degree of M
$\deg_M(q)$	the local degree of M at q
$C_M(\alpha)$	complementary matrix corresponding to index set α (M is sometimes omitted)
$K(M)$	the closed cone of q for which $\mathrm{SOL}(q, M) \neq \emptyset$
$\mathcal{K}(M)$	see **6.1.5**
$\mathcal{L}(M)$	see **6.2.3**
$\mathrm{ind}(\mathrm{pos}\, C_M(\alpha))$	the index of the complementary cone $\mathrm{pos}\, C_M(\alpha)$
ind_M	index (can be applied to a point, an orthant, a solution to the LCP, or a complementary cone) see **6.1.3**
(B, C)	the matrix splitting of M as $B + C$

NUMBERING SYSTEM

The chapters of this book are numbered from 1 to 7; their sections are denoted by decimal numbers of the type 2.3 (which means Section 3 of Chapter 2). Many sections are further divided into subsections. The latter are not numbered, but each has a heading.

All definitions, results, exercises, notes, and miscellaneous items are numbered consecutively within each section in the form **1.3.5, 1.3.6**, meaning Items 5 and 6 in Section 3 of Chapter 1. With the exception of the exercises and the notes, all items are also identified by their types (e.g. **1.4.1 Proposition.**, **1.4.2 Remark.**). When an item is being referred to in the text, it is called out as Algorithm **5.2.1**, Theorem **4.1.7**, Exercise **2.10.9**, etc. At times, only the number is used to refer to an item.

Equations are numbered consecutively in each section by (1), (2), etc. Any reference to an equation in the same section is by this number only, while equations in another section are identified by chapter, section, and equation. Thus, (3.1.4), means Equation (4) in Section 1 of Chapter 3.

Chapter 1

INTRODUCTION

1.1 Problem Statement

The linear complementarity problem consists in finding a vector in a finite-dimensional real vector space that satisfies a certain system of inequalities. Specifically, given a vector $q \in R^n$ and a matrix $M \in R^{n \times n}$, the *linear complementarity problem*, abbreviated LCP, is to find a vector $z \in R^n$ such that

$$z \geq 0 \tag{1}$$

$$q + Mz \geq 0 \tag{2}$$

$$z^{\mathrm{T}}(q + Mz) = 0 \tag{3}$$

or to show that no such vector z exists. We denote the above LCP by the pair (q, M).

Special instances of the linear complementarity problem can be found in the mathematical literature as early as 1940, but the problem received little attention until the mid 1960's at which time it became an object of study in its own right. Some of this early history is discussed in the Notes and References at the end of the chapter.

Our aim in this brief section is to record some of the more essential terminology upon which the further development of the subject relies.

A vector z satisfying the inequalities in (1) and (2) is said to be *feasible.* If a feasible vector z strictly satisfies the inequalities in (1) and (2), then it is said to be *strictly feasible.* We say that the LCP (q, M) is *(strictly) feasible* if a (strictly) feasible vector exists. The set of feasible vectors of the LCP (q, M) is called its *feasible region* and is denoted $\text{FEA}(q, M)$. Let

$$w = q + Mz\,. \tag{4}$$

A feasible vector z of the LCP (q, M) satisfies condition (3) if and only if

$$z_i w_i = 0 \qquad \text{for all } i = 1, \ldots, n. \tag{5}$$

Condition (5) is often used in place of (3). The variables z_i and w_i are called a *complementary pair* and are said to be *complements* of each other.

A vector z satisfying (5) is called *complementary.* The LCP is therefore to find a vector that is both feasible and complementary; such a vector is called a *solution* of the LCP. The LCP (q, M) is said to be *solvable* if it has a solution. The solution set of (q, M) is denoted $\text{SOL}(q, M)$. Observe that if $q \geq 0$, the LCP (q, M) is always solvable with the zero vector being a trivial solution.

The definition of w given above is often used in another way of expressing the LCP (q, M), namely as the problem of finding nonnegative vectors w and z in R^n that satisfy (4) and (5). To facilitate future reference to this equivalent formulation, we write the conditions as

$$w \geq 0, \; z \geq 0 \tag{6}$$

$$w = q + Mz \tag{7}$$

$$z^{\mathrm{T}} w = 0. \tag{8}$$

This way of representing the problem is useful in discussing algorithms for the solution of the LCP.

For any LCP (q, M), there is a positive integer n such that $q \in R^n$ and $M \in R^{n \times n}$. Most of the time this parameter is understood, but when we wish to call attention to the size of the problem, we speak of an LCP *of* order n where n is the dimension of the space to which q belongs, etc. This usage occasionally facilitates problem specification.

The special case of the LCP (q, M) with $q = 0$ is worth noting. This problem is called the *homogeneous* LCP associated with M. A special property of the LCP $(0, M)$ is that if $z \in \text{SOL}(0, M)$, then $\lambda z \in \text{SOL}(0, M)$ for all scalars $\lambda \geq 0$. The homogeneous LCP is trivially solved by the zero vector. The question of whether or not this special problem possesses any *nonzero* solutions has great theoretical and algorithmic importance.

1.2 Source Problems

Historically, the LCP was conceived as a unifying formulation for the linear and quadratic programming problems as well as for the bimatrix game problem. In fact, quadratic programs have always been—and continue to be—an extremely important source of applications for the LCP. Several highly effective algorithms for solving quadratic programs are based on the LCP formulation. As far as the bimatrix game problem is concerned, the LCP formulation was instrumental in the discovery of a superb constructive tool for the computation of an equilibrium point. In this section, we shall describe these classical applications and several others. In each of these applications, we shall also point out some special properties of the matrix M in the associated LCP.

For purposes of this chapter, the applications of the linear complementarity problem are too numerous to be listed individually. We have chosen the following examples to illustrate the diversity. Each of these problems has been studied extensively, and large numbers of references are available. In each case, our discussion is fairly brief. The reader is advised to consult the Notes and References (Section 1.7) for further information on these and other applications.

A word of caution

In several of the source problems described below and in the subsequent discussion within this chapter, we have freely used some basic results from linear and quadratic programming, and the theory of convex polyhedra, as well as some elementary matrix-theoretic and real analysis concepts. For those readers who are not familiar with these topics, we have prepared a brief review in Chapter 2 which contains a summary of the background material needed for the entire book. It may be advisable (for these readers) to proceed to this review chapter before reading the remainder of the present chapter. Several other source problems require familiarity with some topics not within the main scope of this book, the reader can consult Section 1.7 for references that provide more details and related work.

Quadratic programming

Consider the quadratic program (QP)

$$\begin{array}{ll} \text{minimize} & f(x) = c^{\mathrm{T}}x + \frac{1}{2}x^{\mathrm{T}}Qx \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array} \tag{1}$$

where $Q \in R^{n\times n}$ is symmetric, $c \in R^n$, $A \in R^{m\times n}$ and $b \in R^m$. (The case $Q = 0$ gives rise to a linear program.) If x is a locally optimal solution of the program (1), then there exists a vector $y \in R^m$ such that the pair (x, y) satisfies the Karush-Kuhn-Tucker conditions

$$\begin{array}{lll} u = c + Qx - A^{\mathrm{T}}y \geq 0, & x \geq 0, & x^{\mathrm{T}}u = 0, \\ v = -b + Ax \geq 0, & y \geq 0, & y^{\mathrm{T}}v = 0. \end{array} \tag{2}$$

If, in addition, Q is positive semi-definite, i.e., if the objective function $f(x)$ is convex, then the conditions in (2) are, in fact, sufficient for the vector x to be a globally optimal solution of (1).

The conditions in (2) define the LCP (q, M) where

$$q = \begin{bmatrix} c \\ -b \end{bmatrix} \text{ and } M = \begin{bmatrix} Q & -A^{\mathrm{T}} \\ A & 0 \end{bmatrix}. \tag{3}$$

Notice that the matrix M is not symmetric (unless A is vacuous or equal to zero), even though Q is symmetric; instead, M has a property known as *bisymmetry.* (In general, a square matrix N is *bisymmetric* if (perhaps after permuting the same set of rows and columns) it can be brought to the form

$$N = \begin{bmatrix} G & -A^{\mathrm{T}} \\ A & H \end{bmatrix}$$

where both G and H are symmetric.) If Q is *positive semi-definite* as in *convex quadratic programming*, then so is M. (In general, a square matrix M is *positive semi-definite* if $z^{\mathrm{T}}Mz \geq 0$ for every vector z.)

An important special case of the quadratic program (1) is where the only constraints are nonnegativity restrictions on the variables x. In this case, the program (1) takes the simple form

$$\begin{array}{ll} \text{minimize} & f(x) = c^{\mathrm{T}}x + \frac{1}{2}x^{\mathrm{T}}Qx \\ \text{subject to} & x \geq 0. \end{array} \tag{4}$$

If Q is positive semi-definite, the program (4) is completely equivalent to the LCP (c, Q), where Q is symmetric (by assumption). For an arbitrary symmetric Q, the LCP (c, Q) is equivalent to the stationary point problem of (4). As we shall see later on, a quadratic program with only nonnegativity constraints serves as an important bridge between an LCP with a symmetric matrix and a general quadratic program with arbitrary linear constraints.

A significant number of applications in engineering and the physical sciences lead to a convex quadratic programming model of the special type (4) which, as we have already pointed out, is equivalent to the LCP (c, Q). These applications include the contact problem, the porous flow problem, the obstacle problem, the journal bearing problem, the elastic-plastic torsion problem as well as many other free-boundary problems. A common feature of these problems is that they are all posed in an infinite-dimensional function space setting (see Section 5.1 for some examples). The quadratic program (4) to which they give rise is obtained from their finite-dimensional discretization. Consequently, the size of the resulting program tends to be very large. The LCP plays an important role in the numerical solution of these applied problems. This book contains two lengthy chapters on algorithms; the second of these chapters centers on methods that can be used for the numerical solution of very large linear complementarity problems.

Bimatrix games

A *bimatrix game* $\Gamma(A, B)$ consists of two players (called Player I and Player II) each of whom has a finite number of actions (called *pure strategies*) from which to choose. In this type of game, it is *not* necessarily the case that what one player gains, the other player loses. For this reason, the term bimatrix game ordinarily connotes a *finite, two-person, nonzero-sum game.*

Let us imagine that Player I has m pure strategies and Player II has n pure strategies. The symbols A and B in the notation $\Gamma(A, B)$ stand for $m \times n$ matrices whose elements represent *costs* incurred by the two players. Thus, when Player I chooses pure strategy i and Player II chooses pure strategy j, they incur the costs a_{ij} and b_{ij}, respectively. There is no requirement that these costs sum to zero.

A *mixed* (or *randomized*) *strategy* for Player I is an m-vector x whose i-th component x_i represents the probability of choosing pure strategy i. Thus, $x \geq 0$ and $\sum_{i=1}^{m} x_i = 1$. A mixed strategy for Player II is defined analogously. Accordingly, if x and y are a pair of mixed strategies for Players I and II, respectively, then their *expected costs* are given by $x^{\mathrm{T}}Ay$ and $x^{\mathrm{T}}By$, respectively. A pair of mixed strategies (x^*, y^*) with $x^* \in R^m$ and $y^* \in R^n$ is said to be a *Nash equilibrium* if

$$(x^*)^{\mathrm{T}}Ay^* \leq x^{\mathrm{T}}Ay^* \qquad \text{for all } x \geq 0 \text{ and } \textstyle\sum_{i=1}^{m} x_i = 1$$

$$(x^*)^{\mathrm{T}}By^* \leq (x^*)^{\mathrm{T}}By \quad \text{for all } y \geq 0 \text{ and } \textstyle\sum_{j=1}^{n} y_j = 1.$$

In other words, (x^*, y^*) is a Nash equilibrium if neither player can gain (in terms of lowering the expected cost) by unilaterally changing his strategy. A fundamental result in game theory states that such a Nash equilibrium always exists.

To convert $\Gamma(A, B)$ to a linear complementarity problem, we assume that A and B are (entrywise) positive matrices. This assumption is totally unrestrictive because by adding the same sufficiently large positive scalar to all the costs a_{ij} and b_{ij}, they can be made positive. This modification does not affect the equilibrium solutions in any way. Having done this, we consider the LCP

$$\begin{aligned} u &= -e_m + Ay \geq 0, \quad x \geq 0, \quad x^{\mathrm{T}}u = 0 \\ v &= -e_n + B^{\mathrm{T}}x \geq 0, \quad y \geq 0, \quad y^{\mathrm{T}}v = 0, \end{aligned} \tag{5}$$

where, for the moment, e_m and e_n are m- and n-vectors whose components are all ones. It is not difficult to see that if (x^*, y^*) is a Nash equilibrium, then (x', y') is a solution to (5) where

$$x' = x^*/(x^*)^{\mathrm{T}}By^* \quad \text{and} \quad y' = y^*/(x^*)^{\mathrm{T}}Ay^*. \tag{6}$$

Conversely, it is clear that if (x', y') is a solution of (5), then neither x' nor y' can be zero. Thus, (x^*, y^*) is a Nash equilibrium where

$$x^* = x'/e_m^{\mathrm{T}}x' \quad \text{and} \quad y^* = y'/e_n^{\mathrm{T}}y'.$$

The positivity of A and B ensures that the vectors x' and y' in (6) are nonnegative. As we shall see later, the same assumption is also useful in the process of solving the LCP (5).

The vector q and the matrix M defining the LCP (5) are given by

$$q = \begin{bmatrix} -e_m \\ -e_n \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & A \\ B^{\mathrm{T}} & 0 \end{bmatrix}. \tag{7}$$

In this LCP (q, M), the matrix M is nonnegative and structured, and the vector q is very special.

Market equilibrium

A *market equilibrium* is the state of an economy in which the demands of consumers and the supplies of producers are balanced at the prevailing price level. Consider a particular market equilibrium problem in which the supply side is described by a linear programming model to capture the technological or engineering details of production activities. The market demand function is generated by econometric models with commodity prices as the primary independent variables. Mathematically, the model is to find vectors p^* and r^* so that the conditions stated below are satisfied:

(i) supply side

$$\begin{aligned} \text{minimize} \quad & c^{\mathrm{T}}x \\ \text{subject to} \quad & Ax \geq b && (8) \\ & Bx \geq r^* && (9) \\ & x \geq 0 \end{aligned}$$

where c is the cost vector for the supply activities, x is the vector of production activity levels, condition (8) represents the technological constraints on production and (9) the demand requirement constraints;

(ii) demand side

$$r^* = Q(p^*) = Dp^* + d \tag{10}$$

where $Q(\cdot)$ is the market demand function with p^* and r^* representing the vectors of demand prices and quantities, respectively, $Q(\cdot)$ is assumed to be an affine function;

(iii) equilibrating conditions

$$p^* = \pi^* \tag{11}$$

where π^* denotes the (dual) vector of shadow prices (i.e., the market supply prices)corresponding to the constraint (9).

To convert the above model into a linear complementarity problem, we note that a vector x^* is an optimal solution of the supply side linear program if and only if there exists a vector v^* such that

$$\begin{aligned} y^* &= c - A^{\mathrm{T}}v^* - B^{\mathrm{T}}\pi^* \geq 0, & x^* &\geq 0, & (y^*)^{\mathrm{T}}x^* &= 0, \\ u^* &= -b + Ax^* \geq 0, & v^* &\geq 0, & (u^*)^{\mathrm{T}}v^* &= 0, \\ \delta^* &= -r^* + Bx^* \geq 0, & \pi^* &\geq 0, & (\delta^*)^{\mathrm{T}}\pi^* &= 0. \end{aligned} \tag{12}$$

Substituting the demand function (10) for r^* and invoking the equilibrating condition (11), we deduce that the conditions in (12) constitute the LCP (q, M) where

$$q = \begin{bmatrix} c \\ -b \\ -d \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & -A^{\mathrm{T}} & -B^{\mathrm{T}} \\ A & 0 & 0 \\ B & 0 & -D \end{bmatrix}. \tag{13}$$

Observe that the matrix M in (13) is bisymmetric if the matrix D is symmetric. In this case, the above LCP becomes the Karush-Kuhn-Tucker conditions of the quadratic program:

$$\begin{aligned} \text{maximize} \quad & d^{\mathrm{T}}p + \tfrac{1}{2}p^{\mathrm{T}}Dp + b^{\mathrm{T}}v \\ \text{subject to} \quad & A^{\mathrm{T}}v + B^{\mathrm{T}}p \leq c \\ & p \geq 0, \;\; v \geq 0. \end{aligned} \tag{14}$$

On the other hand, if D is asymmetric, then M is not bisymmetric and the connection between the market equilibrium model and the quadratic program (14) fails to exist. The question of whether D is symmetric is related to the *integrability* of the demand function $Q(\cdot)$. Regardless of the symmetry of D, the matrix M is positive semi-definite if $-D$ is so.

Optimal invariant capital stock

Consider an economy with constant technology and nonreproducible resource availability in which an initial activity level is to be determined such that the maximization of the discounted sum of future utility flows over an infinite horizon can be achieved by reconstituting that activity level at the end of each period. The technology is given by $A \in R^{m\times n}$, $B \in R_+^{m\times n}$ and $b \in R^m$ where:

- A_{ij} denotes the amount of good i used to operate activity j at unit level;
- B_{ij} denotes the amount of good i produced by operating activity j at unit level;
- b_i denotes the amount of resource i exogenously provided in each time period ($b_i < 0$ denotes a resource withdrawn for subsistence).

The utility function $U : R^n \to R$ is assumed to be linear, $U(x) = c^{\mathrm{T}}x$. Let $x_t \in R^n$ $(t = 1, 2, \ldots)$ denote the vector of activity levels in period t. The model then chooses $x \in R^n$ so that $x_t = x$ $(t = 1, 2, \ldots)$ solves the problem $P(Bx)$ where, for a given vector $b_0 \in R^m$, $P(b_0)$ denotes the problem of finding a sequence of activity levels $\{x_t\}_1^\infty$ in order to

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_{t=1}^{\infty} \alpha^{t-1} U(x_t) \\ \text{subject to} \quad & Ax_1 \le b_0 + b \\ & Ax_t \le Bx_{t-1} + b, \quad t = 2, 3, \ldots \\ & x_t \ge 0, \qquad\qquad\quad t = 1, 2, \ldots \end{aligned}$$

where $\alpha \in (0, 1)$ is the discount rate. The vector x so obtained then provides an optimal capital stock invariant under discounted optimization.

It can be shown (see Note 1.7.7) that a vector x is an optimal invariant capital stock if it is an optimal solution of the linear program

$$\begin{aligned} \text{maximize} \quad & c^{\mathrm{T}}x \\ \text{subject to} \quad & Ax \le Bx + b \\ & x \ge 0. \end{aligned} \tag{15}$$

Further, the vector y is a set of optimal invariant dual price proportions if x and y satisfy the stationary dual feasibility conditions

$$A^{\mathrm{T}}y \geq \alpha B^{\mathrm{T}}y + c, \quad y \geq 0, \tag{16}$$

in addition to the condition

$$y^{\mathrm{T}}(b - (A - B)x) = x^{\mathrm{T}}((A^{\mathrm{T}} - \alpha B^{\mathrm{T}})y - c) = 0\,.$$

Introducing slack variables u and v into the constraints (15) and (16), we see that (x, y) provides both a primal and dual solution to the optimal invariant capital stock problem if x solves (15) and if (x, y, u, v) satisfy

$$\begin{aligned} u &= b + (B - A)x \geq 0, & y \geq 0, \quad u^{\mathrm{T}}y = 0, \\ v &= -c + (A^{\mathrm{T}} - \alpha B^{\mathrm{T}})y \geq 0, & x \geq 0, \quad v^{\mathrm{T}}x = 0. \end{aligned}$$

The latter conditions define the LCP (q, M) where

$$q = \begin{bmatrix} b \\ -c \end{bmatrix}, \quad \text{and} \quad M = \begin{bmatrix} 0 & B - A \\ A^{\mathrm{T}} - \alpha B^{\mathrm{T}} & 0 \end{bmatrix}. \tag{17}$$

The matrix M in (17) is neither nonnegative nor positive semi-definite. It is also not symmetric. Nevertheless, M can be written as

$$M = \begin{bmatrix} 0 & B - A \\ A^{\mathrm{T}} - B^{\mathrm{T}} & 0 \end{bmatrix} + (1 - \alpha) \begin{bmatrix} 0 & 0 \\ B^{\mathrm{T}} & 0 \end{bmatrix}$$

which is the sum of a skew-symmetric matrix (the first summand) and a nonnegative matrix (the second summand). Developments in Chapters 3 and 4 will show why this is significant.

Optimal stopping

Consider a Markov chain with finite state space $E = \{1, \ldots, n\}$ and transition probability matrix P. The chain is observed as long as desired. At each time t, one has the opportunity to stop the process or to continue. If one decides to stop, one is rewarded the payoff r_i, if the process is in state $i \in E$, at which point the "game" is over. If one decides to continue, then the chain progresses to the next stage according to the transition matrix P, and a new decision is then made. The problem is to determine the optimal time to stop so as to maximize the expected payoff.

Letting v_i be the long-run stationary optimal expected payoff if the process starts at the initial state $i \in E$, and v be the vector of such payoffs, we see that v must satisfy the dynamic programming recursion

$$v = \max(\alpha P v, r) \tag{18}$$

where "max (a, b)" denotes the componentwise maximum of two vectors a and b and $\alpha \in (0, 1)$ is the discount factor. In turn, it is easy to deduce that (18) is equivalent to the following conditions:

$$v \geq \alpha P v, \quad v \geq r, \quad \text{and} \quad (v - r)^{\mathrm{T}}(v - \alpha P v) = 0\,. \tag{19}$$

By setting $u = v - r$, we conclude that the vector v satisfies (18) if and only if u solves the LCP (q, M) where

$$q = (I - \alpha P)r \quad \text{and} \quad M = I - \alpha P\,.$$

Here the matrix M has the property that all its off-diagonal entries are nonpositive and that $Me > 0$ where e is the vector of all ones.

Incidentally, once the vector v is determined, the optimal stopping time is when the process first visits the set $\{i \in E : v_i = r_i\}$.

Convex hulls in the plane

An important problem in computational geometry is that of finding the convex hull of a given set of points. In particular, much attention has been paid to the special case of the problem where all the points lie on a plane. Several very efficient algorithms for this special case have been developed.

Given a collection $\{(x_i, y_i)\}_{i=0}^{n+1}$ of points in the plane, we wish to find the extreme points and the facets of the convex hull in the order in which they appear. We can divide this problem into two pieces; we will first find the lower envelope of the given points and then we will find the upper envelope. In finding the lower envelope we may assume that the x_i are distinct. The reason for this is that if $x_i = x_j$ and $y_i \leq y_j$, then we may ignore the point (x_j, y_j) without changing the lower envelope. Thus, assume $x_0 < x_1 < \cdots < x_n < x_{n+1}$.

Let $f(x)$ be the lower envelope which we wish to determine. Thus, $f(x)$ is the pointwise maximum over all convex functions $g(x)$ in which $g(x_i) \leq y_i$ for all $i = 0, \ldots, n+1$. The function $f(x)$ is convex and piecewise

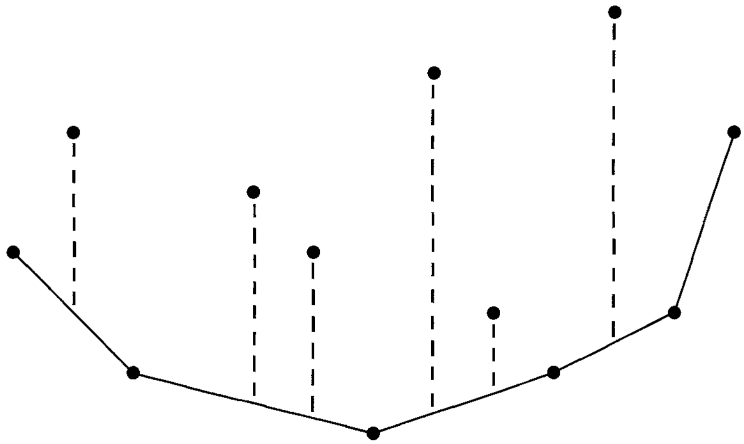

Figure 1.1

linear. The set of breakpoints between the pieces of linearity is a subset of $\{(x_i, y_i)\}_{i=0}^{n+1}$. Let $t_i = f(x_i)$ and let $z_i = y_i - t_i$ for $i = 0, \ldots, n+1$. The quantity z_i represents the vertical distance between the point (x_i, y_i) and the lower envelope. With this in mind we can make several observations (see Figure 1.1). First, $z_0 = z_{n+1} = 0$. Second, suppose the point (x_i, y_i) is a breakpoint. This implies that $t_i = y_i$ and $z_i = 0$. Also, the segment of the lower envelope between (x_{i-1}, t_{i-1}) and (x_i, t_i) has a different slope than the segment between (x_i, t_i) and (x_{i+1}, t_{i+1}). Since $f(x)$ is convex, the former segment must have a smaller slope than the latter segment. This means that strict inequality holds in

$$\frac{t_i - t_{i-1}}{x_i - x_{i-1}} \leq \frac{t_{i+1} - t_i}{x_{i+1} - x_i}. \tag{20}$$

Finally, if $z_i > 0$, then (x_i, y_i) cannot be a breakpoint of $f(x)$. Thus, equality holds in (20).

Bringing the above observations together shows that the vector $z = \{z_i\}_{i=1}^{n}$ must solve the LCP (q, M) where $M \in R^{n \times n}$ and $q \in R^n$ are defined by

$$q_i = \beta_i - \beta_{i-1} \quad \text{and} \quad m_{ij} = \begin{cases} \alpha_{i-1} + \alpha_i & \text{if } j = i, \\ -\alpha_i & \text{if } j = i+1, \\ -\alpha_j & \text{if } j = i-1, \\ 0 & \text{otherwise,} \end{cases} \tag{21}$$

and where

$$\alpha_i = 1/(x_{i+1} - x_i) \text{ and } \beta_i = \alpha_i(y_{i+1} - y_i) \text{ for } i = 0, \ldots, n.$$

It can be shown that the LCP (q, M), as defined above, has a unique solution. (See **3.3.7** and **3.11.10**.) This solution then yields the quantities $\{z_i\}_{i=1}^n$ which define the lower envelope of the convex hull. The upper envelope can be obtained in a similar manner. Thus, by solving two linear complementarity problems (having the same matrix M) we can find the convex hull of a finite set of points in the plane. In fact, the matrix M associated with this LCP has several nice properties which can be exploited to produce very efficient solution procedures. This will be discussed further in Chapter 4.

Nonlinear complementarity and variational inequality problems

The linear complementarity problem is a special case of the *nonlinear complementarity problem*, abbreviated NCP, which is to find an n-vector z such that

$$z \geq 0, \quad f(z) \geq 0, \quad \text{and} \quad z^{\mathrm{T}} f(z) = 0 \tag{22}$$

where f is a given mapping from R^n into itself. If $f(z) = q + Mz$ for all z, then the problem (22) becomes the LCP (q, M). The nonlinear complementarity problem (22) provides a unified formulation for nonlinear programming and many equilibrium problems. In turn, included as special cases of equilibrium problems are the traffic equilibrium problem, the spatial price equilibrium problem, and the n-person Nash-Cournot equilibrium problem. For further information on these problems, see Section 1.7.

There are many algorithms for solving the nonlinear complementarity problem (22). Among these is the family of linear approximation methods. These call for the solution of a sequence of linear complementarity problems, each of which is of the form

$$z \geq 0, \quad w = f(z^k) + A(z^k)(z - z^k) \geq 0, \quad z^{\mathrm{T}}w = 0$$

where z^k is a current iterate and $A(z^k)$ is some suitable approximation to the Jacobian matrix $\nabla f(z^k)$. When $A(z^k) = \nabla f(z^k)$ for each k, we obtain *Newton's method* for solving the NCP (22). More discussion of this method will be given in Section 7.4. The computational experience of many authors has demonstrated the practical success and high efficiency of this sequential linearization approach in the numerical solution of a large variety of economic and network equilibrium applications.

A further generalization of the nonlinear complementarity problem is the *variational inequality problem* Given a nonempty subset K of R^n and a mapping f from R^n into itself, the latter problem is to find a vector $x^* \in K$ such that

$$(y - x^*)^{\mathrm{T}} f(x^*) \geq 0 \quad \text{for all } y \in K. \tag{23}$$

We denote this problem by $\mathrm{VI}(K, f)$. It is not difficult to show that when K is the nonnegative orthant, then a vector x^* solves the above variational problem if and only if it satisfies the complementarity conditions in (22). In fact, the same equivalence result holds between the variational inequality problem with an *arbitrary* cone K and a certain *generalized complementarity problem.* See Proposition **1.5.2**.

Conversely, if K is a polyhedral set, then the above variational inequality problem can be cast as a nonlinear complementarity problem. To see this, write

$$K = \{x \in R^n : Ax \geq b, \; x \geq 0\}.$$

A vector x^* solves the variational inequality problem if and only if it is an optimal solution of the linear program

$$\begin{array}{ll} \text{minimize} & y^{\mathrm{T}} f(x^*) \\ \text{subject to} & Ay \geq b \\ & y \geq 0. \end{array}$$

By the duality theorem of linear programming, it follows that the vector x^* solves the variational inequality problem if and only if there exists a (dual) vector u^* such that

$$\begin{aligned} w^* &= f(x^*) - A^{\mathrm{T}}u^* \geq 0, \quad x^* \geq 0, \quad (w^*)^{\mathrm{T}}x^* = 0, \\ v^* &= -b + Ax^* \geq 0, \qquad\quad u^* \geq 0, \quad (v^*)^{\mathrm{T}}u^* = 0. \end{aligned} \tag{24}$$

The latter conditions are in the form of a nonlinear complementarity problem defined by the mapping:

$$F(x,u) = \begin{bmatrix} f(x) \\ -b \end{bmatrix} + \begin{bmatrix} 0 & -A^{\mathrm{T}} \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}. \tag{25}$$

In particular, when $f(x)$ is an affine mapping, then so is $F(x,u)$, and the system (24) becomes an LCP. Consequently, an affine variational inequality problem VI(K, f), where both K and f are affine, is completely equivalent to a certain LCP. More generally, the above conversion shows that in the case where K is a polyhedral set (as in many applications), the variational inequality problem VI(K, f), is equivalent to a nonlinear complementarity problem and thus can be solved by the aforementioned sequential linear complementarity technique.

Like the LCP, the variational inequality problem admits a rich theory by itself. The connections sketched above serve as an important bridge between these two problems. Indeed, some very efficient methods (such as the aforementioned sequential linear complementarity technique) for solving the variational inequality problem involve solving LCPs; conversely, some important results and methods in the study of the LCP are derived from variational inequality theory.

A closing remark

In this section, we have listed a number of applications for the linear complementarity problem (q, M), and have, in each case, pointed out the important properties of the matrix M. Due to the diversity of these matrix properties, one is led to the study of various matrix classes related to the LCP. The ones represented above are merely a small sample of the many that are known. Indeed, much of the theory of the LCP as well as many algorithms for its solution are based on the assumption that the matrix M belongs to a particular class of matrices. In this book, we shall devote a substantial amount of effort to investigating the relevant matrix classes, to examining their interconnections and to exploring their relationship to the LCP.

1.3 Complementary Matrices and Cones

Central to many aspects of the linear complementarity problem is the idea of a cone.

1.3.1 Definition. A nonempty set X in R^n is a *cone* if, for any $x \in X$ and any $t \geq 0$, we have $tx \in X$. (The origin is an element of every cone.) If a cone X is a convex set, then we say that X is a *convex cone*. The cone X is *pointed* if it contains no line. If X is a pointed convex cone, then an *extreme ray* of X is a set of the form $S = \{tx : t \in R_+\}$ where $x \neq 0$ is a vector in X which cannot be expressed as a convex combination of points in $X \setminus S$.

A matrix $A \in R^{m \times p}$ generates a convex cone (see **1.6.4**) obtained by taking nonnegative linear combinations of the columns of A. This cone, denoted $\operatorname{pos} A$, is given by

$$\operatorname{pos} A = \{q \in R^m : q = Av \text{ for some } v \in R^p_+\}.$$

Vectors $q \in \operatorname{pos} A$ have the property that the system of linear equations $Av = q$ admits a nonnegative solution v. The linear complementarity problem can be looked at in terms of such cones.

The set $\operatorname{pos} A$ is called a *finitely generated cone*, or more simply, a *finite cone.* The columns of the matrix A are called the *generators* of $\operatorname{pos} A$. When A is square and nonsingular, $\operatorname{pos} A$ is called a *simplicial cone.* Notice that in this case, we have

$$\operatorname{pos} A = \{q \in R^m : A^{-1}q \geq 0\}$$

for $A \in R^{m \times m}$; this gives an explicit representation of $\operatorname{pos} A$ in terms of a system of linear inequalities.

In the following discussion, we need certain matrix notation and concepts whose meaning can be found in the Glossary of Notation and in Section 2.2.

Let M be a given $n \times n$ matrix and consider the $n \times 2n$ matrix $(I, -M)$. In solving the LCP (q, M), we seek a vector pair $(w, z) \in R^{2n}$ such that

$$\begin{aligned} Iw - Mz &= q \\ w, z &\geq 0 \\ w_i z_i &= 0 \quad \text{for } i = 1, \ldots, n. \end{aligned} \tag{1}$$

This amounts to expressing q as an element of pos $(I, -M)$, but in a special way.

In general, when $q = Av$ with $v_i \neq 0$, we say the representation *uses* the column $A_{\bullet i}$ of A. Thus, in solving (q, M), we try to represent q as an element of pos $(I, -M)$ so that not both $I_{\bullet i}$ and $-M_{\bullet i}$ are used.

1.3.2 Definition. Given $M \in R^{n\times n}$ and $\alpha \subseteq \{1, \ldots, n\}$, we will define $C_M(\alpha) \in R^{n\times n}$ as

$$C_M(\alpha)_{\bullet i} = \begin{cases} -M_{\bullet i} & \text{if } i \in \alpha, \\ I_{\bullet i} & \text{if } i \notin \alpha. \end{cases} \tag{2}$$

$C_M(\alpha)$ is then called a *complementary matrix* of M (or a *complementary submatrix* of $(I, -M)$). The associated cone, pos $C_M(\alpha)$, is called a *complementary cone* (relative to M). The cone pos $C_M(\alpha)_{\bullet \bar{i}}$ is called a *facet* of the complementary cone $C_M(\alpha)$, where $i \in \{1, \ldots, n\}$. If $C_M(\alpha)$ is nonsingular, it is called a *complementary basis.* When this is the case, the complementary cone pos $C_M(\alpha)$ is said to be *full.* The notation $C(\alpha)$ is often used when the matrix M is clear from context.

In introducing the above terminology, we are taking some liberties with the conventional meaning of the word "submatrix." Strictly speaking, the matrix $C_M(\alpha)$ in (2) need not be a submatrix of $(I, -M)$, although an obvious permutation of its columns would be. Note that a complementary submatrix $C_M(\alpha)$ of M is a complementary basis if and only if the principal submatrix $M_{\alpha\alpha}$ is nonsingular.

For an $n \times n$ matrix M, there are 2^n (not necessarily all distinct) complementary cones. The union of such cones is again a cone and is denoted $K(M)$. It is easy to see that

$$K(M) = \{q : \mathrm{SOL}(q, M) \neq \emptyset\}.$$

Moreover, $K(M)$ always contains $\operatorname{pos} C_M(\emptyset) = R^n_+ = \operatorname{pos} I$ and $\operatorname{pos}(-M)$, and is contained in $\operatorname{pos}(I, -M)$, the latter being the set of all vectors q for which the LCP (q, M) is feasible. Consequently, we have

$$(\operatorname{pos} I \cup \operatorname{pos}(-M)) \subseteq K(M) \subseteq \operatorname{pos}(I, -M).$$

In general, $K(M)$ is not convex for an arbitrary matrix $M \in R^{n \times n}$; its convex hull is $\operatorname{pos}(I, -M)$.

Theoretically, given a vector q, to decide whether the LCP (q, M) has a solution, it suffices to check whether q belongs to one of the complementary cones. The latter is, in turn, equivalent to testing if there exists a solution to the system

$$\begin{aligned} C(\alpha)v &= q \\ v &\geq 0 \end{aligned} \tag{3}$$

for some $\alpha \subseteq \{1, \ldots, n\}$. (If $C(\alpha)$ is nonsingular, the system (3) reduces to the simple condition: $C(\alpha)^{-1}q \geq 0$.) Since the feasibility of the system (3) can be checked by solving a linear program, it is clear that there is a constructive, albeit not necessarily efficient, way of solving the LCP (q, M).

The approach just outlined presents no theoretical difficulty. The practical drawback has to do with the large number of systems (3) that would need to be processed. (In fact, that number already becomes astronomical when n is as small as 50.) As a result, one is led to search for alternate methods having greater efficiency.

1.3.3 Definition. The *support* of a vector $z \in R^n$, denoted $\operatorname{supp} z$, is the index set $\{i : z_i \neq 0\}$.

It is a well known fact that if a linear program has a given optimal solution, then there is an optimal solution that is an extreme point of the feasible set and whose support is contained in that of the given solution. Using the notion of complementary cones, one can easily derive a similar result for the LCP.

1.3.4 Theorem. If $z \in \text{SOL}(q, M)$, with $w = q + Mz$, then there exists a $\tilde{z} \in \text{SOL}(q, M)$, with $\tilde{w} = q + M\tilde{z}$, such that $\tilde{z}$ is an extreme point of $\text{FEA}(q, M)$ and the support of the vector pair (z, w) contains the support of $(\tilde{z}, \tilde{w})$, or more precisely,

$$\operatorname{supp} \tilde{z} \times \operatorname{supp} \tilde{w} \subseteq \operatorname{supp} z \times \operatorname{supp} w.$$

Proof. There must be a complementary matrix $C(\alpha)$ of M such that $v = z + w$ is a solution to (3). As in the aforementioned fact from linear programming, the system (3) has an extreme point solution whose support is contained in the support of $z + w$. This yields the desired result. □

This theorem suggests that since a solvable LCP must have an extreme point solution, a priori knowledge of an "appropriate" linear form (used as an objective function) would turn the LCP into a linear program. The trouble, of course, is that an appropriate linear form is usually not known in advance. However, in a later chapter, we shall consider a class of LCPs where such forms can always be found rather easily.

If the LCP (q, M) arises from the quadratic program (1.2.1), then we can extend the conclusion of Theorem **1.3.4**.

1.3.5 Theorem. If the quadratic program (1.2.1) has a locally optimal solution x, then there exist vectors y, u, and v such that (x, y, u, v) satisfies (1.2.2). Furthermore, there exist (possibly different) vectors $\tilde{x}$, $\tilde{y}$, $\tilde{u}$, $\tilde{v}$ where $\tilde{x}$ has the same objective function value in (1.2.1) as x and such that $(\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v})$ satisfies (1.2.2), forms an extreme point of the feasible region of (1.2.2), and has a support contained in the support of (x, y, u, v).

Proof. The first sentence of the theorem is just a statement of the Karush-Kuhn-Tucker theorem applied to the quadratic program (1.2.1). The second sentence follows almost entirely from Theorem **1.3.4**. The one thing we must prove is that x and $\tilde{x}$ have the same objective function value in (1.2.1).

From the complementarity conditions of (1.2.2), we know

$$\begin{aligned} x^{\mathrm{T}}u = y^{\mathrm{T}}v = 0\,, \\ \tilde{x}^{\mathrm{T}}\tilde{u} = \tilde{y}^{\mathrm{T}}\tilde{v} = 0\,. \end{aligned} \tag{4}$$

As the support of $(\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v})$ is contained in the support of (x, y, u, v), and all these vectors are nonnegative, we have

$$\begin{aligned} x^{\mathrm{T}}\tilde{u} = y^{\mathrm{T}}\tilde{v} = 0\,, \\ \tilde{x}^{\mathrm{T}}u = \tilde{y}^{\mathrm{T}}v = 0\,. \end{aligned} \tag{5}$$

Consequently, it follows from (1.2.2), (4) and (5) that

$$\begin{aligned}0 &= (x-\tilde{x})^{\mathrm{T}}(u-\tilde{u}) = (x-\tilde{x})^{\mathrm{T}}Q(x-\tilde{x}) - (x-\tilde{x})^{\mathrm{T}}A^{\mathrm{T}}(y-\tilde{y}) \\ &= (x-\tilde{x})^{\mathrm{T}}Q(x-\tilde{x}) - (v-\tilde{v})^{\mathrm{T}}(y-\tilde{y}) \\ &= (x-\tilde{x})^{\mathrm{T}}Q(x-\tilde{x}).\end{aligned}$$

Thus, for $\lambda \in R$ we have

$$f(x+\lambda(\tilde{x}-x)) = f(x) + \lambda(c+Qx)^{\mathrm{T}}(\tilde{x}-x),$$

which is a linear function in λ. Clearly, as the support of (x, y, u, v) contains the support of $(\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v})$, there is some $\epsilon > 0$ such that $x + \lambda(\tilde{x} - x)$ is feasible for (1.2.1) if $|\lambda| < \epsilon$. Thus, as $f(x + \lambda(\tilde{x} - x))$ is a linear function in λ and x is a local optimum of (1.2.1), it follows that $f(x + \lambda(\tilde{x} - x))$ is a constant function in λ. Letting $\lambda = 1$ gives $f(x) = f(\tilde{x})$ as desired. □

1.3.6 Remark. Notice, if x is a globally optimal solution, then so is $\tilde{x}$.

The following example illustrates the idea of the preceding theorem.

1.3.7 Example. Consider the (convex) quadratic program

$$\begin{array}{ll}\text{minimize} & \frac{1}{2}(x_1+x_2)^2 - 2(x_1+x_2) \\ \text{subject to} & x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0\end{array}$$

which has $(x_1, x_2) = (1, 1)$ as a globally optimal solution. In this instance we have $u_1 = u_2 = 0$, $y = 0$, and $v = 1$. The vector (x, y, u, v) satisfies the Karush-Kuhn-Tucker conditions for the given quadratic program which define the LCP below, namely

$$\begin{bmatrix} u_1 \\ u_2 \\ v \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix},$$

$$x,\ u,\ y,\ v \geq 0, \quad x^{\mathrm{T}}u = y^{\mathrm{T}}v = 0;$$

but this (x, y, u, v) is not an extreme point of the feasible region of the LCP because the columns

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

that correspond to the positive components of the vector (x, y, u, v) are linearly dependent. However, the vector $(\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}) = (2, 0, 0, 0, 0, 1)$ satisfies the same LCP system and is also an extreme point of its feasible region. Moreover, the support of $(\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v})$ is contained in the support of (x, y, u, v).

For a matrix M of order 2, the associated cone $K(M)$ can be rather nicely depicted by drawing in the plane, the individual columns of the identity matrix, the columns of $-M$, and the set of complementary cones formed from these columns. Some examples are illustrated in Figures 1.2 through 1.7. In these figures we label the (column) vectors $I_{\bullet 1}$, $I_{\bullet 2}$, $-M_{\bullet 1}$, and $-M_{\bullet 2}$ as 1, 2, $\bar{1}$, and $\bar{2}$, respectively. The complementary cones are indicated by arcs around the origin. The significance of these figures is as follows.

Figure 1.2 illustrates a case where (q, M) has a *unique solution* for every $q \in R^2$. Figure 1.3 illustrates a case where $K(M) = R^2$, but for some $q \in R^2$ the problem (q, M) has *multiple solutions*. Figure 1.4 illustrates a case where $K(M)$ is a *halfspace*. For vectors q that lie in one of the open halfspaces, the problem (q, M) has a unique solution, while for those which lie in the other open halfspace, (q, M) has no solution at all. All q lying on the line separating these halfspaces give rise to linear complementarity problems having infinitely many solutions. Figure 1.5 illustrates a case where every point $q \in R^2$ belongs to an *even number*—possibly zero—of complementary cones. Figure 1.6 illustrates a case where $K(M) = R^2$ and not all the complementary cones are full-dimensional. Figure 1.7 illustrates a case where $K(M)$ is *nonconvex*.

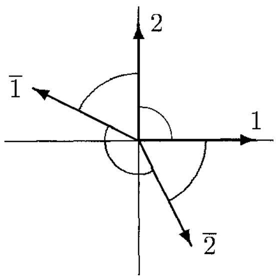

Figure 1.2

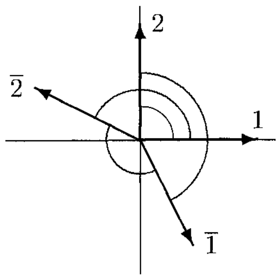

Figure 1.3

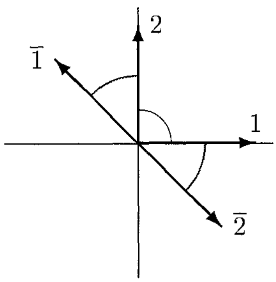

Figure 1.4

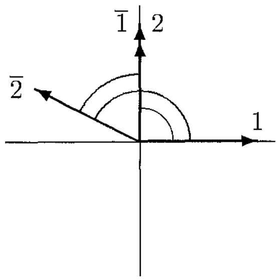

Figure 1.5

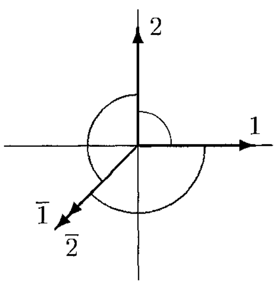

Figure 1.6

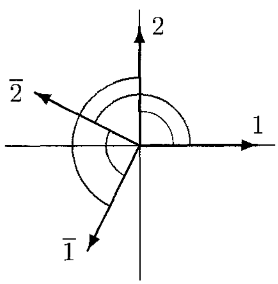

Figure 1.7

1.4 Equivalent Formulations

The linear complementarity problem admits a number of equivalent formulations, some of which have appeared in the previous sections. In what follows, we shall expand on these earlier discussions and introduce several new formulations. Many of these alternative formulations not only provide insights into the LCP, but also form the basis for the development of various methods for its solution.

Quadratic programming formulation

The connection between a quadratic program and the linear complementarity problem has been briefly mentioned in Section 1.2. In particular, we noted there that if the matrix M is symmetric, the LCP (q, M) constitutes the Karush-Kuhn-Tucker optimality conditions of quadratic programs with simple nonnegativity constraints on the variables, that is:

$$\begin{array}{ll} \text{minimize} & f(x) = q^{\mathrm{T}}x + \frac{1}{2}x^{\mathrm{T}}Mx \\ \text{subject to} & x \geq 0. \end{array} \tag{1}$$

If M is asymmetric, this relation between the LCP (q, M) and the quadratic program (1) ceases to hold. In this case, we can associate with the LCP (q, M) the following alternate quadratic program:

$$\begin{array}{ll} \text{minimize} & z^{\mathrm{T}}(q + Mz) \\ \text{subject to} & q + Mz \geq 0 \\ & z \geq 0. \end{array} \tag{2}$$

Notice that the objective function of (2) is always bounded below (by zero) on the feasible set. It is trivial to see that a vector z is a solution of the LCP (q, M) if and only if it is a global minimum of (2) with an objective value of zero.

In the study of the LCP, one normally does not assume that the matrix M is symmetric. The formulation (2) is useful in that it allows one to specialize the results from quadratic programming theory to the general LCP. On the other hand, the formulation (1) is valuable for the reverse reason; namely, it allows one to apply the results for a symmetric LCP to quadratic programs. Combined together, the two formulations (1) and (2) form a two-way bridge connecting the LCP and quadratic programming.

Fixed-point formulations

A *fixed-point* of a mapping $h : R^n \to R^n$ is a vector z such that $z = h(z)$. Obviously, finding a fixed-point of the mapping h is equivalent to finding a zero of the mapping $g(z) = z - h(z)$. Conversely, finding a zero of the mapping $g : R^n \to R^n$ is easily translated into finding a fixed point of the mapping $h(z) = z - g(z)$. We describe several ways of transforming a linear complementarity problem into either a fixed-point or zero-finding problem.

The simplest zero-finding formulation of the LCP (q, M) is gotten by defining

$$g(z) = \min(z, q + Mz) \tag{3}$$

where "min(a,b)" denotes the componentwise minimum of two vectors a and b. Obviously, a vector z is a solution of the LCP (q, M) if and only if $g(z) = 0$. The corresponding fixed-point formulation is defined by the mapping

$$h(z) = z - g(z) = \max(0, -q + (I - M)z). \tag{4}$$

Note that $h(z)$ can be interpreted as the projection of $-q + (I - M)z$ onto the nonnegative orthant.

There are several variations of the formulations (3) and (4). We mention one which is obtained by scaling the vectors z and $w = q + Mz$. Let D and E be two $n \times n$ diagonal matrices with positive diagonal entries. Define

$$\tilde{g}(z) = \min(Dz, E(q + Mz)). \tag{5}$$

Again, a vector $z \in \mathrm{SOL}(q, M)$ if and only if $\tilde{g}(z) = 0$. In this case the associated fixed-point mapping

$$\tilde{h}(z) = z - \tilde{g}(z) \tag{6}$$

can no longer be interpreted as a projection.

Another formulation of the LCP as a zero-finding problem involves the use of a strictly increasing function. We state this in the following result.

1.4.1 Proposition. Let $\theta : R \to R$ be any strictly increasing function such that $\theta(0) = 0$. A vector z^* will then solve the LCP (q, M) if and only if $H(z^*) = 0$ where $H : R^n \to R^n$ is defined by

$$H_i(z) = \theta(\,|(q + Mz)_i - z_i|\,) - \theta((q + Mz)_i) - \theta(z_i), \quad i = 1, \ldots, n. \tag{7}$$

Proof. Write $w^* = q + Mz^*$. Suppose that $H(z^*) = 0$. If $z_i^* < 0$ for some i, then since θ is strictly increasing, we have

$$0 > \theta(z_i^*) = \theta(\,|w_i^* - z_i^*|\,) - \theta(w_i^*) \geq -\theta(w_i^*).$$

Thus, $w_i^* > 0$ which implies $w_i^* - z_i^* > w_i^* > 0$. Consequently,

$$\theta(\,|w_i^* - z_i^*|\,) = \theta(w_i^* - z_i^*) > \theta(w_i^*)$$

contradicting the fact that $H_i(z^*) = 0$. Therefore, we must have $z^* \geq 0$. Similarly, one can deduce that $w^* \geq 0$. If $z_i^* > 0$ and $w_i^* > 0$ for some i, then by the strict increasing property of θ, we must have $H_i(z^*) < 0$, again a contradiction. Consequently, z^* solves the LCP (q, M). The converse is easy to prove. □

1.4.2 Remark. It is not difficult to see that by taking $\theta(t) = t$, the mapping H defined by (7) reduces to $H(z) = -2\min(z, q + Mz) = -2g(z)$ where g is the mapping in (3).

In general, the mappings g and H defined in (3) and (7) respectively are not Fréchet-differentiable. (See Section 2.1 for a review of differentiability concepts.) In connection with the differentiability of these mappings, we introduce the following terminology.

1.4.3 Definition. Let $H_{q,M}(z) = \min(z, q + Mz)$. A vector $z \in R^n$ is said to be *nondegenerate* with respect to $H_{q,M}$ if $z_i \neq (q + Mz)_i$ for each $i \in \{1, \ldots, n\}$.

If z is a solution of the LCP (q, M) which is a nondegenerate vector with respect to $H_{q,M}$, we call z a *nondegenerate solution* and say that *strict complementarity* holds at z. Note that if $z \in \text{SOL}(q, M)$, then z is a nondegenerate solution if and only if $z + q + Mz > 0$, i.e., if exactly n of the $2n$ components of the pair (z, w) are positive, where $w = q + Mz$. In this case $\operatorname{supp} z$ and $\operatorname{supp} w$ are complementary index sets in $\{1, \ldots, n\}$.

Clearly, if z is a nondegenerate vector with respect to $H_{q,M}$, then all vectors sufficiently close to z will also be nondegenerate in the same sense. In this case, the functions g and H in (3) and (7) become differentiable in a neighborhood of z (in the case of H, it is assumed from the start that θ is a differentiable function). An implication of this observation is that if the LCP (q, M) has a nondegenerate solution z, then some (locally convergent) gradient-based zero-finding algorithm (like Newton's method) can be used to approximate the solution z by finding a zero of the (now differentiable) mappings g and H.

In the above fixed-point (or zero-finding) formulations of the LCP, the coordinate space in which the LCP is defined is not altered. In particular, the results presented all state that a vector z solves the LCP if and only if the same vector z is a zero of a certain mapping. We next discuss an alternate formulation of the LCP which involves a change of variables. Define the mapping $f(x) = \sum_{i=1}^{n} f_i(x_i)$ where each

$$f_i(x_i) = \begin{cases} I_{\cdot i} x_i & \text{if } x_i \geq 0, \\ M_{\cdot i} x_i & \text{if } x_i \leq 0. \end{cases} \tag{8}$$

In other words, $f(x) = x^+ - Mx^-$. We have the following result.

1.4.4 Proposition. If $z^* \in \mathrm{SOL}(q, M)$, then the vector

$$x^* = w^* - z^* ,$$

where $w^* = q + Mz^*$, satisfies $f(x^*) = q$. Conversely, if x^* is a vector satisfying $f(x^*) = q$, then the vector $z^* = (x^*)^-$ solves the LCP (q, M).

Proof. Suppose $z^* \in \mathrm{SOL}(q, M)$. Write $w^* = q + Mz^*$. We have

$$\begin{aligned} q &= w^* - Mz^* \\ &= \sum_{i:w_i^*>0} I_{\cdot i} w_i^* - \sum_{i:z_i^*>0} M_{\cdot i} z_i^* \\ &= \sum_{i:x_i^*>0} f_i(x_i^*) + \sum_{i:x_i^*<0} f_i(x_i^*) \\ &= f(x^*). \end{aligned}$$

The reverse statement can be proved in a similar way. □

In essence, the above transformation of the LCP (q, M) into the system of equations $f(x) = q$ involves the combination of the $2n$ variables (z_i, w_i) into the n variables x_i by means of the complementarity condition $z_i w_i = 0$.

The transformation is quite similar to the idea of expressing q as an element of a complementary cone relative to M.

As a result of the equivalence given in Proposition **1.4.4**, $K(M)$ is the range of the above function $f : R^n \to R^n$ and $\text{SOL}(0, M)$ is its kernel. For this reason, we refer to $K(M)$ and $\text{SOL}(0, M)$ as the *complementary range* and *complementary kernel* of M, respectively.

Piecewise linear functions

The functions $g(z)$ in (3) and $f(z)$ in (8) are functions of a special kind; they are examples of piecewise affine functions. We formally introduce this class of functions in the following definition.

1.4.5 Definition. A function $f : \mathcal{D} \to R^m$ where $\mathcal{D} \subseteq R^n$ is said to be *piecewise linear (affine)* if f is continuous and the domain $\mathcal{D}$ is equal to the union of a finite number of convex polyhedra P_i, called the *pieces* of f, on each of which f is a linear (affine) function.

Thus, the preceding discussion has established that the LCP can be formulated as an equivalent square system of equations defined by a piecewise affine function. Such an equivalent formulation of the LCP has several analytical and computational benefits. For instance, in Section 5.8, a solution method for the LCP (q, M) will be developed that is based on the formulation

$$\min(z, q + Mz) = 0; \tag{9}$$

much of the geometric development in Chapter 6 will make heavy use of the formulation

$$q + Mz^- - z^+ = 0. \tag{10}$$

The relationship between the class of piecewise linear (affine) functions and the LCP extends beyond the formulation of the latter as a system of piecewise affine equations. As a matter of fact, the reverse of this formulation is also valid; that is to say, if $f : R^n \to R^n$ is an arbitrary piecewise affine function, then (under a mild "nonsingularity" assumption) the system $f(z) = 0$ is equivalent to a certain LCP (whose order is typically larger than n). Since the proof requires knowledge of conjugate dual functions and is not particularly relevant to the topics of this book, we refer the reader to **1.7.13** for more information.

Besides providing a useful formulation of the LCP, the class of piecewise affine functions is related to the LCP in another way. The next result shows that the solution of the LCP (q, M), if it exists and is unique, is a piecewise linear function of the constant vector q when the matrix M is kept fixed. Before stating this result, we point out that the class of matrices M for which (q, M) has a unique solution for all q plays a central role in the study of the LCP. This class of matrices will be formally defined and analyzed in Section 3.3.

1.4.6 Proposition. Let $M \in R^{n \times n}$ be such that the LCP (q, M) has a unique solution for all vectors $q \in R^n$. Then, the unique solution of the LCP (q, M) is a piecewise linear function in $q \in R^n$.

The proof of **1.4.6** consists of two parts: one part is to show the continuity of the (unique) solution as a function in the constant vector, and the other part is to exhibit the pieces and to establish the linearity of the solution on each piece. In turn, these can be demonstrated by means of a characterizing property of the matrix M satisfying the assumption of the proposition (see **3.3.7**). In what follows, we give a proof of the second part by exhibiting the required pieces of linearity, and refer the reader to Lemma **7.3.10** for the proof of continuity of the solution function.

Partial proof of 1.4.6. Suppose that M is such that the LCP (q, M) has a unique solution for all q. It follows from **3.3.7** that for each index set $\alpha \subseteq \{1, \ldots, n\}$, the principal submatrix $M_{\alpha\alpha}$ is nonsingular. For each such α, let

$$P_\alpha = \{q \in R^n : (M_{\alpha\alpha})^{-1} q_\alpha \leq 0,\ q_{\bar{\alpha}} - M_{\bar{\alpha}\alpha}(M_{\alpha\alpha})^{-1} q_\alpha \geq 0\}.$$

Then, this collection of convex polyhedral cones P_α for α ranging over all index subsets of $\{1, \ldots, n\}$ constitutes the required pieces of linearity of the solution function. As a matter of fact, if $z(q)$ is the unique solution to (q, M), then clearly $q \in P_{\operatorname{supp} z(q)}$ and, thus,

$$\bigcup_\alpha P_\alpha = R^n .$$

Moreover, it is easy to deduce the form of the solution function on the polyhedral cones P_α, In particular, we have

$$z(q)_\alpha = -(M_{\alpha\alpha})^{-1} q_\alpha,\ z(q)_{\bar{\alpha}} = 0 \quad \text{for } q \in P_\alpha. \ \square$$

1.5 Generalizations

The LCP admits a number of interesting generalizations. Two of these have already been introduced, namely, the nonlinear complementarity and the variational inequality problems. In particular, the affine variational inequality problem, i.e., the problem VI(K, f) where K is polyhedral and f is affine, is like a twin problem of the LCP in many respects.

In this section, we discuss a few more generalizations of the LCP. Our discussion is brief in each case. The intention is simply to draw the reader's attention to these generalized problems and to relate them to the LCP. Like the nonlinear complementarity and variational inequality problems, several of these generalizations have an extensive theory of their own, and it would not be possible for us to present them separately in this book.

Mixed LCPs

We start with the mixed linear complementarity problem which is defined as follows. Let A and B be real square matrices of order n and m respectively. Let $C \in R^{n\times m}$, $D \in R^{m\times n}$, $a \in R^n$ and $b \in R^m$ be given. The *mixed linear complementarity problem* is to find vectors $u \in R^n$ and $v \in R^m$ such that

$$\begin{aligned} a + Au + Cv &= 0 \\ b + Du + Bv &\geq 0 \\ v &\geq 0 \\ v^T(b + Du + Bv) &= 0. \end{aligned} \tag{1}$$

Thus, the mixed LCP is a mixture of the LCP with a system of linear equations. Note that the variable u which corresponds to the equation $a + Au + Cv = 0$ is not restricted to be nonnegative.

In the mixed LCP (1), if the matrix A is nonsingular, we may solve for the vector u, obtaining

$$u = -A^{-1}(a + Cv).$$

By eliminating u in the remaining conditions of the problem (1), we can convert this mixed LCP into the standard LCP (q, M) with

$$q = b - DA^{-1}a, \quad M = B - DA^{-1}C.$$

Consequently, in the case where A is nonsingular, the mixed LCP given in (1) can, in principle, be treated like a standard LCP; and hence, from a theoretical point of view, there is no particular advantage in a separate treatment of the mixed LCP. Nevertheless, computationally speaking, it may not be always advisable to actually solve the problem (1) by converting it to the equivalent LCP (q, M).

The mixed LCP provides a natural setting for the Karush-Kuhn-Tucker conditions of a quadratic program with general equality and inequality constraints. Indeed, consider the quadratic program

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x + \frac{1}{2}x^{\mathrm{T}}Qx \\ \text{subject to} & Ax \geq b \\ & Cx = d \end{array}$$

where $Q \in R^{n\times n}$ is symmetric, $c \in R^n$, $A \in R^{m\times n}$, $b \in R^m$, $C \in R^{l\times n}$ and $d \in R^l$. The Karush-Kuhn-Tucker conditions for this program are

$$\begin{aligned} 0 &= c + Qx - C^{\mathrm{T}}z - A^{\mathrm{T}}y, \\ 0 &= -d + Cx \\ v &= -b + Ax \geq 0, \quad y \geq 0, \quad y^{\mathrm{T}}v = 0 \end{aligned}$$

which we easily recognize as a mixed LCP with x and z as the free variables and y as the nonnegative variable.

Besides being an interesting generalization of the standard problem, the mixed LCP (1) plays an important role in the study of the standard LCP; several interesting properties of the latter LCP are characterized in terms of a certain mixed LCP derived from the given problem.

Another generalization of the LCP, which is somewhat related in spirit to the mixed LCP, is defined as follows. Let M and N be two $n\times n$ matrices, and let q be an n-vector. This generalized complementarity problem is to find vectors x and w such that

$$q + Mx - Nw = 0, \quad (x, w) \geq 0, \quad x^{\mathrm{T}}w = 0. \tag{2}$$

Clearly, the standard problem (q, M) corresponds to the case $N = I$. In general, if N is a nonsingular matrix, then the problem (2) can be converted into the equivalent problem $(N^{-1}q, N^{-1}M)$.

Complementarity problems over cones

We next discuss a generalization of the nonlinear complementarity problem which turns out to be a special case of the variational inequality problem. This generalization involves the replacement of the nonnegative orthant, which is the principal domain of definition for the NCP, by an arbitrary cone. In order to define the complementarity problem over a cone, we first introduce a useful concept associated with an arbitrary subset of R^n.

1.5.1 Definition. Let K be a nonempty set in R^n. The *dual cone* of K, denoted by K^*, is defined as the set

$$K^* = \{y \in R^n : x^{\mathrm{T}}y \geq 0 \text{ for all } x \in K\}.$$

A vector y in this dual cone K^* is characterized by the property that it does not make an obtuse angle with any vector in K. It is easy to see that K^* must be a (closed) convex cone for an arbitrary set K; indeed, K^* is the intersection of (possibly infinitely many) closed halfspaces:

$$K^* = \bigcap_{x \in K} \{y \in R^n : x^{\mathrm{T}}y \geq 0\}.$$

We now define the *complementarity problem over a cone*. Given a cone K in R^n and a mapping f from R^n into itself, this problem, denoted by $\mathrm{CP}(K, f)$, is to find a vector $z \in K$ such that

$$f(z) \in K^*, \quad \text{and} \quad z^{\mathrm{T}}f(z) = 0.$$

Geometrically, this problem seeks a vector z belonging to the given cone K with the property that its image under the mapping f lies in the dual cone of K and which is orthogonal to z.

Since the nonnegative orthant is *self-dual*, i.e., $(R^n_+)^* = R^n_+$, it is easy to see that the problem $\mathrm{CP}(R^n_+, f)$ reduces to the NCP given in (1.2.22). What is less evident is the fact that with K being a cone, the two problems, $\mathrm{CP}(K, f)$ and $\mathrm{VI}(K, f)$, have the same solution set. We state this result more formally as follows.

1.5.2 Proposition. Let K be a cone in R^n, and let f be a mapping from R^n into itself. Then, a vector z^* solves $\mathrm{CP}(K, f)$ if and only if z^* solves $\mathrm{VI}(K, f)$.

Proof. Suppose that z^* solves VI(K, f). Since $0 \in K$, by substituting $y = 0$ into the inequality (1.2.23), we have $f(z^*)^{\mathrm{T}} z^* \leq 0$. On the other hand, since $z^* \in K$ and K is a cone, $2z^* \in K$. By substituting $y = 2z^*$ into (1.2.23), we obtain $f(z^*)^{\mathrm{T}} z^* \geq 0$. Consequently, $f(z^*)^{\mathrm{T}} z^* = 0$. To complete the proof that z^* solves CP(K, f), it remains to be shown that $f(z^*)^{\mathrm{T}} y \geq 0$ for all $y \in K$. But this follows easily from (1.2.23) because we have already proved $f(z^*)^{\mathrm{T}} z^* = 0$. The converse is trivial. □

1.5.3 Remark. The only property of K used in the above proof is the cone feature, i.e., the implication $[z \in K \Rightarrow \tau z \in K]$ for all $\tau \in R_+$. In particular, K need not be convex.

The vertical generalization

While the complementarity problem over a cone may be considered a geometric generalization of the LCP, the next generalization is somewhat more algebraic. Let M be a (rectangular) matrix of order $m \times n$ with $m \geq n$, and let q be an m-vector. Suppose that M and q are partitioned in the following form

$$M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix}, \quad q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix},$$

where each $M_i \in R^{m_i \times n}$ and $q_i \in R^{m_i}$ with $\sum_{i=1}^{n} m_i = m$. The *vertical linear complementarity problem* is to find a vector $z \in R^n$ such that

$$\begin{aligned} q + Mz &\geq 0 \\ z &\geq 0 \\ z_i \, \Pi_{j=1}^{m_i} (q_i + M_i z)_j &= 0 \quad i = 1, \ldots, n. \end{aligned} \tag{3}$$

When $m_i = 1$ for each i, this reduces to the standard LCP (q, M).

Just like the standard LCP, the above vertical LCP also bears a close relationship to a certain system of piecewise affine equations. Indeed, define the mapping $H : R^n \to R^n$ with the i-th component function given by

$$H_i(z) = \min(z_i, (q_i + M_i z)_1, \ldots, (q_i + M_i z)_{m_i}), \quad i = 1, \ldots, n.$$

It is not difficult to see that this mapping H is piecewise affine; moreover, a vector z solves the problem (3) if and only if z is a zero of H.

The above vertical LCP points to one difference in the two formulations, (1.4.9) versus (1.4.10) of the LCP (q, M) as a system of piecewise affine equations. The "min" formulation can be easily extended to the vertical LCP in which the complementarity relationship in each component may involve any finite number of affine functions, whereas the (z^+, z^-) formulation is more akin to the standard LCP and not so easily amenable for extension to handle more complicated complementarity conditions.

In principle, we could define a *horizontal linear complementarity problem* involving a (rectangular) matrix $M \in R^{n\times m}$ with $n \leq m$ and a vector $q \in R^n$. We omit this generalization, but point out that the problem (2) belongs to this category of generalized complementarity problems.

The implicit complementarity problem

The next generalization of the LCP is called the *implicit complementarity problem* and is defined as follows. Let $A \in R^{n\times n}$, $a \in R^n$ and $h : R^n \to R^n$ be given. This problem is to find a vector $z \in R^n$ such that

$$\begin{aligned} a + Az &\geq 0 \\ z &\geq h(z) \\ (a + Az)^{\mathrm{T}}(z - h(z)) &= 0. \end{aligned} \tag{4}$$

Clearly, this problem includes as special cases the standard LCP (which corresponds to $h = 0$) and the nonlinear complementarity problem (which corresponds to $a = 0$, $A = I$ and $h(z) = z - f(z)$). Another instance of the problem in (4) is the optimal stopping problem discussed in Section 1.2) (cf. the conditions in (1.2.19) which correspond to $a = 0$, $A = I - P$ and $h(z) = r$). There, by means of a simple translation of variables, the problem was converted into a standard LCP.

Generalizing the change-of-variable idea in the optimal stopping problem, we can define a (multivalued) mapping associated with the implicit complementarity problem (4) that ties it closer to the standard LCP. This mapping $\mathcal{S} : R^n \to R^n$ is defined as follows. For each given vector $u \in R^n$, $\mathcal{S}(u)$ is the (possibly empty) solution set of the complementarity system

$$\begin{aligned} a + Az &\geq 0 \\ z &\geq h(u) \\ (a + Az)^{\mathrm{T}}(z - h(u)) &= 0. \end{aligned} \tag{5}$$

By defining $z = z' + h(u)$, it is easy to see that the latter problem is equivalent to the LCP $(a + Ah(u), A)$ with z' as the variable. As u varies, $(a + Ah(u), A)$ becomes an example of a *multivariate parametric linear complementarity problem*; the vector u is considered as the *parameter* of this problem.

The implicit complementarity problem has some important applications in the study of free-boundary problems arising from mechanics and physics where the problem is typically posed in an infinite-dimensional setting. The finite-dimensional version as defined above is the outcome of a *discretization process* which is designed as an approximation of the infinite-dimensional problem for the purpose of numerical solution. The terminology "implicit complementarity" was coined as a result of the mapping $\mathcal{S}(\cdot)$ which, in most cases, is only implicitly defined and does not have an explicit expression. It is obvious that a vector z is a solution of (4) if and only if z is a fixed point of $\mathcal{S}$, i.e., if

$$z \in \mathcal{S}(z). \tag{6}$$

Of course, when the system (5) has a unique solution for each vector u, then $\mathcal{S}$ becomes a single-valued function, and the relation (6) reduces to the nonlinear equation

$$z = \mathcal{S}(z).$$

In the literature, the study of the implicit complementarity problem given in (4) has been facilitated by consideration of this last equation and its multivalued analog (6).

1.6 Exercises

1.6.1 An important problem in mathematical programming is that of finding a point in a polyhedron

$$P = \{x \in R^n : Ax \geq b\}$$

which is closest to a given vector $y \in R^n$. This problem may be stated as the quadratic program:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}(x-y)^{\mathrm{T}}(x-y) \\ \text{subject to} & Ax \geq b. \end{array}$$

(a) Formulate this program as an equivalent LCP (q, M) with a symmetric positive semi-definite matrix M and a certain vector q.

(b) Conversely, let (q, M) be a given LCP with a symmetric positive semi-definite matrix M. Show that (q, M) is equivalent to the problem of finding a point in a certain polyhedron that is closest to the origin.

1.6.2 A variant of the nonnegatively constrained quadratic program given in (1.2.4) is the box constrained quadratic program

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x + \frac{1}{2}x^{\mathrm{T}}Qx \\ \text{subject to} & a \geq x \geq 0 \end{array} \tag{1}$$

where $Q \in R^{n\times n}$ is symmetric, c and $a \in R^n$, and $a > 0$. Let (q, M) be the LCP formulation of the Karush-Kuhn-Tucker conditions of the program (1). Show that (q, M) is strictly feasible. Does (q, M) always have a solution? Why?

1.6.3 Let $\{(x_i, y_i)\}_{i=0}^{n+1}$ be a collection of points in the plane. We wish to fit these $n+2$ points with a piecewise linear convex function $f(x)$ so as to minimize the squared error between $f(x)$ and the given points. Specifically, let $\{(x_i, v_i)\}_{i=0}^{n+1}$ be the breakpoints of $f(x)$, where $v_i = f(x_i)$ for each i. The problem is to find the ordinates $\{v_i\}_{i=0}^{n+1}$ in order to

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\sum_{i=0}^{n+1}(v_i - y_i)^2 \\ \text{subject to} & \dfrac{v_{i+1}-v_i}{x_{i+1}-x_i} \geq \dfrac{v_i - v_{i-1}}{x_i - x_{i-1}}, \quad i = 1, \ldots, n. \end{array}$$

(a) Formulate this quadratic program as an equivalent LCP (q, M') with $M' \in R^{n \times n}$. Identify as many properties of M' as you can.

(b) Let M denote the defining matrix of the LCP obtained from the problem of finding the lower envelope of the convex hull of the same $n+2$ points $\{(x_i, y_i)\}_{i=0}^{n+1}$, see (1.2.21). Show that

$$M' = \alpha_0^2 u_1 u_1^{\mathrm{T}} + M^2 + \alpha_n^2 u_n u_n^{\mathrm{T}}$$

where $\alpha_i = 1/(x_{i+1} - x_i)$ for $i = 0, \ldots, n$, and $u_k = I_{\bullet k}$, the k-th coordinate vector.

1.6.4 Given $A \in R^{m \times p}$, show that $\operatorname{pos} A$ is a convex cone. Show that if x generates an extreme ray of $\operatorname{pos} A$, then x is a positive multiple of a column of A. Show that if the rank of A equals p, then each column of A generates an extreme ray of $\operatorname{pos} A$.

1.6.5 Describe the solution set of the LCP (q, M) with each of the following matrices

$$M_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$$

for arbitrary vectors q. Determine, for each matrix, those vectors q for which the solution set is connected. Determine, for each matrix, those vectors q for which the solution set is convex. Can you say anything else about the structure of the solution sets?

1.6.6 Determine which of the following two statements is true. Give a proof for the true statement and a counterexample for the false statement.

(a) The LCP (q, M) is strictly feasible if there exists a vector z such that

$$z \geq 0, \quad q + Mz > 0.$$

(b) The LCP (q, M) is strictly feasible if there exists a vector z such that

$$z > 0, \quad q + Mz \geq 0.$$

1.6.7 Prove that if the vector $x \in R^n$ is a solution of (15), and if there exists a vector $y \in R^m$ satisfying (16) such that the vector $(y, x) \in R^{m+n}$ solves the LCP (q, M) where q and M are given in (1.2.17), then x is an optimal invariant capital stock.

1.6.8 Consider the problem $\text{CP}(K, f)$ where $K = \text{pos}\, A$ and $f(z) = q + Mz$. Here, $A \in R^{n \times m}$, $M \in R^{n \times n}$ and $q \in R^n$.

(a) Show that this complementarity problem is equivalent to an LCP of order m.

(b) More generally, consider the problem $\text{VI}(K, f)$ where f is as given above and $K = \text{pos}\, A + H$ where H is the convex hull of p vectors in R^n. Show that this variational inequality problem is equivalent to a mixed LCP of order $m + p + 1$ with one equality constraint and one free variable corresponding to it.

1.6.9 The cone $K(M)$ plays an important role in linear complementarity theory. This exercise concerns two elementary properties of $K(M)$.

(a) Prove or disprove: if $M \in R^{2\times 2}$ and if $K(M)$ is convex, then so is $K(M + \varepsilon I)$ for all $\varepsilon > 0$ sufficiently small.

(b) Suppose $M \in R^{n\times n}$ is nonsingular. How are the two cones $K(M)$ and $K(M^{-1})$ related to one another? For a given $q \in K(M)$, state how this relation pertains to the two LCPs (q, M) and (q', M^{-1}) for some q'?

1.7 Notes and References

1.7.1 The name of the problem we are studying in this book has undergone several changes. It has been called the "composite problem," the "fundamental problem," and the "complementary pivot problem." In 1965, the current name "linear complementarity problem" was proposed by Cottle. It was later used in a paper by Cottle, Habetler and Lemke (1970a). Probably the earliest publication containing an explicitly stated LCP is one by Du Val (1940). This paper, part of the literature of algebraic geometry, used a problem of the form (q, M) to find the *least element* (in the vector sense) of the linear inequality system $q + Mz \geq 0$, $z \geq 0$. Ordinarily, problems of this sort have no solution, but when the matrix M has special properties, a solution exists and is unique. The theory pertaining to problems with these properties is developed in Section **3.11**.

1.7.2 In the very first sentence of the chapter, we described the linear complementarity problem as a system of *inequalities* and then proceeded to write condition (1.1.3) as an *equation!* Two clarifications of this statement can be given. First, an equation in real numbers is always equivalent to a *pair* of inequalities. Second, and more to the point, the inequalities (1.1.1) and (1.1.2) imply $z^{\mathrm{T}}(q+Mz) \geq 0$; so once these conditions are met, (1.1.3) is satisfied if the inequality $z^{\mathrm{T}}(q+Mz) \leq 0$ holds. Imposing the latter in place of (1.1.3), one has a genuine inequality system.

1.7.3 Linear complementarity problems in the context of convex quadratic programming can be found in the work of Hildreth (1954, 1957), Barankin and Dorfman (1955, 1956, 1958). They also appear in the paper by Frank and Wolfe (1956). All these papers make use of the seminal paper of Kuhn and Tucker (1951) and the (then unknown) Master's Thesis of Karush (1939). [For an interesting historical discussion of the latter work, see Kuhn (1976).]

1.7.4 In elementary linear programming, one learns about finite, two-person, zero-sum games and how to solve them by the simplex method. Such games are usually called *matrix games*. When the zero-sum feature is dropped, one gets what are called *bimatrix games*. In place of the minimax criterion used in matrix games, the theory of bimatrix games uses the concept of a *Nash equilibrium point*. See Nash (1950, 1951). Nash's proof of the existence of equilibrium points in non-cooperative games was based upon the (nonconstructive) Brouwer fixed point theorem. The Lemke-Howson method (Algorithm **4.4.21**), which first appeared in Howson (1963) and Lemke and Howson (1964), is an efficient constructive procedure for obtaining a Nash equilibrium point of a bimatrix game. The Lemke-Howson algorithm was not the first constructive procedure to have been proposed. Others—such as Vorob'ev (1958), Kuhn (1961), and Mangasarian (1964)—had suggested algorithms for the problem, but none of these proposals match the simplicity and elegance of the Lemke-Howson method. We cover the Lemke-Howson algorithm in Chapter 4. Balas (1981) discussed an interesting application of bimatrix games in the context of sizing the strategic petroleum reserve.

1.7.5 The synthesis of linear programming, quadratic programming and bimatrix games as instances of the "fundamental problem" (see **1.7.1**) was presented in Cottle and Dantzig (1968). Earlier publications such as Cottle

(1964b) and Dantzig and Cottle (1967) also emphasized this synthesis, but they included no discussion of the bimatrix game problem.

1.7.6 The investigation of market equilibrium problems has a relatively long tradition. A paper by Samuelson (1952) did much to encourage the analysis of (partial) equilibrium problems for spatially separated markets through mathematical programming methods. Thus, linear and quadratic programming—and eventually linear complementarity—entered the picture. Some of this history is recounted in the book by Takayama and Judge (1971). The monograph by Ahn (1979) discussed applications of market equilibrium concepts in policy analysis. A recent collection of papers on the formulation and solution of (computable) equilibrium problems illustrates the applicability of the LCP in the field of economics. See Manne (1985). The issue of integrability is discussed in Carey (1977).

1.7.7 The formulation of the optimal invariant capital stock problem presented here is based on a paper by Dantzig and Manne (1974), which was, in turn, inspired by Hansen and Koopmans (1972). For further work on the subject, see Jones (1977, 1982).

1.7.8 For discussions of the optimal stopping problem, see Çinlar (1975) and Cohen (1975). Problems of the sort we have described possess special properties that lend themselves to solution by methods presented in this book.

1.7.9 Our discussion of finding the convex hull of a set of points in the plane is based on a note by Chandrasekaran (1989). An algorithm that achieves $O(n \log n)$ time complexity was published by Graham (1972). LCPs of this form will be studied in Chapter 4. Exercise **4.11.10** shows that this convex-hull problem can be solved in $O(n)$ time by Algorithm **4.7.3**, provided that the x_i's are pre-sorted.

1.7.10 The variational inequality problem first appeared in Stampacchia (1964). The system we now know as the nonlinear complementarity problem was identified by Cottle (1964a). Soon afterwards, the relationship between these two problems was pointed out by Karamardian (1972). Generalizations of the complementarity problem were also pioneered by Karamardian (1971). For collections of papers dealing with variational inequalities and complementarity problems see Cottle, Giannessi and Lions (1980) and Cottle, Kyparisis and Pang (1990). The latter volume contains an extensive survey of (finite-dimensional) variational inequality and nonlinear complementarity problems by Harker and Pang (1990a).

1.7.11 The sequential linearization approach for solving variational inequality and nonlinear complementarity problems is related to an idea of Wilson (1962) who used it in the context of nonlinear programming. Starting with technical reports by Josephy (1979a, 1979b, 1979c), the linearization algorithms have been studied rather extensively from the theoretical and practical points of view. Josephy's treatment of these algorithms was carried out in the framework of "generalized equations" as introduced by Robinson (1979, 1980). Related work can be found in Pang and Chan (1982) and Eaves (1983). Included among the more noteworthy applications of these algorithms to the computation of general economic equilibria, are the work of Mathiesen (1985a, 1985b), Rutherford (1986) and Stone (1988).

1.7.12 The interpretation of the LCP in terms of complementary cones can be traced to Samelson, Thrall and Wesler (1958), a paper that has great importance for other reasons as well. (See Section 3.3.) The topic was significantly enlarged by Murty (1972).

1.7.13 The formulation set forth in Proposition **1.4.1** is due to Mangasarian (1976c). It was originally developed for the nonlinear complementarity problem. Algorithmic work based on this formulation includes Subramanian (1985), Watson (1979), and Watson, Bixler and Poore (1989). The function $H_{q,M}$ defined in (1.4.3) was used by Aganagić (1978b). The source of this formulation is somewhat obscure. The mapping used in Proposition **1.4.4** can be traced back to a paper of Minty (1962) dealing with monotone operator theory. The piecewise linear formulation (1.4.10) has been used extensively. See, for example, Eaves (1976), Eaves and Scarf (1976), and Garcia, Gould and Turnbull (1981, 1983). The equivalence of the LCP and a system of piecewise linear equations was the subject of discussion in the two papers of Eaves and Lemke (1981, 1983).

1.7.14 The "vertical generalization" (see (1.5.3)) was introduced in Cottle and Dantzig (1970), but was not designated as such. Rather little has been done with this model. An application in tribology by Oh (1986) and another in stochastic control by Sun (1989) have been reported in the literature, however. De Moor (1988) utilizes the "horizontal generalization" of the LCP in his study on modeling piecewise linear electrical networks.

1.7.15 The mixed LCP (1.5.1) is a special case of a complementarity problem over the cone $R^n \times R^m_+$. In the case where the matrix A is singular, the conversion of the problem (1.5.1) into a standard LCP is not an entirely straightforward matter.

1.7.16 Capuzzo Dolcetta (1972) was among the first to investigate the implicit complementarity problem. For other studies, see Pang (1981d, 1982) and Yao (1990).

Chapter 2

BACKGROUND

Our main purpose in this chapter is to collect the essential background materials needed for the rest of the book. Four major topics are covered: linear algebra and matrix theory, elements of real analysis, linear inequalities and programming, and quadratic programming theory. Although much of the development given in this chapter could easily be made more general, our discussion on each topic is by necessity, brief, and intentionally limited to those aspects relevant to the linear complementarity problem. There are excellent textbooks written on the subjects reviewed here. Among those on linear algebra and matrix theory are the two volumes by Horn and Johnson (1985, 1990). The classic by Ortega and Rheinboldt (1970) remains an excellent reference on functions of several variables. Many books are available for the theory of linear programming and inequalities; the one by Dantzig (1963) is a classical treatise, whereas the one by Murty (1983) gives a more contemporary treatment. Finally, the forthcoming book by Cottle on quadratic programming is intended to provide a comprehensive discussion of the subject.

2.1 Real Analysis

In this section, we review elements of real analysis that are important for the linear complementarity problem. Topics covered include elementary point-set topology, basic properties of multivariate functions, some mapping theorems, concepts of multivalued mappings, etc. The proofs of all the results stated in this section are omitted. The reader is asked to supply some of the proofs in the exercises at the end of the chapter.

Vector norms

Although the topics of topology and real analysis can each be developed in an abstract framework without relying on the finite-dimensional nature and other specialized properties of the Euclidean space R^n, we base our discussion in this review on the concept of a vector norm in R^n. The norm of a vector provides a measure of the magnitude of the vector, and it can be used to define the distance between two vectors in R^n.

2.1.1 Definition. A *vector norm* is a function $\|\cdot\| : R^n \to R_+$ satisfying the properties for all vectors $x, y \in R^n$,

(a) $\|x\| = 0$ if and only if $x = 0$, (Positivity)

(b) $\|\lambda x\| = |\lambda|\|x\|$ for all scalars λ, (Homogeneity)

(c) $\|x + y\| \leq \|x\| + \|y\|$, (Triangle inequality).

Given a vector norm $\|\cdot\|$ in R^n, the *unit sphere* is the set

$$\mathcal{S} = \{x \in R^n : \|x\| = 1\},$$

and the *(closed) unit ball* is

$$\mathcal{B} = \{x \in R^n : \|x\| \leq 1\}.$$

Clearly, $0 \in \mathcal{B} \setminus \mathcal{S}$.

2.1.2 Example. Some frequently used norms are

$\|x\|_2 = (\sum_{i=1}^n x_i^2)^{1/2}$	Euclidean norm or (l_2-norm)
$\|x\|_1 = \sum_{i=1}^n \|x_i\|$	sum norm or (l_1-norm)
$\|x\|_\infty = \max_{i=1}^n \|x_i\|$	max norm or (l_∞-norm).

The reader can easily verify that these are vector norms satisfying the three axioms (a), (b) and (c) in Definition **2.1.1**.

The Euclidean norm is derived from the usual Euclidean inner product; that is,

$$\|x\|_2^2 = x^{\mathrm{T}}x.$$

Moreover, the well-known *Cauchy-Schwartz inequality* holds for any two vectors x and y in R^n,

$$|x^{\mathrm{T}}y| \leq \|x\|_2 \; \|y\|_2$$

with equality holding if and only if x and y are linearly dependent vectors.

The Euclidean norm admits a generalization. Let A be an arbitrary $n \times n$ symmetric positive definite matrix (see Section 2.2 for a review of matrices of this kind). The *elliptic norm* or *A-norm* is defined as

$$\|x\|_A = (x^{\mathrm{T}}Ax)^{1/2}, \quad x \in R^n. \tag{1}$$

The reader is asked to verify that this defines a vector norm, see Exercise **2.10.5**. When A is the identity matrix, the A-norm becomes the Euclidean norm.

All vector norms on R^n are *equivalent*; this means that if $\|\cdot\|$ and $\|\cdot\|'$ are two given norms, then there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\|x\| \leq \|x\|' \leq c_2\|x\| \tag{2}$$

for all $x \in R^n$. For the three norms—the Euclidean norm, the sum norm and the max norm—these constants are easy to obtain.

It follows immediately from (2) that if $\{x^\nu\}$ is a sequence of vectors in R^n, then

$$\lim_{\nu\to\infty} \|x^\nu\| = 0 \quad \Leftrightarrow \quad \lim_{\nu\to\infty} \|x^\nu\|' = 0. \tag{3}$$

In view of this equivalence, one may use any vector norm in defining the concept of convergence of a sequence of vectors.

2.1.3 Definition. A sequence of vectors $\{x^\nu\} \subset R^n$ is said to *converge* to the vector $x^* \in R^n$ if

$$\lim_{\nu\to\infty} \|x^\nu - x^*\| = 0$$

for some vector norm $\|\cdot\|$. The vector x^* is called the *limit* of the sequence $\{x^\nu\}$; we write

$$x^* = \lim_{\nu\to\infty} x^\nu.$$

The sequence $\{x^\nu\}$ is said to be *convergent* if it has a limit. The limit of a convergent subsequence of the sequence $\{x^\nu\}$ is called an *accumulation point* (or *limit point*) of $\{x^\nu\}$.

2.1.4 Remark. The limit of a convergent sequence must be unique.

According to the equivalence (3), it follows that a sequence of vectors converges to a limit under a given vector norm if and only if it converges to the same limit under any other norm. This statement can actually be used to characterize the equivalence of vectors norms on R^n. More precisely, it can be shown (see Exercise **2.10.6**) that two vector norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if and only if the condition (3) holds for all sequences $\{x^\nu\}$ in R^n.

2.1.5 Definition. A vector norm $\|\cdot\|$ on R^n is said to be *monotone* if for all $x, y \in R^n$,

$$|x_i| \le |y_i| \text{ for all } i \quad \Rightarrow \quad \|x\| \le \|y\|,$$

and *absolute* if for all $x \in R^n$, $\|x\| = \|\,|x|\,\|$ where $|x|$ is the vector whose i-th component is equal to $|x_i|$.

2.1.6 Proposition. A vector norm $\|\cdot\|$ on R^n is monotone if and only if it is absolute. □

The Euclidean norm, the sum norm and the max norm are all monotone. The elliptic norms are in general not monotone (see Exercise **2.10.5**).

Point set topology

In this subsection, $\|\cdot\|$ denotes a given vector norm defined on R^n. We start our review of various properties of point sets in R^n with the following definition.

2.1.7 Definition. A *neighborhood* of a point $x \in R^n$ is the *open ball* $B(x, r)$ with *center* at x and *radius* $r > 0$, that is, the set

$$B(x, r) = \{y \in R^n : \|y - x\| < r\}.$$

A *closed neighborhood* of x is the (closed) ball $\text{cl}\, B(x,r)$ defined by

$$\text{cl}\, B(x,r) = \{y \in R^n : \|y - x\| \leq r\}.$$

In many parts of this book, we shall omit the pair (x,r) from the notation $B(x,r)$; it is understood that whenever we speak of a neighborhood of a point, there is always a radius associated with the neighborhood. Also, the closed neighborhood of a point is seldom used. Vectors in a neighborhood of a point x are said to be *close to* x. The phrase "vectors *sufficiently close* (or *close enough*) to a point x" refers to those vectors that lie in a certain neighborhood of x whose radius is, typically, very small.

2.1.8 Definition. A subset $X \subseteq R^n$ is said to be

(a) *open* if for every point $x \in X$, there exists a neighorhood V of x such that $V \subseteq X$,

(b) *closed* if its complement X^c is open,

(c) *bounded* if there exists a scalar $r > 0$ such that $\|x\| \leq r$ for all $x \in X$,

(d) *compact* if X is both closed and bounded.

2.1.9 Remark. It is important to point out that although we have defined the various properties of the set X in terms of a given vector norm, these properties are really independent of the norm used because of the equivalence of all vector norms on R^n. Throughout the subsequent definitions, we shall continue to adopt this practice which the reader is asked to keep in mind.

The closedness of a set can be characterized in terms of sequences. Specifically, a set $X \subseteq R^n$ is closed if and only if the limit of every convergent sequence $\{x^\nu\} \subseteq X$ lies in X. A related characterization exists for the compactness of a set; namely, a set $X \subseteq R^n$ is compact if and only if every sequence $\{x^\nu\} \subseteq X$ contains a convergent subsequence whose limit belongs to X.

In general, a sufficient condition for a sequence $\{x^\nu\} \subset R^n$ to have an accumulation point is that it is bounded. There are various criteria for the sequence $\{x^\nu\}$ to actually converge. For instance, if $\{x^\nu\}$ is bounded and has a unique accumulation point, then it converges. The following result, due to Ostrowski, generalizes this fact. The reader is asked to prove this result in an exercise.

2.1.10 Theorem. Suppose that the sequence $\{x^\nu\} \subset R^n$ satisfies the following three properties:

(a) it is bounded,

(b) $\lim_{\nu\to\infty} \|x^{\nu+1} - x^\nu\| = 0$,

(c) it has only a finite number of accumulation points.

Then, $\{x^\nu\}$ converges. □

We review a few more concepts associated with a subset of R^n.

2.1.11 Definition. Let $S \subseteq R^n$. The *(topological) interior* of S, denoted $\operatorname{int} S$, is the subset of S consisting of vectors x for which there exists a neighborhood N (of x) contained entirely in S. The *(topological) closure* of S, denoted $\operatorname{cl} S$, consists of all vectors which are limits of sequences of vectors from S. The *(topological) boundary* of S, denoted $\operatorname{bd} S$, is the set $\operatorname{cl} S \setminus \operatorname{int} S$. A subset T of a set S is *dense* in S if $S \subseteq \operatorname{cl} T$.

Clearly, we have

$$\operatorname{int} S \subseteq S \subseteq \operatorname{cl} S;$$

moreover, $\operatorname{int} S = S$ if and only if S is open, and $\operatorname{cl} S = S$ if and only if S is closed. In general, the union (intersection) of a finite family of closed (open) sets is closed (open) , and the union (intersection) of an arbitrary family of open (closed) sets is open (closed).

Some subsets of R^n have an important property called *convexity*. At this stage our discussion of convexity is extremely brief, providing only the bare essentials. Much more will be done in Sections 2.6 and 2.7.

2.1.12 Definition. If x and y are points in R^n and $\lambda \in [0,1]$, the point

$$z = \lambda x + (1-\lambda)y$$

is a *convex combination* of x and y. More generally, if $x^1, \ldots, x^k$ are points and $\lambda_1, \ldots, \lambda_k$ are nonnegative numbers such that $\lambda_1 + \cdots + \lambda_k = 1$, then

$$z = \lambda_1 x^1 + \cdots + \lambda_k x^k$$

is a *convex combination* of $x^1, \ldots, x^k$.

In this definition, when $k = 2$ and $x^1 \neq x^2$, the set of all convex combinations of x^1 and x^2 is called the *line segment* determined by x^1 and x^2, the *line segment* between x^1 and x^2, and the *line segment* joining x^1 and x^2.

A point x is a convex combination of $x^1, \ldots, x^k$ if and only if

$$\begin{bmatrix} x \\ 1 \end{bmatrix} = \sum_{i=1}^{k} \lambda_i \begin{bmatrix} x^i \\ 1 \end{bmatrix} \qquad \lambda_i \geq 0,\ i = 1, \ldots, k. \tag{4}$$

The usefulness of this representation will be illustrated in the sequel.

2.1.13 Definition. A set C is *convex* if and only if it contains the line segment between each pair of its points.

2.1.14 Examples. We list a few simple convex sets of interest.

(a) The empty set, $\emptyset$.

(b) Any linear subspace of R^n, that is, a nonempty subset $L \subseteq R^n$ satisfying the conditions

  (i) $x + y \in L$ for all $x, y \in L$,

  (ii) $\lambda x \in L$ for all $\lambda \in R$ and all $x \in L$.

(c) The intersection of any collection of convex sets.

(d) Any *translate* of a convex set, i.e.,

$$C + \{b\} = \{y : y = x + b,\ x \in C\}$$

where C is a convex set. In particular, any *affine subspace* $L + \{b\}$ where L is a linear subspace is convex.

(e) Sets of the form

$$H = \{x : a^{\mathrm{T}}x = b\} \quad a \neq 0,\ b \in R.$$

Such sets are called *hyperplanes*. They are affine subspaces and are linear subspaces if and only if $b = 0$.

(f) Sets of the form

$$S = \{x : a^{\mathrm{T}}x \geq b\} \text{ and } S = \{x : a^{\mathrm{T}}x \leq b\} \text{ where } a \neq 0.$$

These are called *(closed) halfspaces*. When the inequalities are strict ($>$ or $<$), they are called *(open) halfspaces*.

In Section 2.6 we study convex sets that are intersections of closed halfspaces. Such sets play a central role in the theory of mathematical programming. There are various generalizations of a convex set; some of which we shall make use of later are defined below.

2.1.15 Definition. A set $C \subseteq R^n$ is *path-connected* if for any two points $x, y \in C$, there exists a continuous function $p : [0, 1] \to C$ such that $p(0) = x$ and $p(1) = y$. This function is called a *path* joining x and y. For any set $S \subseteq R^n$ and any $x \in S$, the *path component* of S containing x is the union of all path connected sets C such that $x \in C \subseteq S$.

2.1.16 Definition. A set $C \subseteq R^n$ is *connected* if there do not exist disjoint open sets $U, V \subseteq R^n$ such that $U \cap C \neq \emptyset$, $V \cap C \neq \emptyset$, and $C \subseteq U \cup V$. For any set $S \subseteq R^n$ and any $x \in S$, the *(connected) component* of S containing x is the union of all connected sets C such that $x \in C \subseteq S$.

We now summarize some standard facts concerning connectedness.

2.1.17 Proposition. Let $S \subseteq R^n$ and $x \in S$ be given.

(a) The component of S containing x is connected.

(b) The path component of S containing x is path connected.

(c) The path component of S containing x is contained in the component of S containing x.

(d) If S is open, then the path component of S containing x equals the component of S containing x.

(e) For each $y \in S$, the component (path component) of S containing y is either equal to or disjoint from the component (path component) of S containing x.

□

Functions of several variables

Although the linear complementarity problem is defined by an affine function, nonlinear functions also play an important role in the study of this problem. For this reason, we review some basic properties of general vector-valued functions. Throughout this book, we use the terms: functions, mappings and maps, interchangeably.

2.1.18 Definition. Let $\mathcal{D}$ be an open set in R^n. A vector-valued function $f : \mathcal{D} \to R^m$ defined on the domain $\mathcal{D}$ is said to be

(a) *continuous* at the point $x \in \mathcal{D}$ if

$$\lim_{y \to x} f(y) = f(x),$$

that is to say, if for every $\varepsilon > 0$, there is a neighborhood $N \subseteq \mathcal{D}$ of x such that

$$y \in N \quad \Rightarrow \quad \|f(y) - f(x)\| \leq \varepsilon;$$

(b) *Lipschitz continuous* at $x \in \mathcal{D}$ if there exist a constant $L > 0$ and a neighborhood $N \subseteq \mathcal{D}$ of x such that

$$u, v \in N \quad \Rightarrow \quad \|f(u) - f(v)\| \leq L\|u - v\|;$$

the constant L is called a *Lipschitz modulus* of f;

(c) *directionally differentiable* at $x \in \mathcal{D}$ if for every vector $d \in R^n$, the limit

$$\lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t}$$

exists; in this case, the above limit is denoted $f'(x, d)$ and is called the *directional derivative* of f at x along the direction d;

(d) *Fréchet* (or simply *F*-) *differentiable* at x if there exists a matrix of order $m \times n$, denoted $\nabla f(x)$ and called the *Fréchet* (or simply, the *F*-) *derivative* of f at x, such that

$$\lim_{y \to x} \frac{f(y) - f(x) - \nabla f(x)(y - x)}{\|y - x\|} = 0.$$

This derivative $\nabla f(x)$ is also called the *Jacobian matrix* of f at x.

If property (a), (c) or (d) holds at all points $x \in \mathcal{D}$, then we say that f possesses the property in $\mathcal{D}$. If property (b) holds at all points $x \in \mathcal{D}$, then f is said to be *locally Lipschitzian* on $\mathcal{D}$. If there exists a constant $L > 0$ such that for all x, y in $\mathcal{D}$,

$$\|f(x) - f(y)\| \leq L\|x - y\|,$$

then f is said to be *(globally) Lipschitzian* on $\mathcal{D}$.

Notice that a locally Lipschitzian function is not necessarily globally Lipschitzian. A simple example is the real-valued function $f(x) = 1/x$ on the interval $(0, 1)$.

If the function $f : \mathcal{D} \to R^m$ satisfies any one of the properties in Definition **2.1.18**, then so does each of its component functions $f_i : \mathcal{D} \to R$ for $i = 1, \ldots, m$. In general, if the mapping f is F-differentiable at x, then it is directionally differentiable and continuous at x; moreover, we have $f'(x, d) = \nabla f(x) d$ for all $d \in R^n$. Nevertheless, if f is merely directionally differentiable, it is not necessarily true that f is continuous.

2.1.19 Remark. A word of caution about notation is in order. If the real-valued function $\theta : \mathcal{D} \to R$ is Fréchet differentiable, its Fréchet derivative at a point $x \in \mathcal{D}$ is called the *gradient vector* and denoted by the column vector $\nabla\theta(x) \in R^n$. This convention is somewhat inconsistent with that used for a vector-valued function. In particular, if $f : \mathcal{D} \to R^m$ is vector-valued with component functions $f_i : \mathcal{D} \to R$ $(i = 1, \ldots, m)$, then we have

$$\nabla f(x) = \begin{bmatrix} \nabla f_1(x)^{\mathrm{T}} \\ \nabla f_2(x)^{\mathrm{T}} \\ \vdots \\ \nabla f_m(x)^{\mathrm{T}} \end{bmatrix}.$$

The reason for this unconventional use of notation for the Jacobian matrix is to avoid writing the transpose when we multiply the $m \times n$ matrix $\nabla f(x)$ with a vector $d \in R^n$.

A notion about the F-derivative that is sometimes useful is introduced below.

2.1.20 Definition. Let $f : \mathcal{D} \to R^m$ be F-differentiable at $x \in \mathcal{D}$. The F-derivative $\nabla f(x)$ is said to be *strong* if

$$\lim_{(u,v) \to (x,x)} \frac{f(u) - f(v) - \nabla f(x)(u - v)}{\|u - v\|} = 0.$$

It can be shown that if f is F-differentiable in a neighborhood of a point $x \in \mathcal{D}$, then the F-derivative $\nabla f(x)$ is strong if and only if the derivative mapping $\nabla f : N \to R^{m \times n}$ (where N is a suitable neighborhood of x) is continuous at x. In general, if this mapping is continuous in a domain $\mathcal{D}$, then we say that f is *continuously differentiable* on $\mathcal{D}$.

The derivative $\nabla f(x)$ is sometimes called the *first-order derivative* of f. It follows from Definition **2.1.18** that for all vectors y sufficiently close to x, we have

$$f(y) = f(x) + \nabla f(x)(y - x) + o(\|y - x\|) \tag{5}$$

where $o(t)$ is a (vector) quantity with the property:

$$\lim_{t \to 0} \frac{o(t)}{t} = 0.$$

The noteworthy point about the expression (5) is that in a suitable neighborhood of x, the function f can be approximated by the affine mapping $g(y) = f(x) + \nabla f(x)(y - x)$ and the accuracy of the approximation is of the first order. This affine map g is called the *linearization* of f at the point x.

Among the many basic results for functions of several variables, the *mean value theorems* are among the most useful. There are several forms of these theorems; the *integral form* says that if $f : \mathcal{D} \to R^m$ is continuously differentiable on the convex domain $\mathcal{D} \subseteq R^n$, then for any two vectors x and y in $\mathcal{D}$, we have

$$f(y) = f(x) + \int_0^1 \nabla f(x + t(y - x))(y - x)dt.$$

As an immediate consequence of this expression, we obtain

$$\|f(y) - f(x)\| \leq \sup_{0 \leq t \leq 1} \|\nabla f(x + t(y - x))\| \; \|y - x\|$$

(see Section 2.2 for a review of matrix norms). Another useful inequality is the following: for any $x, y, z \in \mathcal{D}$,

$$\|f(y) - f(z) - \nabla f(x)(y - z)\| \leq \sup_{0 \leq t \leq 1} \|\nabla f(z + t(y - z)) - \nabla f(x)\| \; \|y - z\|. \tag{6}$$

In general, one can define higher-order derivatives of a mapping f and write down a corresponding local approximation of f. Instead of doing this in its full generality, we consider a *twice continuously differentiable* real-valued function $\theta : \mathcal{D} \to R$. Let $\nabla^2 \theta(x)$ denote the *Hessian matrix* of

θ at $x \in \mathcal{D}$, i.e., $\nabla^2\theta(x)$ is the $n \times n$ matrix with entries $\partial^2\theta(x)/\partial x_i \partial x_j$ for $i, j = 1, \ldots, n$. For such a function, the second-order analog of (5) is

$$\theta(y) = \theta(x) + \nabla\theta(x)^{\mathrm{T}}(y - x) + \tfrac{1}{2}(y - x)^{\mathrm{T}}\nabla^2\theta(x)(y - x) + \mathrm{o}(\|y - x\|^2) \quad (7)$$

which is valid for all vectors y sufficiently close to x. For an arbitrary vector $y \in \mathcal{D}$, the *second-order Taylor expansion* yields

$$\theta(y) = \theta(x) + \nabla\theta(x)^{\mathrm{T}}(y - x) + \int_0^1 (1 - t)(y - x)^{\mathrm{T}}\nabla^2\theta(x + t(y - x))(y - x)dt$$

in which the error term $o(\cdot)$ is no longer present. It should be pointed out that if θ is a quadratic function, i.e., if $\theta(x) = q^{\mathrm{T}}x + \frac{1}{2}x^{\mathrm{T}}Mx$ for some vector $q \in R^n$ and matrix $M \in R^{n \times n}$ that is symmetric, then the quadratic approximation of θ (i.e., the right-hand term in (7) without the $\mathrm{o}(\cdot)$ term) gives an exact representation of $\theta(y)$ in terms of $\theta(x)$ for all vectors x and y; in other words, for such a function θ, we have

$$\theta(y) = \theta(x) + \nabla\theta(x)(y - x) + \tfrac{1}{2}(y - x)^{\mathrm{T}}M(y - x)$$

and $\nabla\theta(x) = q + Mx$.

If $\theta : R^n \to R$ is a continuous real-valued function and if X is a compact subset of R^n, then θ attains its maximum and minimum on X; that is to say, there exist vectors u and v, both in X, such that for all $x \in X$,

$$\theta(u) \leq \theta(x) \leq \theta(v).$$

This is the well-known *Weierstrass theorem.* The vector u (v) is a *global minimum (maximum)* of θ on X.

More generally, if X is an arbitrary subset of R^n, and if u is a vector in X with the property that for some neighborhood N of u, $\theta(u) \leq \theta(x)$ for all $x \in N \cap X$, then u is called a *local minimum* of θ on X. A *local maximum* is defined analogously. When $X = R^n$, the term "unconstrained" is sometimes attached to these concepts.

If θ is directionally differentiable in X, then any vector $z \in X$ satisfying the condition that

$$\theta'(z, y - z) \geq 0 \quad \text{for all } y \in X$$

is called a *stationary point* of θ. The *minimum principle* states that if X is convex, then any local minimum of θ on X must be a stationary point, but not conversely. When θ is F-differentiable, then the problem of finding a stationary point of θ on X is easily seen to be the variational inequality problem VI$(X, \nabla\theta)$; moreover, if X is open in R^n, then a stationary point of θ coincides with a zero of the F-derivative $\nabla\theta$, i.e., such a vector must satisfy the equation $\nabla\theta(z) = 0$.

In the study of the LCP, mappings defined on subsets of R^n and with their range contained in R^n play a rather important role. In the sequel, we review some further properties of these mappings.

2.1.21 Definition. Let $f : \mathcal{D} \to \mathcal{R}$ be a mapping with domain $\mathcal{D} \subseteq R^n$ and range $\mathcal{R} \subseteq R^n$. Then f is said to be

(a) *injective*, or an *injection*, if $f(x) = f(y)$ for $x, y \in \mathcal{D}$, then $x = y$;

(b) *surjective*, or a *surjection*, if $f(\mathcal{D}) = \mathcal{R}$;

(c) *bijective*, or a *bijection*, if f is both injective and surjective;

(d) a *local homeomorphism* at $x \in \mathcal{D}$ if there exist (open) neighborhoods U of x and V of $f(x)$ such that (i) $V = f(U)$, (ii) f restricted to U is continuous and injective, and (iii) the inverse of f restricted to V, $f^{-1} : V \to U$, is also continuous;

(e) a *local homeomorphism* on $\mathcal{D}$ if f is a local homeomorphism at every point in $\mathcal{D}$;

(f) a *global homeomorphism* from $\mathcal{D}$ onto $f(\mathcal{D})$ if f is injective and f and its inverse f^{-1} are continuous on $\mathcal{D}$ and $f(\mathcal{D})$ respectively.

The following result, which contains a partial statement of the *inverse function theorem*, gives a sufficient condition for a mapping to be a local homeomorphism.

2.1.22 Theorem. Let $f : \mathcal{D} \to R^n$ have a strong F-derivative at the point $x \in \operatorname{int} \mathcal{D} \subseteq R^n$. If $\nabla f(x)$ is nonsingular, then f is a local homeomorphism at x. $\square$

Using this result, one can derive the following consequence, which is related to the *open mapping theorem* and the *invariance of domain theorem*.

2.1.23 Corollary. Let $\mathcal{D}$ be an open set in R^n and $f : \mathcal{D} \to R^n$ be a continuously differentiable mapping. Suppose that $\nabla f(x)$ is nonsingular for all $x \in \mathcal{D}$. Then $f(\mathcal{D})$ is an open set in R^n. □

An important application of the inverse function theorem is the *implicit function theorem*. Since we do not make direct use of the latter result, we choose not to review it here.

A fixed-point theorem

We have seen in Section 1.4 that the LCP can be formulated as a fixed point problem. As a matter of fact, fixed-point theory plays an important role throughout the study of the LCP. The following fundamental result, which provides sufficient conditions for the existence of a fixed point of a continuous mapping, is commonly known as Brouwer's fixed-point theorem.

2.1.24 Theorem. Let $X \neq \emptyset$ be a compact convex subset of R^n and $f : X \to X$ be continuous. Then f has a fixed point in X. □

A contemporary reference which summarizes many applications of this famous theorem to mathematical economics and game theory is the monograph of Border (1985). For further discussion of Brouwer's fixed-point theorem, see **2.11.1**. Later in Section 3.7, we shall use **2.1.24** to prove the existence of a solution to a certain fundamental LCP. At this point, we simply mention that the fixed-point formulations of the LCP described in 1.4 do not lend themselves easily to a fruitful application of **2.1.24**; the difficulty lies in the identification of the compact set X. Indeed, a rather different approach is needed.

Multivalued mappings

As in many topics within mathematical programming, the concept of a *multivalued mapping* is also relevant in LCP theory. In general, a mapping $f : \mathcal{D} \subseteq R^n \to R^m$ is said to be *multivalued* if for each vector $x \in \mathcal{D}$, the image $f(x)$ is a subset (possibly the empty set) of R^m. If $f(x)$ is a singleton for all x, we identify the set $f(x)$ with its sole element, and f becomes the usual kind of mapping that we have been dealing with up to now; occassionally, the term "single-valued" is attached to the word "mapping" to describe a mapping of the latter kind.

Examples of of multivalued mappings abound in the study of the LCP. As an illustration, consider the solution mapping SOL : $R^n \times R^{n\times n} \to R^n$ that assigns to each pair $(q, M) \in R^n \times R^{n\times n}$ the set SOL(q, M) of solutions to the LCP (q, M). This is clearly a multivalued mapping in general. Variations of this mapping can also be defined.

In the following definition, we summarize several important concepts associated with a multivalued mapping. These concepts generalize some well known continuity properties for (single-valued) mappings.

2.1.25 Definition. Let $f : \mathcal{D} \subseteq R^n \to R^m$ be a multivalued mapping. The *graph* of f is the set

$$\mathcal{G}(f) = \{(x, y) \in \mathcal{D} \times R^m : y \in f(x)\}.$$

This mapping f is said to be

(a) *closed* if its graph $\mathcal{G}(f)$ is a closed set in $R^n \times R^m$;

(b) *upper semi-continuous* at $x \in \mathcal{D}$ if for every open set $U \subseteq R^m$ containing the image $f(x)$, there exists an open set $V \subseteq R^n$ containing x such that $f(x) \in U$ for every $x \in V \cap \mathcal{D}$;

(b′) *upper semi-continuous* in $\mathcal{D}$ if f is upper semi-continuous at all points in $\mathcal{D}$;

(c) *lower semi-continuous* at $x \in \mathcal{D}$ if for every open set $U \subseteq R^m$ for which $f(x) \cap U \neq \emptyset$, there exists an open set $V \subseteq R^n$ containing x such that $f(y) \cap U \neq \emptyset$ for every $y \in V \cap \mathcal{D}$;

(c′) *lower semi-continuous* in $\mathcal{D}$ if f is lower semi-continuous at all points in $\mathcal{D}$.

Convex functions

In Section 1.2, we have seen how a convex quadratic program is directly equivalent to a certain LCP. The validity of this result is partly the consequence of certain basic properties of a general convex function. In order to provide the background for the stated equivalence (see also Section 2.8), we present a brief review of the class of convex functions.

2.1.26 Definition. A real-valued function $\theta : \mathcal{D} \to R$ defined on the convex set $\mathcal{D} \subseteq R^n$ is said to be *convex* if for any two vectors x and y in $\mathcal{D}$ and any scalar $\lambda \in [0,1]$,

$$\theta(\lambda x + (1-\lambda)y) \leq \lambda\theta(x) + (1-\lambda)\theta(y).$$

The function θ is said to be *strictly convex* on $\mathcal{D}$ if strict inequality holds for all vectors $x \neq y$ in $\mathcal{D}$ and all $\lambda \in (0,1)$; θ is said to be *strongly convex* on $\mathcal{D}$ if there exists a constant $c > 0$ such that

$$\theta(\lambda x + (1-\lambda)y) - \lambda\theta(x) - (1-\lambda)\theta(y) \leq -c\lambda(1-\lambda)\|x-y\|_2^2$$

for all vectors $x, y \in \mathcal{D}$ and all scalars $\lambda \in [0,1]$.

If the domain $\mathcal{D}$ is also open, and θ is F-differentiable on $\mathcal{D}$, then θ is convex on $\mathcal{D}$ if and only if the *gradient inequality*

$$\theta(x) - \theta(y) \geq \nabla\theta(y)^{\mathrm{T}}(x-y)$$

holds for all vectors $x, y \in \mathcal{D}$. In this case, if X is any convex subset of $\mathcal{D}$, then any (constrained) stationary point of θ on X must be a global minimum point of the mathematical program:

$$\begin{array}{ll} \text{minimize} & \theta(x) \\ \text{subject to} & x \in X. \end{array} \tag{8}$$

If θ is strongly convex and X is closed, then there is a unique global minimum of the above program. More generally, if θ is strictly convex, then the program (8) has at most one global minimum.

Let θ be a twice differentiable function defined on the open convex set $\mathcal{D} \subseteq R^n$. Then θ is convex on $\mathcal{D}$ if and only if the Hessian matrix $\nabla^2\theta(x)$ is positive semi-definite for all $x \in \mathcal{D}$; if $\nabla^2\theta(x)$ is positive definite for all $x \in \mathcal{D}$, then $\theta(x)$ is strictly convex. Finally, θ is strongly convex on $\mathcal{D}$ if and only if there exists a constant $c' > 0$ such that for all vectors $x \in \mathcal{D}$ and all vectors $y \in R^n$,

$$y^{\mathrm{T}}\nabla^2\theta(x)y \geq c'\|y\|_2^2.$$

In particular, it follows that if θ is a quadratic function, then θ is strictly convex on R^n if and only if it is strongly convex there; the latter property is further equivalent to the positive definiteness of the (constant) Hessian matrix $\nabla^2\theta(x)$.

2.2 Matrix Analysis

In this section, we review elements of matrix analysis that are important for the linear complementarity problem. Topics covered include principal submatrices, vector and matrix norms, spectral properties, and special matrices and their characteristics. As in the previous section, we omit the proofs of all the results and ask the reader to supply some of them in the exercises.

Principal rearrangements and submatrices

As we saw in Section 1.3, submatrices, especially the principal ones, play an important role in the study of the LCP. For this reason, we formally define these concepts and some related notions.

2.2.1 Definition. Let $A \in R^{m\times n}$ be given. For index sets $\alpha \subseteq \{1,\ldots,m\}$ and $\beta \subseteq \{1,\ldots,n\}$, the submatrix $A_{\alpha\beta}$ of A is the matrix whose entries lie in the rows of A indexed by α and the columns indexed by β. If $\alpha = \{1,\ldots,m\}$, we denote the submatrix $A_{\alpha\beta}$ by $A_{\cdot\beta}$; similarly, if $\beta = \{1,\ldots,n\}$, we denote $A_{\alpha\beta}$ by $A_{\alpha\cdot}$. If $m = n$ and $\alpha = \beta$, the submatrix $A_{\alpha\alpha}$ is called a *principal submatrix* of A; the determinant of $A_{\alpha\alpha}$ is called a *principal minor* of A.

2.2.2 Definition. Let $A \in R^{n\times n}$ be given. For a given integer k $(1 \leq k \leq n)$, the principal submatrix $A_{\alpha\alpha}$ where $\alpha = \{1,\ldots,k\}$ is called a *leading principal submatrix* of A. The determinant of a leading principal submatrix of A is called a *leading principal minor* of A.

Let $A \in R^{n\times n}$ and $\alpha \subseteq \{1,\ldots,n\}$ be given. Then, there exists a permutation matrix P such that the principal submatrix $A_{\alpha\alpha}$ appears as a leading principal submatrix in the matrix PAP^{T} which is called a *principal rearrangement* of A. To see what P is, let $\alpha = \{k_1,\ldots,k_m\}$ and its complement $\bar{\alpha} = \{k_{m+1},\ldots,k_n\}$; then P is the permutation matrix with $p_{ik_i} = 1$ for all i. The matrix PAP^{T} can be written in the partitioned form:

$$PAP^{\mathrm{T}} = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}.$$

Thus, given any principal submatrix $A_{\alpha\alpha}$ of A, we can always *principally rearrange* the rows and columns of A in such a way that $A_{\alpha\alpha}$ becomes a leading principal submatrix in the resulting rearranged matrix.

2.2.3 Example. Let A be a 4×4 matrix. Suppose we want to move the principal submatrix

$$\begin{bmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{bmatrix}$$

to the upper-left-hand corner and the principal submatrix

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

to the lower-right-hand corner. The permutation matrix which accomplishes this principal rearrangement is given by

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

It is known that the principal minors of a matrix $A \in R^{n\times n}$ appear as coefficients in the *characteristic polynomial* of A which is defined as

$$p_A(t) = \det(tI - A), \quad t \in R.$$

More generally, by the fact that the determinant of a matrix is a multilinear function of its entries, one can prove the determinantal formula: for an arbitrary diagonal matrix $D \in R^{n\times n}$,

$$\det(D + A) = \sum_{\alpha} \det D_{\alpha\alpha} \det A_{\bar{\alpha}\bar{\alpha}} \tag{1}$$

where the summation ranges over subsets α of $\{1, \ldots, n\}$ (with complement $\bar{\alpha}$). The reader is asked to prove this formula in Exercise **2.10.4**.

Given the LCP (q, M), principally rearranging the matrix M and the vector q will only rearrange the components of a solution to the problem, and will not change the solution in any other way. More specifically, if P is the permutation matrix corresponding to the rearrangement, then x solves (q, M) if and only if Px solves (Pq, PMP^{T}), the latter being the rearranged LCP.

2.2.4 Definition. A matrix $A \in R^{n\times n}$ for $n > 1$ is *reducible* if it can be principally rearranged into the form:

$$\begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ 0 & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \tag{2}$$

for some nonempty proper subset $\alpha \subseteq \{1, \ldots, n\}$. If $n = 1$ then A is *reducible* if and only if $A = 0$. The matrix A is *irreducible* if it is not reducible.

Irreducibility can be characterized in the following way: $A \in R^{n\times n}$ is irreducible if and only if for any two distinct indices $1 \leq i, j \leq n$, there is a sequence of nonzero elements of A of the form

$$\{a_{ii_1}, a_{i_1i_2}, \ldots, a_{i_mj}\}.$$

The matrix M arising from the convex-hull problem (see (1.2.21) in Section 1.2) is irreducible but is *very sparse*, i.e., contains relatively few nonzero entries. Incidentally, the structure of this matrix is known as *tridiagonality*; more precisely, a matrix $A \in R^{n\times n}$ is *tridiagonal* if $a_{ij} = 0$ for all $|i - j| > 1$.

It is possible for a reducible matrix to be very *dense*, i.e., to contain a large proportion of nonzero entries. For example, in the partitioned matrix A given in (2), the index set α may contain just a few indices and the three submatrices $A_{\alpha\alpha}, A_{\alpha\bar{\alpha}}, A_{\bar{\alpha}\bar{\alpha}}$ could be completely dense. In general, the property of reducibility of a matrix is not related to its density, but rather to the *decomposability* of systems of equations involving the matrix of concern.

Matrix norms

2.2.5 Definition. Let A be a given $m \times n$ matrix, and let $\|\cdot\|$ and $\|\cdot\|'$ be two vector norms on R^m and R^n respectively. The *matrix norm induced* by these vector norms is

$$\|A\| = \max_{\|x\|'=1} \|Ax\| \tag{3}$$

or equivalently,

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|'}. \tag{4}$$

2.2.6 Remark. If A is a square matrix, we say, unless otherwise stated, that the matrix norm $\|A\|$ is induced by the vector norm $\|\cdot\|$ to mean $\|\cdot\|' = \|\cdot\|$ in the above definition.

The maximum in (3) is well-defined and actually attained because of the compactness of the unit ball $\{x \in R^n : \|x\|' = 1\}$ and the continuity of the norm $\|Ax\|$ as a function in x. (See Section 2.1 for a review of the continuity of a function; see also Exercise **2.10.8**.) The equivalence between (3) and (4) is rather obvious.

It can be shown that the function $A \mapsto \|A\|$ indeed defines a norm on the $m \times n$ matrices in the sense that the three axioms in Definition **2.1.1** are satisfied. Moreover, by the definition, we have

$$\|Ax\| \leq \|A\| \; \|x\|'$$

for all $x \in R^n$. It follows from this last inequality that if A and B are given $m \times n$ and $n \times p$ matrices and $\|\cdot\|, \|\cdot\|'$ and $\|\cdot\|''$ are vector norms defined on R^m, R^n and R^p respectively, then

$$\|AB\| \leq \|A\|' \; \|B\|'' \tag{5}$$

where the matrix norms are induced by the corresponding vector norms. In particular, if A is square, then $\|A^2\| \leq \|A\|^2$.

2.2.7 Remark. One may define an *abstract* matrix norm on $R^{n \times n}$ as a function $R^{n \times n} \to R_+$ satisfying the axioms (a), (b), and (c) in Definition **2.1.1** and the *submultiplicativity* condition (5). There are matrix norms which are not induced by vector norms. For our purpose in this book, all matrix norms are induced by vector norms.

If A is an $n \times n$ matrix, we write $\|A\|_p$ to denote the matrix norm induced by the vector norm $\|\cdot\|_p$ for $p = 1, 2, \infty$; that is

$$\|A\|_p = \max_{\|x\|_p=1} \|Ax\|_p.$$

It is not difficult to show

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$$

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|;$$

these are called the *maximum column sum matrix norm* and *maximum row sum matrix norm* respectively.

Eigenvalues and eigenvectors

So far, we have been dealing with vectors and matrices whose entries are real numbers. Although complex numbers are rarely encountered in the study of the linear complementarity problem, it would be appropriate for us to temporarily enlarge the field of real numbers to include the complex scalars in order to give a meaningful discussion on the subject of eigenvalues and eigenvectors. Norms of complex vectors and matrices are defined in the same way as in the real case.

2.2.8 Definition. Let A be an $n \times n$ complex matrix. A complex number λ is an *eigenvalue* of A if there exists a nonzero, complex n-vector v, called an *eigenvector* of A *associated with* λ, such that

$$Av = \lambda v.$$

Every complex, square matrix A of order n has exactly n eigenvalues; they are the n roots of the *characteristic equation*

$$\det(\lambda I - A) = 0.$$

If λ is a multiple root of this equation, then λ is an eigenvalue of "multiplicity" greater than one. If λ is a simple root of the characteristic equation, we say that it is *algebraically simple.*[1]

[1] In general, there are two kinds of multiplicity of an eigenvalue: algebraic and geometric. These multiplicity concepts are rarely used in the study of the linear complementarity problem.

2.2.9 Definition. The set of all eigenvalues of a complex square matrix A is called its *spectrum* and is denoted by $\sigma(A)$. The *spectral radius* of A is the nonnegative real number

$$\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}.$$

When A is a real square matrix, its spectrum may contain complex elements. If A is a real square matrix and λ is a real eigenvalue (i.e., $\lambda \in R$), then A has a real eigenvector $v \in R^n$ associated with λ. All eigenvalues of a real symmetric matrix are real. Moreover, the largest and smallest eigenvalues of such a matrix A play a special role in connection with the quadratic form $x^{\mathrm{T}}Ax$. We summarize these statements in the result below.

2.2.10 Proposition. Let $A \in R^{n\times n}$ be symmetric. Then,

(a) all eigenvalues of A are real, and there exists an orthonormal basis of R^n which is comprised of eigenvectors of A;

(b) there exists an *orthogonal* matrix $U \in R^{n\times n}$ $(U^{\mathrm{T}}U = I)$ whose columns are eigenvectors of A such that

$$A = UDU^{\mathrm{T}} \tag{6}$$

where D is the diagonal matrix whose diagonal entries are the eigenvalues of A (expression (6) is called a *spectral decomposition* of A);

(c) if the eigenvalues of A are arranged as $\lambda_1 \leq \ldots \leq \lambda_n$, then

$$\lambda_1 x^{\mathrm{T}}x \leq x^{\mathrm{T}}Ax \leq \lambda_n x^{\mathrm{T}}x$$

for all $x \in R^n$. □

The following result summarizes two important properties of the spectral radius of a square matrix, see also Exercise **2.10.9**.

2.2.11 Proposition. Let A be a given complex square matrix. Then,

(a) $\rho(A) \leq \|A\|$ for any matrix norm;

(b) for every $\varepsilon > 0$, there exists at least one matrix norm $\|\cdot\|$ (induced by some vector norm) such that

$$\|A\| \leq \rho(A) + \varepsilon. \;\square$$

The matrix norm $\|A\|_2$ induced by the Euclidean vector norm is sometimes called the *spectral norm.* This terminology is due to the following result.

2.2.12 Proposition. Let $A \in R^{n\times n}$ be given. Then

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = [\rho(A^{\mathrm{T}}A)]^{1/2}.$$

In particular, if A is symmetric, then $\|A\|_2 = \rho(A)$. □

2.2.13 Definition. A complex square matrix A is said to be *convergent* if

$$\lim_{k\to\infty} A^k = 0;$$

i.e., if all the entries of A^k approach zero as $k \to \infty$.

Convergent matrices play an important role in the convergence analysis of iterative methods, see Section 2.5. These matrices can be characterized in terms of their spectral radii.

2.2.14 Proposition. A complex square matrix A is convergent if and only if $\rho(A) < 1$. If A is convergent, then $I - A$ is nonsingular, and

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k. \ \square$$

Special matrix classes

Matrix classes play an important role in the study of the linear complementarity problem. For this reason, it is is useful to review some fundamental matrix classes which will form the basis for subsequent generalizations.

2.2.15 Definition. A matrix $A \in R^{n\times n}$ is said to be *positive semi-definite* if $x^{\mathrm{T}}Ax \geq 0$ for all $x \in R^n$. It is *positive definite* if $x^{\mathrm{T}}Ax > 0$ for all $x \in R^n \setminus \{0\}$.

In many matrix theory textbooks, a positive definite (semi-definite) matrix is restricted to be symmetric; however, the matrices encountered in the context of the LCP are frequently asymmetric (see e.g. (1.2.3)), hence our extended definition. Notice that an asymmetric matrix A is positive definite (semi-definite) if its symmetric part, i.e., $\frac{1}{2}(A + A^{\mathrm{T}})$, is so.

Clearly, principal submatrices of positive definite (semi-definite) matrices are positive definite (semi-definite). Thus, the property of positive definiteness (semi-definiteness) of a matrix is *inherited* by its principal submatrices.

Symmetric positive definite matrices can be characterized in a number of ways. We list several of these characterizations in the result below.

2.2.16 Proposition. Let $A \in R^{n \times n}$ be symmetric. The statements below are equivalent.

(a) A is positive definite.

(b) All principal minors of A are positive.

(c) All leading principal minors of A are positive.

(d) All eigenvalues of A are positive.

(e) There exists a nonsingular matrix G such that $A = G^{\mathrm{T}}G$.

Moreover, if A is symmetric positive definite, there exists a unique symmetric positive definite matrix denoted by $A^{1/2}$ such that $A = (A^{1/2})^2$. This matrix $A^{1/2}$ is called the *square root* of A. □

We have the following similar result for a symmetric positive semi-definite matrix.

2.2.17 Proposition. Let $A \in R^{n \times n}$ be symmetric. The statements below are equivalent.

(a) A is positive semi-definite.

(b) All principal minors of A are nonnegative.

(d) All eigenvalues of A are nonnegative.

(e) There exists a matrix G such that $A = G^{\mathrm{T}}G$.

(f) For every $\varepsilon > 0$, $A + \varepsilon I$ is positive definite.

Moreover, if A is symmetric positive semi-definite, there exists a unique symmetric positive semi-definite matrix denoted by $A^{1/2}$ such that $A = (A^{1/2})^2$. This matrix $A^{1/2}$ is called the *square root* of A. □

2.2.18 Remark. The analogue of part (c) in Proposition **2.2.16** fails to characterize positive semi-definiteness. This explains the absence of part (c) in **2.2.17**. For example, the matrix

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

satisfies the analogue of part (c) in **2.2.16** but fails part (b).

Next, we define the class of diagonally dominant matrices.

2.2.19 Definition. A matrix $A \in R^{n\times n}$ is *row diagonally dominant* if

$$|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|, \quad i = 1, \ldots, n, \tag{7}$$

and *strictly row diagonally dominant* if strict inequality holds for all i. The matrix A is *irreducibly row diagonally dominant* if it is irreducible, diagonally dominant and strict inequality holds in (7) for at least one i.

The matrix A is *column diagonally dominant* if its transpose is row diagonally dominant. The concepts of *column strictly diagonally dominant* and *column irreducibly diagonally dominant* are defined analogously.

Clearly, the property of (strict) diagonal dominance of a matrix is inherited by it principal submatrices. Other properties of these matrices are listed in the result below.

2.2.20 Proposition. Let $A \in R^{n\times n}$. The statements below hold.

(a) If A is either (row or column) strictly or irreducibly diagonally dominant, then A is nonsingular.

(b) If A is symmetric, strictly or irreducibly diagonally dominant, and has positive diagonal entries, then A is positive definite.

(c) If A is symmetric, diagonally dominant and has nonnegative diagonal entries, then A is positive semi-definite. □

Finally, we shall review the class of nonnegative matrices. A matrix A is *nonnegative* if all its entries are nonnegative real numbers, and is *positive* if all its entries are positive. We note that an $n \times n$ nonnegative matrix is irreducible if and only if $(I + A)^{n-1}$ is positive. In particular, a square, positive matrix must be irreducible.

The eigen-theory of nonnegative matrices, also known as the *Perron-Frobenius theory*, has a wide range of applications; in particular, it plays an important role in the convergence of iterative methods as well as in other topics within matrix theory. We summarize some properties pertaining to these special matrices.

2.2.21 Theorem. Let $A \in R^{n \times n}$ be positive. Then,

(a) $\rho(A) > 0$;

(b) $\rho(A)$ is an algebraically simple eigenvalue of A, greater than the magnitude of any other eigenvalue;

(c) A has a positive eigenvector corresponding to $\rho(A)$.

If A is nonnegative and irreducible, then,

(a′) $\rho(A) > 0$;

(b′) $\rho(A)$ is an algebraically simple eigenvalue of A; moreover, if λ is another eigenvalue with $|\lambda| = \rho(A)$, then λ is also algebraically simple;

(c′) A has a positive eigenvector corresponding to $\rho(A)$; this eigenvector is the only nonnegative eigenvector of A (up to scalar multiples).

If A is nonnegative, then, $\rho(A)$ is an eigenvalue of A and A has a nonnegative eigenvector corresponding to $\rho(A)$. □

The above *Perron-Frobenius theorem* has many consequences. We give one comparison result for the spectral radii of two nonnegative matrices.

2.2.22 Corollary. If $A, B \in R^{n \times n}$ and $0 \leq A \leq B$, then $\rho(A) \leq \rho(B)$; moreover if A is irreducible, then $\rho(A) = \rho(B)$ implies $A = B$. □

2.3 Pivotal Algebra

In this section, we focus on the mechanics of pivoting and some important facts about the effects that the process of pivoting has upon certain matrices. The development here continues to center around the linear complementarity problem. As a matter of fact, we deal with the specific system of linear equations

$$w = q + Mz \tag{1}$$

where $q \in R^n$ and $M \in R^{n \times n}$. Of course, one recognizes (1) as a defining equation for the LCP (q, M). The alternative representation

$$Iw - Mz = q \tag{2}$$

is also used, but to a lesser extent.

Just as in elementary treatments of linear programming, it is helpful to record the data of an equation like (1) in a *tableau* or *schema*

$$\begin{array}{c|c|c|} & 1 & z \\ \hline w & q & M \\ \hline \end{array} \tag{3}$$

Elementwise, this tableau looks like

$$\begin{array}{c|c|cccc|} & 1 & z_1 & \cdots & z_n \\ \hline w_1 & q_1 & m_{11} & \cdots & m_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ w_n & q_n & m_{n1} & \cdots & m_{nn} \\ \hline \end{array}$$

In the system of equations represented by (1), we think of $w_1, \dots, w_n$ as *basic variables* and $z_1, \dots, z_n$ as *nonbasic variables.* These terms (which are borrowed from linear programming) have the synonyms *dependent variables* and *independent variables*, respectively. The same terminology applies to (2) and (3).

The essential point about the basic variables is that the columns associated with them are linearly independent as is clearly the case in (1) and (2). The matrix of columns associated with a maximal set of basic variables is called a *basis* (in the matrix from which the columns are drawn).

For the most part, the basis matrices used here will be square and nonsingular. When this is so, the system can be put into *canonical form* with respect to any given basis. That is, we may write the system with the vector of basic variables appearing by itself on one side of the equation.

For example, the system (1) is in canonical form with respect to the basic variables w (the associated basis being the identity matrix I). Similarly, if the matrix M is nonsingular, the system (1) can be put into canonical form with respect to the variables z. Doing so yields

$$z = -M^{-1}q + M^{-1}w. \tag{4}$$

In this representation of the system, the z-variables are basic and the w-variables are nonbasic. *Pivoting* refers to changing the basic/nonbasic roles of variables and to the process whereby the system of equations is put into canonical form with respect to the new basis. Thus, the status of a variable as basic or nonbasic is not necessarily fixed.

An exchange of two variables—one basic, the other nonbasic—is called a *simple pivot.* Consider the system (1) and assume that $m_{rs} \neq 0$. Then (and only then) the basic variable w_r can be exchanged with the nonbasic variable z_s. This amounts to a two-step procedure: first, solving the w_r equation for z_s in terms of w_r and the other variables; second, using this new equation to substitute for z_s in the *other* equations. Thus, in the first step we have

$$z_s = -\frac{q_r}{m_{rs}} + \sum_{j \neq s} \frac{-m_{rj}}{m_{rs}} z_j + \frac{1}{m_{rs}} w_r. \tag{5}$$

This expression can then be used to eliminate z_s from the other equations of the system, thereby making the dependence of each basic variable on the independent nonbasic variables explicit. This second step leads to

$$w_i = q_i - q_r \frac{m_{is}}{m_{rs}} + \sum_{j \neq s} (m_{ij} - m_{rj} \frac{m_{is}}{m_{rs}}) z_j + \frac{m_{is}}{m_{rs}} w_r, \quad i \neq r. \tag{6}$$

In this pivoting process, the nonzero scalar m_{rs} is called the *pivot element* and the operation is referred to as *pivoting on* m_{rs}. We denote the pivotal exchange by $\langle w_r, z_s \rangle$. When $r = s$ and the matrix M is square, the operation $\langle w_r, z_r \rangle$ is called a *simple principal pivot* in recognition of the fact that the pivot element m_{rr} is a principal submatrix of M.

Simple pivoting leads to a transformation of the problem data. To be precise, we have

$$
\begin{aligned}
q'_r &= -q_r/m_{rs} & & \\
q'_i &= q_i - (m_{is}/m_{rs})q_r & & (i \neq r) \\
m'_{rs} &= 1/m_{rs} & & \\
m'_{is} &= m_{is}/m_{rs} & & (i \neq r) \\
m'_{rj} &= -m_{rj}/m_{rs} & & (j \neq s) \\
m'_{ij} &= m_{ij} - (m_{is}/m_{rs})m_{rj} & & (i \neq r,\ j \neq s).
\end{aligned}
\tag{7}
$$

Block pivoting is done in the analogous way. In a block pivot operation, a set of basic variables is exchanged for a set of nonbasic variables of the same cardinality. The analogue of a pivot element is a nonsingular submatrix, called the *pivot matrix* or *pivot block*. The case where the pivot block is a principal submatrix of M is called a *block principal pivot* and the operation is called *principal pivoting*.

Principal pivotal transforms

Consider a system like (1) where $q \in R^n$ and $M \in R^{n \times n}$. Let α be a subset of $\{1, \ldots, n\}$ and suppose the principal submatrix $M_{\alpha\alpha}$ is nonsingular. By means of a principal rearrangement, we may assume that $M_{\alpha\alpha}$ is a leading principal submatrix of M. Now write the equation $w = q + Mz$ in partitioned form:

$$
\begin{aligned}
w_\alpha &= q_\alpha + M_{\alpha\alpha} z_\alpha + M_{\alpha\bar{\alpha}} z_{\bar{\alpha}} \\
w_{\bar{\alpha}} &= q_{\bar{\alpha}} + M_{\bar{\alpha}\alpha} z_\alpha + M_{\bar{\alpha}\bar{\alpha}} z_{\bar{\alpha}}.
\end{aligned}
\tag{8}
$$

Since $M_{\alpha\alpha}$ is nonsingular by hypothesis, we may exchange the roles of w_α and z_α thereby obtaining a system of the form

$$
\begin{aligned}
z_\alpha &= q'_\alpha + M'_{\alpha\alpha} w_\alpha + M'_{\alpha\bar{\alpha}} z_{\bar{\alpha}} \\
w_{\bar{\alpha}} &= q'_{\bar{\alpha}} + M'_{\bar{\alpha}\alpha} w_\alpha + M'_{\bar{\alpha}\bar{\alpha}} z_{\bar{\alpha}}
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
q'_\alpha &= -M_{\alpha\alpha}^{-1} q_\alpha \\
q'_{\bar{\alpha}} &= q_{\bar{\alpha}} - M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1} q_\alpha
\end{aligned}
\tag{10}
$$

and

$$\begin{aligned} M'_{\alpha\alpha} &= M_{\alpha\alpha}^{-1} & M'_{\alpha\bar{\alpha}} &= -M_{\alpha\alpha}^{-1}M_{\alpha\bar{\alpha}} \\ M'_{\bar{\alpha}\alpha} &= M_{\bar{\alpha}\alpha}M_{\alpha\alpha}^{-1} & M'_{\bar{\alpha}\bar{\alpha}} &= M_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha}M_{\alpha\alpha}^{-1}M_{\alpha\bar{\alpha}}. \end{aligned} \tag{11}$$

The resulting matrix

$$M' = \begin{bmatrix} M'_{\alpha\alpha} & M'_{\alpha\bar{\alpha}} \\ M'_{\bar{\alpha}\alpha} & M'_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \tag{12}$$

is called a *principal pivotal transform* of M with respect to the index set α (and the nonsingular principal submatrix $M_{\alpha\alpha}$). Similarly, the vector

$$q' = \begin{bmatrix} q'_{\alpha} \\ q'_{\bar{\alpha}} \end{bmatrix}$$

is called a *principal pivotal transform* of q with respect to the same index set (and principal submatrix). The system (9) is the result of the principal pivot operation applied to the original system (8) with $M_{\alpha\alpha}$ as the pivot block.

To indicate that the system (9)

$$w' = q' + M'z'$$

is obtained from (1) by a principal pivot transformation with respect to the index set α (and the nonsingular principal submatrix $M_{\alpha\alpha}$), we write

$$(q', M') = \wp_{\alpha}(q, M).$$

We also use this to express the fact that the LCP(q', M') is obtained from the LCP (q, M) by pivoting on $M_{\alpha\alpha}$. In the homogeneous case, where $q = 0$, we abbreviate the notation to

$$M' = \wp_{\alpha}(M).$$

This notation is useful in stating the following theorem. Before we come to it, though, we wish to introduce the notion of a *sign-changing matrix*. Such a matrix is diagonal, and each of its diagonal entries equals ± 1. For any $\alpha \subseteq \{1, \dots, n\}$ let $E_{\bar{\alpha}}$ denote the sign-changing matrix such that $e_{ii} = -1$ if and only if $i \in \bar{\alpha}$.

2.3.1 Example. To illustrate the effect of pre- and post-multiplication by $E_{\bar{\alpha}}$ as in the next theorem, we consider the case of an $n \times n$ matrix M and a leading index set α. Then

$$E_{\bar{\alpha}} M E_{\bar{\alpha}} = E_{\bar{\alpha}} \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} E_{\bar{\alpha}} = \begin{bmatrix} M_{\alpha\alpha} & -M_{\alpha\bar{\alpha}} \\ -M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}.$$

In general, for an arbitrary index set α, we have

$$(E_{\bar{\alpha}} M E_{\bar{\alpha}})_{ij} = \begin{cases} m_{ij} & \text{if } i \in \alpha, \text{ and } j \in \alpha \\ m_{ij} & \text{if } i \in \bar{\alpha}, \text{ and } j \in \bar{\alpha} \\ -m_{ij} & \text{if } i \in \alpha, \text{ and } j \in \bar{\alpha} \\ -m_{ij} & \text{if } i \in \bar{\alpha}, \text{ and } j \in \alpha \end{cases}.$$

2.3.2 Theorem. Let $M \in R^{n \times n}$ have the nonsingular principal submatrix $M_{\alpha\alpha}$. Then

$$(\wp_\alpha(M))^{\mathrm{T}} = E_{\bar{\alpha}}(\wp_\alpha(M^{\mathrm{T}}))E_{\bar{\alpha}}. \tag{13}$$

Proof. This formula follows from an essentially routine calculation that makes use of the following facts:

1. Pre- and post-multiplication by $E_{\bar{\alpha}}$ changes the signs of the off-diagonal blocks but not the diagonal blocks.

2. $(M^{\mathrm{T}})_{\alpha\alpha} = (M_{\alpha\alpha})^{\mathrm{T}} =: M^{\mathrm{T}}{}_{\alpha\alpha}$.

3. $(M^{-1}_{\alpha\alpha})^{\mathrm{T}} = (M^{\mathrm{T}}_{\alpha\alpha})^{-1}$.

4. $(M_{\alpha\beta})^{\mathrm{T}} = (M^{\mathrm{T}})_{\beta\alpha}$.

(Note: $M^{-1}_{\alpha\alpha} = (M_{\alpha\alpha})^{-1}$.) □

It is useful to note that principal pivots can destroy symmetry. More precisely, if the matrix M is symmetric, a principal pivotal tranform of M need no longer be symmetric (cf. (11) and (12)). Actually, the principal pivotal transform M' of a symmetric matrix M has the bisymmetric property (defined in Section 1.2). Up to a principal rearrangement, it is of the form (12) where

$$M'_{\alpha\alpha} = (M'_{\alpha\alpha})^{\mathrm{T}}, \quad M'_{\bar{\alpha}\bar{\alpha}} = (M'_{\bar{\alpha}\bar{\alpha}})^{\mathrm{T}}, \quad M'_{\alpha\bar{\alpha}} = -(M'_{\bar{\alpha}\alpha})^{\mathrm{T}}.$$

It can be shown, in general, that every principal pivotal transform of a bisymmetric matrix is again bisymmetric. Furthermore, as we shall show in Section 4.1, principal pivoting preserves positive semi-definiteness, definiteness, and several other properties of interest.

It is important to realize that a given linear complementarity problem (q, M) and its principal pivotal transforms $\wp_\alpha(q, M)$ are *equivalent* in the sense that from a (feasible) solution of (q, M) one can obtain a corresponding (feasible) solution of $\wp_\alpha(q, M)$.

2.3.3 Proposition. Let $(q', M') = \wp_\alpha(q, M)$. If $\bar{z}$ is a (feasible) solution of the LCP (q, M) and $\bar{w} = q + M\bar{z}$, then a (feasible) solution of (q', M') is given by

$$z'_i = \begin{cases} \bar{w}_i & \text{if } i \in \alpha \\ \bar{z}_i & \text{if } i \in \bar{\alpha}. \end{cases} \qquad \square$$

This simple observation can sometimes be advantageously used in connection with existence results or with algorithms.

Principal pivoting is closely connected with complementary matrices and cones. (See Section 1.3.) For instance, consider the complementary matrix

$$B = \begin{bmatrix} -M_{\alpha\alpha} & 0 \\ -M_{\bar{\alpha}\alpha} & I \end{bmatrix}$$

with respect to the index set α. If $q \in R^n$ is such that $B^{-1}q \geq 0$, then q is an element of the complementary cone $\operatorname{pos} B$. Thus, the LCP (q, M) has the solution (w, z) with

$$w = (0, q_{\bar{\alpha}} - M_{\bar{\alpha}\alpha}M_{\alpha\alpha}^{-1}q_\alpha) \quad \text{and} \quad z = (-M_{\alpha\alpha}^{-1}q_\alpha, 0).$$

The thing to notice is that if we (block) pivot on $M_{\alpha\alpha}$ in (1) and then set the nonbasic variables of the resulting system to zero (i.e., $w_\alpha = 0$ and $z_{\bar{\alpha}} = 0$), we obtain exactly the above displayed solution of the LCP.

In summary, if a principal pivotal transform of (1) gives a vector q' (see (10)) that is nonnegative, then we immediately obtain a solution to the LCP (q, M). Hence, principal pivoting can be looked upon as a strategy for solving the LCP (q, M) by attempting to achieve a nonnegative principal pivotal transform of the vector q. This is not to say that the strategy always works, but when it does, a solution of the LCP is easily identified. Much of Section 4.1 is devoted to developing results that support this solution strategy.

The Schur complement

The matrix $M'_{\bar{\alpha}\bar{\alpha}} = M_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha}M_{\alpha\alpha}^{-1}M_{\alpha\bar{\alpha}}$ in (11) is an instance of a *Schur complement,* a type of matrix that arises in many contexts in numerical linear algebra. It is of considerable importance in the linear complementarity problem. The notion of a Schur complement is actually a bit more general than the one just illustrated.

2.3.4 Definition. Let A_{11} be a nonsingular submatrix of the $R^{m\times n}$ matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Then $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the *Schur complement of* A_{11} *in* A, denoted (A/A_{11}).

Notice that the matrix A in this definition need not be square as our matrices M of the LCP are. Actually, the case of a square matrix helps to motivate the quotient-like notation just introduced.

Expression (12) shows that the Schur complement of $M_{\alpha\alpha}$ in the square matrix M appears as a principal submatrix of the principal pivotal transform $\wp_\alpha(M)$.

The next result shows how the nonsingularity of a square matrix is related to the nonsingularity of the Schur complement of any one of its nonsingular principal submatrices, and how the inverse of M (if it exists) can be expressed in terms of the Schur complement.

2.3.5 Proposition. If M is a square matrix having the block partitioned form

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

where $\det M_{11} \neq 0$, then

$$\det M/\det M_{11} = \det(M_{22} - M_{21}M_{11}^{-1}M_{12}) = \det(M/M_{11}). \qquad (14)$$

If, in addition, $N = (M/M_{11})$ is nonsingular, then

$$M^{-1} = \begin{bmatrix} M_{11}^{-1} + M_{11}^{-1}M_{12}N^{-1}M_{21}M_{11}^{-1} & -M_{11}^{-1}M_{12}N^{-1} \\ -N^{-1}M_{21}M_{11}^{-1} & N^{-1} \end{bmatrix}. \tag{15}$$

Proof. Direct multiplication shows that

$$\begin{bmatrix} I & 0 \\ -M_{21}M_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ 0 & N \end{bmatrix}. \tag{16}$$

Taking determinants on both sides of the above equation, we obtain

$$\det M = (\det M_{11})(\det N),$$

which clearly implies (14).

If both M_{11} and N are nonsingular, then M^{-1} exists by (14). To derive the inverse formula (15), we note that

$$\begin{bmatrix} I & -M_{12}N^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ 0 & N \end{bmatrix} = \begin{bmatrix} M_{11} & 0 \\ 0 & N \end{bmatrix}.$$

This equation, together with (16), easily establish the desired inverse expression (15). □

Equation (14) is called *Schur's determinantal formula.* It is clear that under the hypotheses of **2.3.5**, M is nonsingular if and only if (M/M_{11}) is nonsingular.

We come next to another property of the Schur complement.

2.3.6 Proposition. Let M be a square matrix with block partitioned form

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

where

$$M_{11} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

If M_{11} and A_{11} are nonsingular, then

$$(M/M_{11}) = ((M/A_{11})/(M_{11}/A_{11})). \tag{17}$$

Proof. Consider the systems

$$\begin{aligned} u_1 &= M_{11}v_1 + M_{12}v_2 \\ u_2 &= M_{21}v_1 + M_{22}v_2 \end{aligned} \tag{18}$$

and

$$\begin{aligned} y_1 &= A_{11}z_1 + A_{12}z_2 + A_{13}z_3 \\ y_2 &= A_{21}z_1 + A_{22}z_2 + A_{23}z_3 \\ y_3 &= A_{31}z_1 + A_{32}z_2 + A_{33}z_3. \end{aligned} \tag{19}$$

In conformity with the relationships between the blocks M_{ij} $(i, j = 1, 2)$ and $A_{k,l}$ $(k, l = 1, 2, 3)$, we have

$$\begin{aligned} u_1 &= (y_1, y_2) \qquad & v_1 &= (z_1, z_2) \\ u_2 &= y_3 & v_2 &= z_3. \end{aligned}$$

The Schur complement (M/M_{11}) is produced by $\langle u_1, v_1 \rangle$, i.e., by pivoting on M_{11} in (18). The same effect is produced by performing two block pivots in (19), namely $\langle y_1, z_1 \rangle$ followed by $\langle y_2, z_2 \rangle$. The pivot block for $\langle y_2, z_2 \rangle$ is the leading principal submatrix of (M/A_{11}) given by (M_{11}/A_{11}). Thus, we have (17). □

Equation (17) is called the *quotient formula* for the Schur complement. One feature to notice is how it *formally* resembles the algebraic rule used for simplifying a complex fraction.

Principal pivoting in conjunction with the quotient formula leads to a technique for determining whether the leading principal minors of a square matrix are positive (or are nonzero). Let the given matrix be M. If m_{11} is nonpositive (zero), we stop. Otherwise we pivot on m_{11}. The leading entry of (M/m_{11}) is the leading principal minor of order 2 divided by m_{11}. Checking this ratio, we decide whether or not to continue. For example, suppose we are interested in positive leading principal minors. Since $\det m_{11} = m_{11}$, we can ascertain by inspection whether the leading principal minor of order 1 is positive. Suppose it is. Next, we want to find out whether the leading principal minor of order 2 is positive. We note that when $m_{11} > 0$:

$$\det \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} > 0$$

if and only if

$$(M/m_{11})_{11} = (1/\det m_{11}) \det \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} > 0.$$

If we obtain an affirmative answer, we then proceed to test the leading principal minor of order 3. This is done by performing a simple principal pivot on the leading entry of the previously obtained Schur complement, that is, (M/m_{11}). The leading entry of the new Schur complement is the leading principal minor of order 3 divided by the leading principal minor of order 2. The procedure continues in this manner. If M is of order n, then at most $n - 1$ simple principal pivots are required to complete the test. With each successive step, the order of the matrix in which the pivoting is done is one less than in the preceding step.

To make this more precise, let M_k denote the leading $k \times k$ principal submatrix of M, and let

$$M^{(k)} = (M/M_k).$$

This is the same as the Schur complement on hand after k simple principal pivots as described above. The leading entry of $M^{(k)}$ is

$$m_{11}^{(k)} = (M_{k+1}/M_k),$$

and we have

$$m_{11}^{(k)} = \det m_{11}^{(k)} = \frac{\det M_{k+1}}{\det M_k}.$$

In theory, this test can be used to determine whether a symmetric matrix $M \in R^{n \times n}$ is positive definite. In practice, however, one can run into the problem of round-off error when implementing the procedure on a computer.

Lexicographic ordering of vectors

An unfortunate fact of life is that the usual linear ordering of the real number system does not hold in higher dimensions. For instance, we can say that $(1,0) \geq (0,0)$ and $(0,1) \geq (0,0)$ but there is no such relationship between $(1,0)$ and $(0,1)$. The notion of lexicographic ordering of vectors remedies this problem in a particular way.

2.3.7 Definition. The nonzero vector $x \in R^n$ is *lexicographically positive (negative)* if and only if its first (i.e., lowest indexed) nonzero component is positive (negative). If x is lexicographically positive (negative) we write $x \succ 0$ ($x \prec 0$). A vector x is *lexicographically nonnegative (nonpositive)* if and only if it is either zero or lexicographically positive (negative). The symbols for the latter conditions are $x \succeq 0$ and $x \preceq 0$, respectively.

It is clear from this definition that every vector in R^n is either lexicographically positive, lexicographically negative, or zero. This is not the case with the ordinary partial ordering of vectors through the relations $\geq$ and $\leq$.

2.3.8 Definition. Let x and y be arbitrary vectors in R^n. Then x is *lexicographically greater than (less than)* y if and only if $x - y \succ 0$ ($x - y \prec 0$). In the former case, we write $x \succ y$ whereas in the latter we write $x \prec y$.

In general, for vectors x and y in R^n we have precisely one of the following three cases: $x \prec y$, $x \succ y$, or $x = y$. By analogy with the previously defined notation, we have the two possibilities: either $x \preceq y$ or $x \succeq y$.

2.3.9 Proposition. Every nonempty finite subset of R^n has a unique lexicographic maximum and a unique lexicographic minimum. □

In this proposition it is understood that the elements of the finite set are all distinct from each other. Finding the lexicographic maximum (or minimum) of such a set is a simple matter. Let $\{b^1, \ldots, b^m\}$ be a finite set of vectors in R^n, no two of which are equal. The first step is to compute

$$\mu_1 = \arg\max\{b_1^i : i = 1, \ldots, m\}.$$

If μ_1 is a singleton, say r, then $b^r \succ b^i$ for all $i \neq r$, so b^r is the lexicographic maximum of the set. If μ_1 contains more than one element, compute

$$\mu_2 = \arg\max\{b_2^i : i \in \mu_1\}.$$

The process is repeated in this manner component by component until a (necessarily unique) lexicographic maximum is found. Needless to say, the search for the lexicographic minimum is done analogously.

A common procedure in simplex-like methods of mathematical programming is the *minimum ratio test.* It arises in connection with systems of equations of the form

$$x_{\alpha_i} + a_{is}x_s = b_i \quad i = 1, \ldots, m \tag{20}$$

in which $b_i \geq 0$ for all i. We call (the nonbasic variable) x_s a *driving variable* as it "drives" the values of (the basic variables) x_{α_i}. One wants to know the largest value of x_s for which $x_{\alpha_i} \geq 0$ for all i; this value is given by

$$\hat{x}_s = \sup\{x_s : b_i - a_{is}x_s \geq 0, \text{ for all } i = 1, \ldots, m\}.$$

It is evident that $\hat{x}_s = \infty$ if and only if $a_{is} \leq 0$ for all i. On the other hand, if $a_{is} > 0$ for some i, then

$$\hat{x}_s = \min\{\frac{b_i}{a_{is}} : a_{is} > 0\}.$$

Finding r such that $a_{rs} > 0$ and $\hat{x}_s = b_r \,/\, a_{rs}$ is the minimum ratio test. We call x_{α_r} a *blocking variable* as it *blocks* the increase of the driving variable x_s. However, if $\hat{x}_s = \infty$, we say that x_s is *unblocked.* Making x_s a basic variable in place of x_{α_r} preserves the nonnegativity of the constants on the right-hand side of equation (20).

There is an analogous development in terms of lexicographic ordering. Let B denote a nonsingular $m \times m$ matrix with the property that the $m \times (m+1)$ matrix $[b, B]$ has lexicographically positive rows. (For instance, when $b \geq 0$ one can choose $B = I$, the identity matrix of order m.) Suppose the pivoting is to be done so as to preserve the lexicographic positivity of the transform of $[b, B]$. Again, under the assumption that $a_{is} > 0$ for some i, this can be accomplished by finding

$$\text{lexico}\min\ \{\frac{1}{a_{is}}\,[b_i,\, B_{i\bullet}] : a_{is} > 0\}.$$

Doing this identifies an index r such that

$$\frac{1}{a_{rs}}\,[b_r,\, B_{r\bullet}] \prec \frac{1}{a_{is}}\,[b_i,\, B_{i\bullet}] \qquad \text{for all } i \neq r.$$

Pivoting on a_{rs} transforms the matrix $[b,\ B]$ into another matrix $[b',\ B']$ for which

$$[b'_r,\ B'_{r\bullet}] = \frac{1}{a_{rs}}[b_r,\ B_{r\bullet}]$$

and

$$[b'_i,\ B'_{i\bullet}] = \frac{1}{a_{rs}}[a_{rs}b_i - a_{is}b_r,\ a_{rs}B_{i\bullet} - a_{is}B_{r\bullet}] \quad \text{for all } i \neq r.$$

2.3.10 Proposition. The rows of $[b',\ B']$ are linearly independent and lexicographically positive. □

2.4 Matrix Factorization

Many solution methods for the linear complementarity problem are based on the solution of linear equations. As a matter of fact, the pivoting process discussed in Section 2.3 is, in practice, implemented by solving systems of linear equations. For this reason, it is important to understand the algorithms and underlying theory of this aspect of linear algebra.

In this section, we focus on the topic of matrix factorizations which forms the theoretical foundation of the finite elimination methods for solving linear equations. There are many kinds of matrix factorizations; we review only the LU and Cholesky factorizations. The reader can find an extensive discussion of these and other matrix computations in the text by Golub and Van Loan (1989).

2.4.1 Theorem. Let $A \in R^{n\times n}$. Suppose that the first $n-1$ leading principal minors of A are nonzero. Then, there exist a lower triangular matrix L with unit diagonal entries, and an upper triangular matrix U such that

$$A = LU. \quad \square \tag{1}$$

The representation (1) is called the *LU* (or *triangular*) *factorization* of A. The assumption about the principal minors in Theorem **2.4.1** is essential for this factorization to be valid. For example, the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

has no LU factorization. This assumption about the leading principal minors can be dropped provided that one allows permutations of the rows and columns of the matrix A. This is the assertion of the following generalized triangular factorization result.

2.4.2 Theorem. Let $A \in R^{n \times n}$. Then, there exist permutation matrices P and Q, a lower triangular matrix L with unit diagonal entries, and an upper triangular matrix U such that

$$PAQ = LU. \quad \square \tag{2}$$

Both Theorems **2.4.1** and **2.4.2** can be proved *constructively*, i.e., by means of an algorithm which actually produces the desired factorization. The proof itself is quite insightful and reveals several interesting facts about the factorization. In what follows, we sketch the proof of **2.4.1**.

Let E_1 denote the *elementary matrix*

$$E_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -a_{21}/a_{11} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1}/a_{11} & 0 & \dots & 1 \end{bmatrix}$$

which is well defined because a_{11} is nonzero. Premultiplying the matrix A by E_1 yields the product matrix

$$E_1 A = \begin{bmatrix} a_{11} & A_1' \\ 0 & (A/a_{11}) \end{bmatrix}$$

where A_1' denotes the first row of A with a_{11} deleted. Note that in the matrix $E_1 A$, the first column of A is reduced to a unit vector with all entries below the main diagonal equal to zero, and the Schur complement (A/a_{11}) appears in the lower right block; this Schur complement is of order $n-1$ and has the property that its first $n-2$ leading principal minors are nonzero; the latter property follows from the Schur determinantal formula (2.3.14) and the quotient formula (2.3.17).

Repeating the above elimination process $n-1$ times, we deduce there exist elementary matrices $E_1, E_2, \ldots, E_{n-1}$ such that for each $k = 1, \ldots, n-1$,

$$E_k \cdots E_1 A = \begin{bmatrix} A_{11}^k & A_{12}^k \\ 0 & (A/A_k) \end{bmatrix},$$

where A_{11}^k is upper triangular and A_k is the k-th leading principal submatrix of A; moreover, each E_k is a lower triangular elementary matrix with unit diagonal entries and all entries equal to zero except for those below the k-th column. The desired factorization (1) is given by:

$$L = (E_{n-1} \cdots E_1)^{-1}, \quad U = \begin{bmatrix} A_{11}^{n-1} & A_{12}^{n-1} \\ 0 & (A/A_{n-1}) \end{bmatrix}.$$

Several observations can be drawn from the above argument: one, the factorization can be written in the form $L^{-1}A = U$ where L^{-1} is the product of $n-1$ special elementary matrices; two, the lower triangular factor L is itself the product of elementary matrices; and three, if A is written in the partitioned form:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} is nonsingular and if L and U are similarly partitioned:

$$L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix},$$

then $L_{11}U_{11}$ and $L_{22}U_{22}$ are, respectively, the LU factorization of A_{11} and that of the Schur complement (A/A_{11}). It follows from this last observation that the determinantal formulas below are valid:

$$\det A_{11} = \det U_{11} \qquad \det(A/A_{11}) = \det U_{22}; \tag{3}$$

in particular, $\det A = \det U$.

The representation of the matrix L as a product of elementary matrices is useful when dealing with large-scale sparse matrices. This is because when A contains many zero entries, the nonzero entries in each elementary matrix E_k will tend to be relatively few, whereas the product $E_k \ldots E_1$

will typically be much denser. Thus, by storing only the nonzero entries of the individual elementary matrices, one can substantially reduce the total amount of storage and therefore can handle sparse problems of large size more effectively.

The Cholesky factorization

A particular instance in which the assumption of Theorem **2.4.1** is satisfied occurs in the case of a symmetric positive definite matrix. Indeed, if A is such a matrix, then according to Proposition **2.2.16**, all the leading principal minors of A are positive. Thus, **2.4.1** is applicable. Moreover, in this case, the diagonal entries of U are all positive (this follows from the determinantal expressions in (3) and an inductive argument); by factoring out the diagonal entries of U, we can show that the resulting upper triangular factor is actually equal to the transpose of L. We state this factorization more precisely in the result below which provides an additional characterization of a symmetric positive definite matrix (cf. Proposition **2.2.16**).

2.4.3 Proposition. Let $A \in R^{n\times n}$ be symmetric. Then A is positive definite if and only if there exist a unique diagonal matrix D with positive diagonal entries and a unique lower triangular matrix L with unit diagonal entries such that

$$A = LDL^{\mathrm{T}}. \quad \square \tag{4}$$

2.4.4 Definition. The factorization (4) is known as the *Cholesky factorization* of the symmetric positive definite matrix A.

The Cholesky factorization of a symmetric positive definite matrix A is not difficult to compute. Indeed, the following recursion defines the diagonal entries of D and the lower off-diagonal entries of L:

$$d_{11} = a_{11}$$

and for $i = 2, \ldots, n$,

$$l_{ij} = \begin{cases} a_{i1}/d_{11} & \text{if } j = 1 \\ (a_{ij} - \sum_{k<j} l_{ik} d_{kk} l_{jk})/d_{jj} & \text{if } 2 \le j < i \end{cases}$$

$$d_{ii} = a_{ii} - \sum_{k<i} l_{ik}^2 d_{kk}.$$

The Gaussian elimination method

The basic method for solving a system of linear equations

$$Ax = b \tag{5}$$

where $A \in R^{n \times n}$ and $b \in R^n$ are given, is *Gaussian elimination*. This method is based on the triangular factorization of A as the product of a lower triangular and an upper triangular matrix. To simplify our discussion, we assume in the sequel that the determinantal assumption in Theorem **2.4.1** is satisfied, and write $A = LU$ where L and U are as specified in that theorem. Then, the solution of the system (5) can be accomplished in two steps.

1. Solve the lower triangular system

$$Ly = b. \tag{6}$$

2. Solve the upper triangular system

$$Ux = y. \tag{7}$$

Since L is a lower triangular matrix with unit diagonal entries, the vector y satisfying (6) is unique and can be computed by *forward substitution*; similarly, a desired solution x of the system (5), if it exists, can be obtained from (7) by *backward substitution.*

In the practical implementation of the Gaussian elimination process, care must be taken to reduce the adverse effect of round-off errors. In particular, even in cases where A has the LU factorization as specified in Theorem **2.4.1**, it is often necessary to permute some rows and columns of A during the factorization process in order to avoid division by quantities whose magnitude is too small. There are various strategies to accomplish a numerically stable factorization; the discussion of these numerical schemes is beyond the scope of this book, but can be found in many references, see **2.11.8**.

Updating matrix factorizations

In many solution methods for the LCP that require the recursive solution of systems of linear equations (as well as in a host of other instances within the field of mathematical programming), it is often the case that each system of linear equations is modified only slightly from one iteration to the next. As a matter of fact, the change often involves a simple rank-one, or rank-two matrix. Since the change is so minor, one would hope that by means of a relatively easy updating procedure, it is possible to compute the desired factorization of the modified matrix from the known factorization of the preceding matrix. Besides saving computational effort, such a matrix factorization updating scheme often helps to reduce round-off errors and preserve sparsity, thus obtaining the factorization more accurately and effectively.

Since a full description of the various factorization updating schemes is outside the domain of this book, we choose to illustrate the main idea using the Cholesky factorization of a symmetric positive definite matrix modified by a symmetric rank-one matrix.

Let $A \in R^{n\times n}$ be a symmetric positive definite matrix with a given Cholesky factorization

$$A = LDL^{\mathrm{T}}.$$

Let

$$\bar{A} = A + \sigma a a^{\mathrm{T}}$$

be positive definite. We wish to determine the factorization of $\bar{A}$:

$$\bar{A} = \bar{L}\bar{D}\bar{L}^{\mathrm{T}}. \tag{8}$$

The way to accomplish this is to note that

$$\bar{A} = LDL^{\mathrm{T}} + \sigma a a^{\mathrm{T}} = L(D + \sigma \bar{a}\bar{a}^{\mathrm{T}})L^{\mathrm{T}}$$

where $\bar{a} = L^{-1}a$. Due to the special structure, the Cholesky factorization of the matrix $D + \sigma\bar{a}\bar{a}^{\mathrm{T}}$ (which is positive definite because $\bar{A}$ is so) can be easily computed. Writing

$$D + \sigma\bar{a}\bar{a}^{\mathrm{T}} = \tilde{L}\tilde{D}\tilde{L}^{\mathrm{T}},$$

we obtain the desired factorization (8) with

$$\bar{L} = L\tilde{L} \quad \text{and} \quad \bar{D} = \tilde{D}.$$

The algorithm stated below summarizes a streamlined procedure for the computation of $\bar{L}$ and $\bar{D}$ which bypasses the explicit evaluation of $\tilde{L}$.

In the statement of the algorithm, d_i's denote the (given) diagonal entries of the matrix D.

2.4.5 Algorithm. (Gill-Golub-Murray-Saunders)

Step 1. Define $\sigma_1 = \sigma$ and $p^1 = a$.

Step 2. For $j = 1, 2, \ldots, n$, compute

$$\begin{aligned} \bar{d}_j &= d_j + \sigma_j (p^j_j)^2 \\ \beta &= p^j_j \sigma_j / \bar{d}_j \\ \sigma_{j+1} &= d_j \sigma_j / \bar{d}_j, \end{aligned}$$

and for $r = j+1, \ldots, n$,

$$\begin{aligned} p^{j+1}_r &= p^j_r - p^j_j l_{rj} \\ \bar{l}_{rj} &= l_{rj} + \beta\, p^{j+1}_r. \end{aligned}$$

If the given matrix A is sufficiently positive definite, that is, if its smallest eigenvalue is sufficiently large relative to some norm of $\bar{A}$, then the above algorithm is numerically stable and round-off error tends not to cause serious problem. However, if σ is negative and $\bar{A}$ is near singularity, it is possible that round-off error could cause the diagonal elements $\bar{d}_j$ to become very small. In such cases, extreme care is needed, and some other numerical schemes might be preferred.

2.5 Iterative Methods for Equations

In this section, we review some basic iterative methods for solving systems of equations and their convergence theory. In the case of linear equations, the emphasis of the discussion is placed on the class of iterative methods that are derived from a *matrix splitting*. These methods will form the basis for generalization to the linear complementarity problem. In the case of nonlinear equations, we shall describe the basic Newton method and the contraction principle. We conclude this section with a brief review of a general descent method for unconstrained minimization and a globally convergent Newton method for nonlinear equations.

Linear equations

Let $A \in R^{n\times n}$ and $b \in R^n$ be given. Consider the system of linear equations

$$Ax = b. \tag{1}$$

Suppose that A is expressed as the sum of two matrices B and C, i.e.,

$$A = B + C. \tag{2}$$

This representation is called a *splitting* of the matrix A. Given such a splitting of the matrix A, a sequence of vectors $\{x^\nu\} \subset R^n$ can be generated as follows: let x^0 be arbitrary; in general, given x^ν, we obtain the next iterate $x^{\nu+1}$ by solving the system

$$Bx = b - Cx^\nu;$$

thus,

$$x^{\nu+1} = B^{-1}b - B^{-1}Cx^\nu \tag{3}$$

provided that B is nonsingular.

In order for the above iteration to be practically effective, the matrix B must be chosen so that the vector $x^{\nu+1}$ can be easily computed. Some common choices of B are: a diagonal matrix with nonzero diagonal entries, a nonsingular triangular matrix, and a block diagonal matrix with nonsingular diagonal blocks. In particular, if the matrix A is written as

$$A = D + L + U$$

where D, L and U are the diagonal, strictly lower and strictly upper triangular parts of A respectively, and if D is nonsingular, then the choice $B = D$ leads to the *Jacobi iterative method* in which the sequence $\{x^\nu\}$ is defined recursively by: for $\nu = 0, 1, 2, \ldots$,

$$x_i^{\nu+1} = (b_i - \sum_{j\neq i} a_{ij}x_j^\nu)/a_{ii}, \quad i = 1, \ldots, n. \tag{4}$$

When $B = D + L$, we obtain the *Gauss-Seidel method* in which

$$x_i^{\nu+1} = (b_i - \sum_{j<i} a_{ij}x_j^{\nu+1} - \sum_{j>i} a_{ij}x_j^\nu)/a_{ii}, \quad i = 1, \ldots, n.$$

The difference between the Jacobi method and the Gauss-Seidel method is that in the latter, the components $\{x_j^{\nu+1}\}$ for $j < i$ are employed to update the i-th component of $x^{\nu+1}$, whereas in the former, none of the computed components of $x^{\nu+1}$ are used in the computation of the remaining components. Thus, the Gauss-Seidel method is a *sequential* procedure in the sense that the components of $x^{\nu+1}$ are updated successively, and the Jacobi method is a *parallel* scheme, meaning that the components of $x^{\nu+1}$ can be updated simultaneously.

A generalization of the Gauss-Seidel method is the family of *successive overrelaxation* (abbreviated as SOR) methods. In these methods, the matrix B is given by

$$B = \omega^{-1} D + L$$

where ω is a given parameter in the interval $(0, 2)$. With the matrix B as given, the sequence $\{x^\nu\}$ is defined by the recursion:

$$x_i^{\nu+1} = x_i^\nu + \omega a_{ii}^{-1}\Big(b_i - \sum_{j<i} a_{ij} x_j^{\nu+1} - \sum_{j \geq i} a_{ij} x_j^\nu\Big), \qquad i = 1, \ldots, n.$$

When $\omega = 1$, we recover the Gauss-Seidel method.

In general, given the splitting (2) of the matrix A, the question arises as to when the sequence $\{x^\nu\}$ defined iteratively by the expression (3) will converge to a solution of the basic system (1). The following theorem provides a complete characterization of this convergence.

2.5.1 Theorem. Let (2) be a splitting of the matrix $A \in R^{n \times n}$ with B nonsingular. Then for an arbitrary starting vector $x^0 \in R^n$, the sequence $\{x^\nu\}$ generated by (3) converges to the (unique) solution of the system (1) if and only if the matrix $B^{-1}C$ is convergent, i.e., if $\rho(B^{-1}C) < 1$. □

2.5.2 Remark. It follows from Proposition **2.2.14** that if B is nonsingular and if $B^{-1}C$ is convergent, then A must be a nonsingular matrix. Hence, the system (1) has a unique solution.

When the matrix A is symmetric, some specialized convergence results can be established. The following theorem provides a necessary and sufficient condition for the matrix $B^{-1}C$ to be convergent under a positive definiteness assumption on the matrix $B - C$.

2.5.3 Theorem. Let (2) be a splitting of the matrix $A \in R^{n\times n}$ with B nonsingular. Assume that A is symmetric and $B - C$ is positive definite. Then $\rho(B^{-1}C) < 1$ if and only if A is positive definite. □

Specializing this result to the SOR method, we obtain the well known Ostrowski-Reich theorem for the convergence of this method.

2.5.4 Corollary. Let $A \in R^{n\times n}$ be symmetric with a positive diagonal. Then the sequence $\{x^\nu\}$ produced by the SOR method converges for all $\omega \in (0,2)$ if and only if A is positive definite. □

The convergence results discussed so far pertain to nonsingular systems of linear equations. For singular systems, the following result gives an extension of Theorem **2.5.3**.

2.5.5 Theorem. Let (2) be a splitting of the matrix $A \in R^{n\times n}$ with B nonsingular. Assume that A is symmetric and $B - C$ is positive definite. Then for any vector b in the range of A and any starting vector $x^0 \in R^n$, the sequence $\{x^\nu\}$ generated by (3) converges to some solution of the system (1) if and only if A is positive semi-definite. □

Nonlinear equations

In this subsection, we review Newton's method for solving a system of nonlinear equations and the contraction principle for computing a fixed point of a mapping. We start with the former. Consider the system:

$$f(x) = 0 \tag{5}$$

where f is a mapping from R^n into itself. In *Newton's method* for solving this system, the mapping f is assumed to be continuously differentiable. This method generates a sequence of iterates $\{x^\nu\}$ in the following way. Given the iterate x^ν, we linearize f at x^ν and get the next iterate $x^{\nu+1}$ by solving the resulting system of linear equations:

$$f(x^\nu) + \nabla f(x^\nu)(x - x^\nu) = 0.$$

Assuming that the Jacobian matrix $\nabla f(x^\nu)$ is nonsingular, we obtain $x^{\nu+1}$ as

$$x^{\nu+1} = x^\nu - \nabla f(x^\nu)^{-1} f(x^\nu). \tag{6}$$

The convergence of the Newton sequence $\{x^\nu\}$ is ensured by the following result.

2.5.6 Theorem. Suppose that x^* is a zero of f and that f is continuously differentiable in a neighborhood of x^*. Suppose also that the Jacobian matrix $\nabla f(x^*)$ is nonsingular. Then there exists a neighborhood of x^* such that if the initial iterate x^0 is chosen in it, the entire Newton sequence $\{x^\nu\}$ is well defined and converges to x^*. Moreover, if ∇f is Lipschitz continuous at x^*, then the *rate of convergence* is quadratic; i.e., there exists a constant $c > 0$ such that for all ν sufficiently large,

$$\|x^{\nu+1} - x^*\| \leq c\|x^\nu - x^*\|^2. \quad \square$$

Two important features of the above convergence result are worth mentioning. First, **2.5.6** asserts only the *local convergence* of the sequence $\{x^\nu\}$; that is to say, the initial iterate x^0 must be chosen sufficiently close to the solution x^*. Second, under an additional Lipschitzian assumption on the derivative ∇f, the convergence rate of the method is quadratic; the latter property is what makes Newton's method rank among the most effective computational schemes in practice.

Both Newton's method (for nonlinear equations) and the matrix splitting method (for linear equations) are based on the idea of a fixed-point iteration. This is a computational scheme for finding the (unique) fixed point of a function of a special type. More specifically, let $g : R^n \to R^n$ be a mapping whose fixed point is being sought. One may generate a sequence $\{x^\nu\}$ by choosing an arbitrary starting point x^0 and by letting

$$x^{\nu+1} = g(x^\nu), \quad \nu = 0, 1, 2, \ldots. \tag{7}$$

This is known as a *fixed-point iteration.* Clearly, the iteration (3) corresponds to the iteration (7) for finding the fixed point of the affine mapping

$$g(x) = B^{-1}b - B^{-1}Cx. \tag{8}$$

Similarly, the Newton iteration (6) is derived from the mapping

$$g(x) = x - \nabla f(x)^{-1} f(x). \tag{9}$$

Rigorously speaking, the latter mapping g is defined only at those points x at which the Jacobian matrix $\nabla f(x)$ is nonsingular. We shall return to more discussion of this later. For now, we introduce the following important concept.

2.5.7 Definition. A mapping $g : \mathcal{D} \subseteq R^n \to R^n$ is said to be *nonexpansive* if there exist a vector norm $\|\cdot\|$ on R^n and a constant $c \in (0,1]$ such that for all $x, y \in \mathcal{D}$,

$$\|g(x) - g(y)\| \leq c\|x - y\|.$$

If $c < 1$, g is said to be *contractive*, or a *contraction*.

Note that a nonexpansive mapping is Lipschitz continuous with modulus given by c.

2.5.8 Theorem. (Contraction Principle) Suppose that $g : R^n \to R^n$ is a contraction (with constant $c \in (0,1)$ and with respect to the norm $\|\cdot\|$). Then for any starting vector x^0, the sequence $\{x^\nu\}$ generated by the iteration (7) converges to the unique fixed point x^* of g. Moreover, the following error estimate holds: for $\nu = 1, 2, 3, \ldots$,

$$\|x^\nu - x^*\| \leq \frac{c}{1-c}\|x^\nu - x^{\nu-1}\|. \quad \square$$

The contraction principle can be used to demonstrate the convergence of a sequence without the explicit reference to a particular mapping g. More specifically, we say that a sequence of vectors $\{x^\nu\} \subseteq R^n$ is a *contraction* if there exist a vector norm $\|\cdot\|$ on R^n and a constant $c \in (0,1)$ such that for all ν,

$$\|x^{\nu+1} - x^\nu\| \leq c\|x^\nu - x^{\nu-1}\|.$$

Similar to **2.5.8**, one can show that every contraction sequence must converge. Of course, if there is no contraction mapping associated with such a sequence, then the limit of the sequence need not be a fixed point of a mapping.

By Proposition **2.2.11**, one can easily show that if g is an affine mapping, then g is a contraction if and only if $\rho(\nabla g(x)) < 1$. Hence, specializing the above theorem to the mapping (8), one obtains **2.5.1**. To deduce the convergence of Newton's method, we need a local version of **2.5.8**.

2.5.9 Theorem. Suppose that $g : \mathcal{D} \subseteq R^n \to R^n$ is F-differentiable at the fixed point $x^* \in \operatorname{int} \mathcal{D}$. If $\rho(\nabla g(x^*)) < 1$, then there exists a neighborhood

$N \subseteq \mathcal{D}$ of x^* such that for any starting vector x^0 chosen from N, the sequence $\{x^\nu\}$ generated by the iteration (7) remains in N and converges to x^*. □

It should be pointed out that in the above result, the mapping g is assumed to be F-differentiable at just the fixed point x^*; its differentiability at other points is not assumed. The reader is asked to show that the mapping (9), which gives rise to Newton's method, satisfies the assumption of this theorem; see Exercise **2.10.13**.

Descent methods for equations and optimization

A practical drawback of the basic Newton method for solving the system of equations (5) is that the initial iterate needs to be chosen close to a solution in order to ensure the convergence of the sequence generated. There are various ways to overcome this weakness of the method; the general idea is to devise schemes to enlarge its domain of convergence. In what follows, we describe a commonly used approach, known as the *damped Newton method*, for accomplishing this objective. Our choice of this particular method as the subject of discussion serves two purposes. One is as just explained; the other is to make use of this opportunity to review a basic technique for solving optimization problems that is itself useful in the context of the LCP.

The main idea involved in the modified Newton method is to simply *dampen* the Newton step. More specifically, rather than defining the next iterate $x^{\nu+1}$ directly by (6), we let d^ν denote the (unique) solution of the system of linear equations

$$f(x^\nu) + \nabla f(x^\nu)d = 0, \tag{10}$$

and then define $x^{\nu+1}$ as

$$x^{\nu+1} = x^\nu + \tau_\nu d^\nu$$

where $\tau_\nu \in (0, 1]$ is a certain scalar, called the *steplength* or *stepsize*. Typically, this stepsize is chosen so that for a certain real-valued function $\theta : R^n \to R$, we have

$$\theta(x^{\nu+1}) < \theta(x^\nu). \tag{11}$$

Such a function, if it exists, is called a *merit* or *potential function* for the method. A natural question arises: for the vector d^ν defined above, what is an appropriate merit function? Before answering this question, we digress a little to discuss a basic technique for solving optimization problems.

The underlying theme of the damped Newton method outlined above is rooted in the problem of *unconstrained minimization*, i.e., the problem:

$$\text{minimize} \quad \theta(x) : x \in R^n \tag{12}$$

where $\theta : R^n \to R$ is a real-valued function. Central to the latter problem is the notion of descent as defined below.

2.5.10 Definition. Let $\theta : R^n \to R$ be a directionally differentiable function. A vector $d \in R^n$ is called a *descent direction* for θ at a point x if $\theta'(x, d) < 0$.

By the definition of the directional derivative (see **2.1.18**), it follows that if d is a descent direction for θ at x, then

$$\theta(x + \tau d) < \theta(x)$$

for all $\tau > 0$ sufficiently small; this implies that by starting at the vector x and moving along the direction d, then provided that the movement is not too far away from x, a strict decrease in the value of the objective function θ is guaranteed. Presumably, this decrease is desirable as we are trying to minimize θ. Note that if no descent direction exists for θ at x, then x is a stationary point for θ.

In general, the descent property (11) of a sequence of iterates $\{x^\nu\}$ is far from being sufficient for its convergence. An additional notion that is important in this regard is that of *sufficient decrease*. More specifically, if x is a current iterate and d is a descent direction for θ at x, then we want to choose a stepsize $\tau \in (0, 1]$ such that

$$\theta(x + \tau d) - \theta(x) \leq \sigma \tau \theta'(x, d) \tag{13}$$

where $\sigma \in (0, 1)$ is a prescribed scalar that is independent of the pair (x, d). In practice, the required steplength τ is chosen to equal ρ^m where ρ, like σ, is a prescribed scalar in the interval $(0, 1)$, and m is the smallest nonnegative integer for which the inequality (13) holds (with $\tau = \rho^m$). The constant ρ is called a *backtracking factor*.

When the system of equations (5) is cast as a minimization problem, a natural objective function is

$$\theta(x) = \tfrac{1}{2} f(x)^{\mathrm{T}} f(x).$$

Clearly, θ is F-differentiable if f is so; in this case, we have

$$\nabla\theta(x) = \nabla f(x)^{\mathrm{T}} f(x).$$

Hence, if d^ν satisfies the equation (10), then,

$$\nabla\theta(x^\nu)^{\mathrm{T}} d^\nu = -f(x^\nu)^{\mathrm{T}} f(x^\nu)$$

which is negative provided that $f(x^\nu) \neq 0$. Consequently, this vector d^ν is a descent direction for θ at x^ν if the latter is not a zero of f.

In Exercise **2.10.15**, the reader is asked to establish several convergence properties of the the descent method described above for solving the general minimization problem (12), and to specialize the results to prove a global convergence property of the damped Newton method for finding a zero of the mapping f.

2.6 Convex Polyhedra

The linear complementarity problem is usually specified by a system of linear inequalities and an orthogonality condition. (The latter can be construed as a *nonlinear equation.*) In general, finite systems of linear inequalities give rise to sets known as convex polyhedra, the main subject of this section. Our focus here will be mainly on the geometric side of the subject; the algebraic side is treated in the next section. Much of what we do in these two sections is ordinarily included in books and courses dealing with linear programming. Readers already familiar with the latter subject (as we assume many will be) may safely skip ahead, but we advise at least a glance at the terms, notations, and results recorded here.

Convex sets

The subject of convex sets was briefly touched on in Section 2.1. Here we shall concentrate on convex sets of a certain type.

2.6.1 Definition. A *(convex) polyhedron* is the intersection of finitely many closed halfspaces.

A convex polyhedron is also called a *polyhedral convex set.* In situations where the convexity is understood to hold, such a set is simply called a *polyhedron.* As the intersection of closed sets, a convex polyhedron is itself a closed set.

According to the preceding definition, a convex *polyhedron* is given as the (possibly empty) solution set of a finite system of linear inequalities, e.g.,

$$a_i^{\mathrm{T}} x \geq b_i \quad i = 1, \ldots, m.$$

This definition gives an *external* representation of the polyhedron, namely through a finite collection of halfspaces that contain it. One of the fascinating aspects of the theory is how one also can describe a convex polyhedron through an *internal* representation. We shall come to this shortly.

2.6.2 Definition. Let S be a nonempty subset of R^n. The *convex hull* of S, denoted $\operatorname{conv} S$, is the intersection of all convex subsets of R^n that contain S. The *affine hull* of S, denoted $\operatorname{affn} S$, is the intersection of all affine subspaces in R^n that contain S.

2.6.3 Remarks. We note the following facts about the concept of convex hull. (Similar facts hold in the case of affine hull.)

(a) Since the space in which S lies is convex, the convex hull of S is well defined.

(b) A set S is convex if and only if $S = \operatorname{conv} S$.

(c) The convex hull of a set S is defined *externally*, i.e., in terms of sets that *contain* S.

The following theorem gives an internal characterization of a convex set and hence also of the convex hull of a set.

2.6.4 Theorem. A set C is convex if and only if it contains every convex combination of every finite subset of its points. □

It follows from the above theorem that the convex hull of a set $S \subseteq R^n$ is equal to the set of all convex combinations of all finite subsets of points from S. In general, if a set S is not already convex, then taking the convex hull of S is a process of filling in certain missing points in order to obtain a set that is convex.

2.6.5 Definition. A *polytope* is the convex hull of a finite set of points.

The convex hull of two distinct points is a closed line segment; conversely, every closed line segment is the convex hull of its end points. The convex hull of three noncollinear points is a triangle (with its interior points). Such a set is called a 2-*simplex*, whereas a closed line segment is called a 1-*simplex*. To define simplices of higher dimension, one needs the concept of points in general position.

2.6.6 Definition. A set of $k+1$ points $x^0, x^1, \ldots, x^k \in R^n$ is said to be *in general position* provided $x^1 - x^0, \ldots, x^k - x^0$ are linearly independent vectors.

The linear independence condition limits the cardinality of a set of points in general position; in particular, no set of more than $n+1$ points in n-space can be in general position. A set of $k+1$ points in general position determines a unique k-dimensional linear manifold.

2.6.7 Definition. A k-*simplex* is the convex hull of $k+1$ points in general position.

A tetrahedron, for example, is the convex hull of 4 points in general position. Points in general position are, in a certain sense, special. The idea is formalized in the following definition which is of great importance in the study of convex sets generally and convex polyhedra in particular.

2.6.8 Definition. Let C be a convex set. Then $x \in C$ is an *extreme point* of C if and only if for all $x^1, x^2 \in C$ and for all $\lambda \in [0,1]$,

$$x = \lambda x^1 + (1-\lambda)x^2 \text{ for all } x^1 \neq x^2 \quad \Rightarrow \quad \lambda \in \{0,1\}.$$

Geometrically, this definition means that an extreme point cannot lie in the *open* line segment between any two distinct points of C. Nonzero subspaces, linear manifolds, and halfspaces have no extreme points. In general, open sets have no extreme points. Notice that the set of all extreme points of a convex set need not be in general position.

2.6.9 Theorem. Let $S = \{x^1, \ldots, x^m\}$ and let $C = \text{conv}\, S$. Every extreme point of C is an element of S, and C is the convex hull of these extreme points. □

The preceding theorem does not assert that every point of the generating set S is an extreme point of C; such a statement would not be true in general.

2.6.10 Theorem. Every polytope is a compact set. □

We now ask the reader to recall the terms, "uses" and "support (of a vector)" that are defined in Section 1.3. The first occurs in the paragraph just above **1.3.2**. The second is given in **1.3.3**. These terms are clearly related to one another. For example, when a vector $\bar{x}$ satisfies the system of equations given by $Ax = b$, the representation $b = A\bar{x}$ uses the columns of A that are indexed by the elements of $\text{supp}\, \bar{x}$. These terms lend precision to one of the central concepts of linear programming and its extensions.

2.6.11 Definition. A *basic solution* of the system of equations $Ax = b$ is a solution that uses only linearly independent columns of A.

2.6.12 Theorem. If the system $Ax = b$ has a nonnegative solution, then the system has a nonnegative *basic* solution. □

The intimate connection between nonnegative basic solutions and extreme points is summarized by the following result.

2.6.13 Theorem. Let $\bar{x} \in X = \{x : Ax = b,\ x \geq 0\}$. Then $\bar{x}$ is an extreme point of X if and only if $\bar{x}$ is a basic solution of $Ax = b$. □

It follows from the above theorem that a convex polyhedron can have only finitely many extreme points.

We close this subsection by introducing the important concept of relative interior of a set.

2.6.14 Definition. Let S be a subset of R^n. A vector $x \in S$ is called a *relative interior point* of S if there exists an (open) neighborhood N of x such that $N \cap \text{affn}\, S \subseteq S$. The *relative interior* of S, denoted $\text{ri}\, S$, is the set of all relative interior points of S. The set $\text{cl}\, S \setminus \text{ri}\, S$ is called the *relative boundary* of S.

In general, a convex set $S \subseteq R^n$ need not have a nonempty (topological) interior in R^n; nevertheless, it is a known fact that the relative interior of a nonempty convex set must be nonempty.

Cones

Section 1.3 gives an indication of the important role played by cones in the theory of the linear complementarity problem. See **1.3.1** for the basic definition of a cone.

Since cones are nonempty, by definition, they all contain 0. Not all cones are convex, however. Necessary and sufficient conditions for the latter property are easily expressed.

2.6.15 Proposition. Let C be a cone. Then C is convex if and only if

$$C = C + C = \{z : z = x + y, \quad x, y \in C\}. \quad \square$$

If X is a nonempty subset of R^n, the *conical hull* of X is the intersection of all convex cones in R^n that contain X. Analogous to Theorem **2.6.4**, one can show that the conical hull of X is equal to the set of all nonnegative linear combinations of all finite subsets of points from X. A few other examples of cones are listed below.

(a) All linear subspaces.

(b) Finite cones, e.g., $\text{pos}\, A$ where $A \in R^{m \times n}$.

(c) *Polyhedral cones*, i.e., solutions of homogeneous linear inequality systems such as $\{x : Ax \geq 0\}$ and R^n_+.

(d) Intersections of cones.

In Definition **1.5.1**, we have introduced an important cone associated with an arbitrary set S in R^n. This is the dual cone S^* defined by

$$S^* = \{y : y^{\mathrm{T}} x \geq 0 \quad \text{for all } x \in S\}.$$

We have also noted there that regardless of what property the given set S might have, S^* is always a closed convex cone. In what follows, we record two other useful facts about this special cone.

2.6.16 Proposition. Let $S_1, \ldots, S_m$ be subsets of R^n. Then,

(a) $S_1 \subseteq S_2 \;\Rightarrow\; S_1^* \supseteq S_2^*$;

(b) $\left(\sum_{i=1}^m S_i\right)^* \supseteq \bigcap_{i=1}^m S_i^*$, and equality holds if $0 \in \bigcap_{i=1}^m S_i$. □

It is clear that $C \subseteq C^{**}$ (the dual cone of the dual cone). Since C^{**} is necessarily closed and convex, it is immediate that the reverse inclusion $C \supseteq C^{**}$ can hold only if C is a *closed convex* cone. That is,

$$C = C^{**} \quad \Rightarrow \quad C \text{ is a closed and convex cone.}$$

The converse is also true, but proving it requires some machinery in the general case. There is, however, an important special case that is not hard to handle.

2.6.17 Theorem. If C is a finite cone, then $C = C^{**}$. □

As a direct consequence of this theorem and the observations made above, we get the following result.

2.6.18 Corollary. Every finite cone is a closed set. □

When $A = [A_{\bullet 1}, \ldots, A_{\bullet n}]$, each column vector $A_{\bullet j}$ is an element of $\operatorname{pos} A$. Hence $\operatorname{pos} A$ is generated from a finite set of its own elements. This is an *interior description* of the cone. Polyhedral cones are given by *exterior descriptions*. Actually, every polyhedral cone is a finite cone and vice versa. This remarkable fact can be deduced from the assertions below.

(a) Every linear subspace L is a finite cone.

(b) Every polyhedral cone of the form $C = \{x : Ax = 0,\ x \geq 0\}$ is a finite cone.

(c) If L is a linear subspace of R^n, then $L \cap R^n_+$ is a finite cone.

One direction of the "remarkable fact" stated above is a theorem attributed to Minkowski.

2.6.19 Theorem. If $A \in R^{m \times n}$, then the polyhedral cone

$$C = \{x : Ax \geq 0\}$$

is a finite cone. □

The opposite direction is a theorem due to Weyl.

2.6.20 Theorem. If $A \in R^{m\times n}$, then the finite cone $C = \text{pos}\, A$ is a polyhedral cone. □

As a consequence of Theorem **2.6.20**, we can assert the polyhedrality of any polytope.

2.6.21 Corollary. A polytope is a polyhedral convex set. □

The technique for proving this fact uses (2.1.4). Defining

$$C = \text{pos}\left[\begin{bmatrix} x^1 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} x^r \\ 1 \end{bmatrix}\right],$$

we can view $X = \text{conv}\{x^1, \ldots, x^r\} \subset R^n$ as the intersection (in R^{n+1}) of C with the hyperplane $\{(x, \xi) : \xi = 1\}$. Weyl's Theorem **2.6.20** implies there exists some matrix $A \in R^{m\times(n+1)}$ such that

$$\begin{bmatrix} x \\ \xi \end{bmatrix} \in C \quad \Leftrightarrow \quad A_{\bullet 1}x_1 + \cdots + A_{\bullet n}x_n + A_{\bullet n+1}\xi \geq 0.$$

Then with $b = -A_{\bullet n+1}$, it follows that $x \in X$ if and only if $A_{\bullet \overline{n+1}}x \geq b$. This is *not* to say, however, that an arbitrary polyhedron is a polytope.

2.6.22 Remark. The preceding corollary has a natural extension to sets in R^n which are sums of polytopes and finite cones. See Exercise **2.10.16** for a precise statement of the result.

The structure of convex polyhedra

Let $X = \{x : Ax = b,\ x \geq 0\}$ and $Y = \{x : Ax = 0,\ e^{\mathrm{T}}x = 1,\ x \geq 0\}$. If $X \neq \emptyset$, it has a finite number of extreme points, say $x^1, \ldots, x^r$. If $Y \neq \emptyset$, it has a finite number of extreme points, say $y^1, \ldots, y^s$. Let P denote the convex hull of $\{x^1, \ldots, x^r\}$ and C the conical hull of $\{y^1, \ldots, y^s\}$. Then clearly,

$$C = \{x : Ax = 0,\ x \geq 0\} \tag{1}$$

and $P + C \subseteq X$. As a matter of fact, the reverse inclusion also holds. These facts are summarized in the following result.

2.6.23 Theorem. Every nonempty polyhedral set is the sum of a polytope and a finite cone. That is, if $X = \{x : Ax = b,\ x \geq 0\} \neq \emptyset$, there exist a polytope P and a finite cone C such that

$$X = P + C. \tag{2}$$

Moreover, in any such representation, $C = \{x : Ax = 0,\ x \geq 0\}$.□

This is known as *Goldman's resolution theorem*. An immediate consequence of this theorem is that the polyhedron X (represented as above) is bounded if and only if the associated cone C consists of the zero vector alone, or equivalently, if the set Y is empty. In general, the (external) representation of the cone C varies with that of X; a purely topological characterization of C is possible and is given in in Exercise **2.10.18**. The following result is another useful consequence of Theorem **2.6.23**.

2.6.24 Theorem. If X is a polyhedron in R^n and $f : R^n \to R^m$ is an affine transformation, then $f(X)$ is a polyhedron in R^m and hence is a closed set. □

Actually, a slightly more detailed version of Goldman's resolution theorem can be stated, but to do this we need to introduce some terminology.

2.6.25 Definition. Let C be a convex cone in R^n. The set

$$L = C \cap (-C) \tag{3}$$

is called the *lineality space* of C. The dimension of L is called the *lineality* of C. A cone C for which the lineality is 0 is said to be *pointed*.

To illustrate the terms defined above, consider R^2_+ and the "upper half plane," $U = \{(x_1, x_2) : x_2 \geq 0\}$. The former is a pointed cone whereas the latter is not. The lineality space L of the cone U is $\{(x_1, x_2) : x_2 = 0\}$, so its lineality is 1. Notice that the upper half plane U in R^2 is the sum of L and the first quadrant R^2_+, i.e., the sum of its lineality space and a pointed cone.

2.6.26 Corollary. Every nonempty polyhedral set is the sum of a polytope, a finite pointed cone, and a linear subspace. That is, if X is a polyhedral set, there exists a polytope P, a finite pointed cone C and a linear subspace L such that

$$X = P + C + L. \quad \square \tag{4}$$

Bases and basic solutions

Suppose $A \in R^{m \times n}$, and $\operatorname{rank} A = m$. Define

$$X = \{x : Ax = b, \ x \geq 0\}.$$

If $X \neq \emptyset$, then the following hold.

(a) X has at least one extreme point.

(b) $m \leq n$. We may assume $m < n$ for otherwise $X = \{A^{-1}b\}$ which is trivial.

(c) A contains a nonsingular $m \times m$ submatrix, B. Such a matrix is called a *basis* in A. The basis B is called a *feasible basis* if $B^{-1}b \geq 0$.

Let $\sigma = \operatorname{supp} \bar{x}$ where $\bar{x}$ is an extreme point of X. If $\bar{x} \neq 0$, we have $|\sigma| = r > 0$ for some integer r and $A_{\bullet\sigma} \in R^{m \times r}$ has linearly independent columns.

Now what if $r < m$? Since A has rank m, the matrix $A_{\bullet\sigma}$ can be extended to a basis $\hat{B}$ in A. That is, $m - r$ columns can be adjoined to those of $A_{\bullet\sigma}$ so as to make m linearly independent columns of A. (This is a standard result from linear algebra.) Note that the equation $\hat{B}u = b$ has a unique solution: $u = (\hat{B})^{-1}b$. But since $A_{\bullet\sigma}$ is a submatrix of $\hat{B}$ and $A_{\bullet\sigma}\bar{x}_\sigma = b$, it follows that the basic solution of $Ax = b$ corresponding to $\hat{B}$ is just $\bar{x}$. Thus, although there may be many ways to extend the $A_{\bullet\sigma}$ to a full-size basis in A, all the corresponding basic solutions equal $\bar{x}$.

2.6.27 Definition. Let $A \in R^{m \times n}$ have rank m. If $\bar{x}$ is a basic solution to $Ax = b$, then $\bar{x}$ is said to be *nondegenerate* if and only if $|\operatorname{supp} \bar{x}| = m$. Otherwise, it is said to be *degenerate*.

Let X be as defined above; the dimension of X must be $n - m$ if X contains at least one nondegenerate point.

2.6.28 Example. Let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & -4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

The rank of A is 2. There are two bases in A:

$$B = [A_{\bullet 1}, A_{\bullet 2}] = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \tilde{B} = [A_{\bullet 2}, A_{\bullet 3}] = \begin{bmatrix} 0 & -2 \\ 1 & -4 \end{bmatrix}.$$

Note that B is a feasible basis and $\tilde{B}$ is not a feasible basis. The matrix with columns $A_{\bullet 1}$ and $A_{\bullet 3}$ is not a basis as it is singular. The set $X = \{x : Ax = b,\ x \geq 0\}$ has only one extreme point: $\bar{x} = (3, 0, 0)$. The set $Y = \{x : Ax = 0,\ x \geq 0\}$ contains the nonzero vector $x^0 = (2, 0, 1)$. Since $\operatorname{rank} A + \operatorname{nullity} A = 3$, all elements of Y are nonnegative multiples of x^0. In fact (recall the Goldman resolution theorem)

$$\begin{aligned} X &= \{x : x = \bar{x} + \lambda x^0,\ \lambda \geq 0\} \\ &= \{\bar{x}\} + \operatorname{pos} x^0. \end{aligned}$$

Notice that the set X is only one-dimensional.

Edges of polyhedra

2.6.29 Notation. Let x and y be distinct points in R^n. Let

$$\ell(x, y) = \{z : z = \lambda x + (1 - \lambda) y,\ 0 < \lambda < 1\},$$

$$\ell[x, y] = \{z : z = \lambda x + (1 - \lambda) y,\ 0 \leq \lambda \leq 1\}.$$

These sets are, respectively, the *open* and *closed* line segments between x and y.

2.6.30 Definition. Let C be a convex set, and let E be a convex subset of C. Then E is an *extreme subset* of C if and only if

$$\left.\begin{array}{c} x, y \in C \\ \ell(x, y) \cap E \neq \emptyset \end{array}\right\} \quad \Rightarrow \quad x, y \in E.$$

(This means that no point of E belongs to the open line segment spanned by two points of C not both of which belong to E.)

2.6.31 Definition. If X is a polyhedron and H is a hyperplane, then $H \cap X$ is called a *face* of X if $\emptyset \neq H \cap X \subseteq \text{rb}\, X$. A k-dimensional face of X is called a *k-face* of X. If X is p-dimensional, then the 0-faces, 1-faces, and $(p-1)$-faces of X are called, respectively, the *vertices*, *edges*, and *facets* of X.

2.6.32 Remark. A face is always an extreme set. If X is a polyhedron, then every extreme set is a face (see Theorem **2.7.5**). In this book, complementary cones are polyhedral sets of particular interest. We use Definition **1.3.2**, not Definition **2.6.31**, when referring to the facets of a complementary cone. For full complementary cones, the two definitions are equivalent.

The following theorem, known as *Euler's relation*, is a basic result concerning the number of faces of a bounded polyhedral set.

2.6.33 Theorem. If X is an p-dimensional polytope, and if we let f_k denote the number of k-faces of X, then

$$\sum_{k=0}^{p-1} (-1)^k f_k \;=\; 1 - (-1)^p \,.$$

In general, an edge of a polyhedron can be a line segment (two endpoints, both of which are extreme points), a halfline (one endpoint which is an extreme point) or an entire line (no endpoints). In the case of a pointed cone, an edge which is also a halfline is called an *extreme ray*. It can be shown that this definition coincides with the one given in **1.3.1**.

Sometimes it is handy to have a notation for the set of all extreme points of a polyhedral set. For any polyhedral set, X, let $\ddot{X}$ denote the set of all extreme points of X. The set $\ddot{X}$ (which could be empty) is called the *profile* of X.

2.6.34 Definition. Let $x^1, x^2 \in \ddot{X}$. Then x^1 and x^2 are *adjacent* if and only if $\ell[x^1, x^2]$ is an edge of X.

For some matrix $A \in R^{m \times n}$ and $b \in R^m$, let $X = \{x : Ax = b,\ x \geq 0\}$. We are interested in an algebraic characterization of adjacency.

2.6.35 Theorem. Let x^1 and x^2 be distinct elements of $\ddot{X}$. Let

$$\alpha = \{i : x_i^1 > 0 \text{ or } x_i^2 > 0\}.$$

Then x^1 and x^2 are adjacent if and only if the rank of $A_{\bullet\alpha}$ is $|\alpha| - 1$.

Once again, let $X = \{x : Ax = b, x \geq 0\}$ where $A \in R^{m \times n}$ has rank m. For simplicity, let $A = [A_{\bullet\beta}, A_{\bullet\bar{\beta}}]$ where $A_{\bullet\beta}$ is a basis in A. We can write

$$Ax = A_{\bullet\beta}x_\beta + A_{\bullet\bar{\beta}}x_{\bar{\beta}} = b,$$

and hence we have

$$x_\beta = A_{\bullet\beta}^{-1}b - A_{\bullet\beta}^{-1}A_{\bullet\bar{\beta}}x_{\bar{\beta}}.$$

Let $A_{\bullet\bar{\beta}} = A_{\bullet\beta}^{-1}A_{\bullet\bar{\beta}}$ and $\bar{b} = A_{\bullet\beta}^{-1}b$. If $A_{\bullet\beta}$ is a feasible basis in A, (that is, if $A_{\bullet\beta}^{-1}b \geq 0$), then

$$\begin{bmatrix} x_\beta \\ x_{\bar{\beta}} \end{bmatrix} \longleftrightarrow x_{\bar{\beta}}$$

gives a one-to-one correspondence between points in X and points in the polyhedral set

$$\bar{X} = \{x_{\bar{\beta}} : Ax_{\bar{\beta}} \leq \bar{b},\ x_{\bar{\beta}} \geq 0\}.$$

Note that $\bar{b} \geq 0$ implies $x_{\bar{\beta}} = 0$ is an element of $\bar{X}$. Clearly $x_{\bar{\beta}} = 0$ is an extreme point of $\bar{X}$ and the corresponding element of X, namely

$$x = \begin{bmatrix} x_\beta \\ x_{\bar{\beta}} \end{bmatrix} = \begin{bmatrix} \bar{b} \\ 0 \end{bmatrix}$$

is an extreme point of X. There is, in fact, a one-to-one correspondence between extreme points of X and those of $\bar{X}$.

2.7 Linear Inequalities

Our purpose in this section is to record some properties of linear inequality systems. We are particularly interested in what are called "theorems of the alternative." These have important applications in the theory of linear programming, quadratic programming and the linear complementarity problem. One approach to this subject is through "separation theorems." This is the subject we take up first.

Separation of convex sets

We return briefly to more general (i.e., not necessarily polyhedral) convex sets in order to discuss the idea of separation by hyperplanes. Let $H = \{x : a^{\mathrm{T}}x = b\}$ be a hyperplane in R^n. (Hence $a \neq 0$.) Let S_1 and S_2 be two nonempty subsets of R^n. Then H *separates* S_1 and S_2 if $a^{\mathrm{T}}x \geq b$ for all $x \in S_1$ and $a^{\mathrm{T}}x \leq b$ for all $x \in S_2$. *Proper separation* requires that $S_1 \cup S_2 \not\subseteq H$. (This would still allow *one* of the sets to lie in H.) The sets S_1 and S_2 are *strictly separated* by H if $a^{\mathrm{T}}x > b$ for all $x \in S_1$ and $a^{\mathrm{T}}x < b$ for all $x \in S_2$. (This does not prevent points of S_1 and S_2 from becoming arbitrarily close to H.) The sets S_1 and S_2 are *strongly separated* by H if there exists a number $\varepsilon > 0$ such that $a^{\mathrm{T}}x > b + \varepsilon$ for all $x \in S_1$ and $a^{\mathrm{T}}x < b - \varepsilon$ for all $x \in S_2$.

In some situations, the existence of a separating hyperplane can be made to rest on a theorem about the solution of a nearest point problem.

2.7.1 Theorem. Let C be a nonempty closed convex subset of R^n and let $p \in R^n \setminus C$. There exists a unique point $\bar{x} \in C$ such that

$$\|\bar{x} - p\|_2 \leq \|x - p\|_2 \quad \text{for all } x \in C.$$

Furthermore, $\bar{x}$ is the minimizing point (i.e., the point of C closest to p) if and only if

$$(\bar{x} - p)^{\mathrm{T}}(x - \bar{x}) \geq 0 \quad \text{for all } x \in C. \quad \square \tag{1}$$

The vector $\bar{x}$ is called the *projection* of p onto the set C (under the l_2-norm), and is denoted $\Pi_C(p)$. Sometimes, it may be desirable to use a different norm to define the projection vector. Exericse **2.10.22** discusses the case when C is a polyhedron and the l_∞ norm is used. In this case, the characterizing property of the projection vector (1) will not be valid.

The following theorem—which is just one of many to be found in the theory of convex sets—is particularly useful in connection with polyhedra, especially through their representation as solution sets of linear inequality systems. The proof of this theorem follows from a simple manipulation of the variational characterization (1) of the projection vector. The reader is asked to supply the details in Exercise **2.10.20**.

2.7.2 Theorem. If C is a nonempty closed convex set and $p \notin C$, then there exists a vector a and a real number b such that

$$a^{\mathrm{T}}p > b \geq a^{\mathrm{T}}x \quad \text{for all } x \in C. \quad \square$$

The above theorem states a point lying outside of a closed convex set can be properly separated from the set. It is interesting and instructive to consider how this separation theorem applies to the case when the given set is a cone.

2.7.3 Corollary. If C is a closed convex cone and $p \notin C$, then there exists a vector a such that $a^{\mathrm{T}}p > 0 \geq a^{\mathrm{T}}x$ for all $x \in C$. $\square$

In effect, this corollary states that in the present case, the constant b mentioned in the main theorem can be taken as zero. Indeed, with a and b as in the theorem, we have $a^{\mathrm{T}}p > b \geq a^{\mathrm{T}}x$ for all $x \in C$. Since $0 \in C$, it follows that $b \geq 0$. Moreover, since C is a cone, there is no $\tilde{x} \in C$ such that $a^{\mathrm{T}}\tilde{x} > 0$, for otherwise there exists a positive scalar λ such that $a^{\mathrm{T}}(\lambda\tilde{x}) > b$ which is impossible. Thus, $a^{\mathrm{T}}p > b \geq 0 \geq a^{\mathrm{T}}x$ for all $x \in C$.

The above theorem and corollary very naturally lead to the following definition and theorem.

2.7.4 Definition. Let C be a nonempty convex set and let p be a point in $\operatorname{rb} C$. If $a \neq 0$ is a vector and if b is a real number such that $a^{\mathrm{T}}p = b \geq a^{\mathrm{T}}x$ for all $x \in C$, then the hyperplane $\{x : a^{\mathrm{T}}x = b\}$ is called a *supporting* hyperplane to C at p.

2.7.5 Theorem. Let C be a nonempty convex set. If $p \in \operatorname{rb} C$, then there exists a supporting hyperplane to C at p. Further, if C is a nonempty polyhedral set and if F is a k-face of C, then there exists a supporting hyperplane H such that $F = H \cap C$. $\square$

Besides yielding the important separation results stated above, the projection concept plays a central role in many topics of mathematical programming. Later in Section **3.7**, we shall use Theorem **2.7.1** to prove a fundamental existence result of the variational inequality problem. Some interesting properties of the projection function are identified in Exercise **2.10.21**.

Theorems of the alternative

The results designated collectively as *theorems of the alternative* (and also as *transposition theorems*) are among the most useful and best known topics in the theory of linear inequalities. Typically, these theorems involve two related linear inequality systems and assert that precisely one of the systems has a solution. The following theorem illustrates the pattern.

2.7.6 Theorem. Let $A \in R^{m \times n}$ and $b \in R^m$ be given. The system

$$Ax = b, \qquad x \geq 0 \tag{2}$$

has a solution if and only if the system

$$y^{\mathrm{T}}A \leq 0, \qquad y^{\mathrm{T}}b > 0 \tag{3}$$

has no solution. □

Notice that the theorem speaks about two systems built upon the same data and that precisely one of these systems has a solution. Not both systems can have a solution for otherwise it would be possible to satisfy

$$0 < y^{\mathrm{T}}b = y^{\mathrm{T}}(Ax) = (y^{\mathrm{T}}A)x \leq 0$$

which is absurd. The fact (3) has a solution when (2) does not is a direct consequence of **2.7.3**. In this instance the closed convex cone C is $\operatorname{pos} A$. (The closedness of this convex cone is noted in **2.6.18**.) Saying that (2) has no solution is equivalent to saying that $b \notin \operatorname{pos} A$. Corollary **2.7.3** then guarantees that there is a vector y such that $y^{\mathrm{T}}b > 0 \geq y^{\mathrm{T}}z$ for all $z \in \operatorname{pos} A$. The inequality system $y^{\mathrm{T}}A \leq 0$ follows from the fact that all the columns of A belong to $\operatorname{pos} A$.

Actually, Theorem **2.7.6** is a version of the much-used "Farkas's lemma" which was originally expressed in the following form.

2.7.7 Theorem. For $A \in R^{m \times n}$ and $b \in R^m$, if $y^{\mathrm{T}}b \leq 0$ for all y such that $y^{\mathrm{T}}A \leq 0$, then there exists a vector $x \geq 0$ such that $Ax = b$. □

Farkas's lemma is neither the oldest nor the most general result of its kind. Nevertheless, it is powerful enough to imply several other significant theorems of the alternative. We state five of them below. The first two are for inhomogeneous systems.

2.7.8 Theorem. Let $A \in R^{m \times n}$ and $b \in R^m$ be given. The system

$$Ax \geq b \tag{4}$$

has a solution if and only if the system

$$y^{\mathrm{T}}A = 0, \quad y \geq 0, \quad y^{\mathrm{T}}b > 0 \tag{5}$$

has no solution. □

2.7.9 Theorem. Let $A \in R^{m \times n}$ and $b \in R^m$ be given. The system

$$Ax \geq b, \quad x \geq 0 \tag{6}$$

has a solution if and only if the system

$$y^{\mathrm{T}}A \leq 0, \quad y \geq 0, \quad y^{\mathrm{T}}b > 0 \tag{7}$$

has no solution. □

The next three alternative theorems are for homogeneous systems. The first is known as Gordan's theorem.

2.7.10 Theorem. Let $A \in R^{m \times n}$ be given. The system

$$Ax > 0 \tag{8}$$

has a solution if and only if the system

$$y^{\mathrm{T}}A = 0, \quad y \geq 0, \quad y \neq 0 \tag{9}$$

has no solution. □

As an application of Gordan's theorem, we have Ville's theorem.

2.7.11 Theorem. Let $A \in R^{m \times n}$ be given. The system

$$Ax > 0, \quad x > 0 \tag{10}$$

has a solution if and only if the system

$$y^{\mathrm{T}}A \leq 0, \quad y \geq 0, \quad y \neq 0 \tag{11}$$

has no solution. □

Finally, we have Stiemke's theorem.

2.7.12 Theorem. Let $A \in R^{m \times n}$ be given. The system

$$Ax \geq 0, \quad Ax \neq 0 \tag{12}$$

has a solution if and only if the system

$$y^{\mathrm{T}}A = 0, \quad y > 0 \tag{13}$$

has no solution. □

In this book, we will invoke Theorem **2.7.9** several times. As related to the LCP (q, M), it states that if the system

$$q + Mz \geq 0, \quad z \geq 0$$

has no solution, then there exists a vector y such that

$$y^{\mathrm{T}}M \leq 0, \quad y \geq 0, \quad \text{and} \quad y^{\mathrm{T}}q < 0.$$

For certain kinds of problems, the existence of a solution to the latter (alternative) system has theoretical implications that can be used to advantage.

Duality in linear programming

Theorems of the alternative are closely related to the subject of duality in linear programming. We shall review some theorems from this duality theory.

Given a linear programming problem of the form

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x \\ \text{subject to} & Ax \geq b \\ & x \geq 0, \end{array} \tag{14}$$

there is an associated *dual problem*

$$\begin{array}{ll} \text{maximize} & b^{\mathrm{T}}y \\ \text{subject to} & A^{\mathrm{T}}y \leq c \\ & y \geq 0 \end{array} \tag{15}$$

formed from the same data (used differently). In this pairing, the given linear program is called the *primal problem*. The relationship is *involutory* in the sense that the dual of the dual problem is the primal problem.

When x and y are feasible solutions of (14) and (15), respectively, the inequality

$$b^{\mathrm{T}}y \leq c^{\mathrm{T}}x$$

is always valid. Moreover, if this equality holds as an equation, then the feasible solutions x and y are optimal for their respective programs. Sometimes called the *weak duality theorem*, this observation serves as a lemma for establishing the existence of optimal solutions.

2.7.13 Theorem. If the primal problem (14) and the dual problem (15) are both feasible, then there exist a primal-feasible vector $\bar{x}$ and a dual-feasible vector $\bar{y}$ such that $c^{\mathrm{T}}\bar{x} = b^{\mathrm{T}}\bar{y}$. □

This result can be proved by formulating an appropriate linear inequality system and applying a corresponding theorem of the alternative, see **2.10.23**. The same approach works for *strong duality theorem* of linear programming below. An analogous theorem beginning with the dual problem can also be stated.

2.7.14 Theorem. If the primal problem (14) has an optimal solution $\bar{x}$, then the dual has an optimal solution $\bar{y}$ and $c^{\mathrm{T}}\bar{x} = b^{\mathrm{T}}\bar{y}$. □

When $\bar{x}$ and $\bar{y}$ are optimal solutions of the primal problem (14) and its dual (15), respectively, the equality of $c^{\mathrm{T}}\bar{x}$ and $b^{\mathrm{T}}\bar{y}$ implies the *complementary slackness conditions*

$$\bar{y}^{\mathrm{T}}(A\bar{x} - b) = 0 \quad \text{and} \quad \bar{x}^{\mathrm{T}}(A^{\mathrm{T}}\bar{y} - c) = 0.$$

Conversely, if $\bar{x}$ and $\bar{y}$ are feasible solutions of the primal and dual problems, respectively, and the complementary slackness conditions hold, then they are optimal solutions. In fact, solving the primal problem (14) or the dual problem (15) is mathematically equivalent to solving the LCP (q, M) with data

$$q = \begin{bmatrix} c \\ -b \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & -A^{\mathrm{T}} \\ A & 0 \end{bmatrix}.$$

There are other theorems about dual linear programs, and there are other formulations of the primal problem, each one giving rise to a corresponding dual problem. For the sake of brevity, we relegate these matters to the reader.

2.8 Quadratic Programming Theory

By definition, quadratic programming is concerned with the problem of minimizing (or maximizing) a quadratic function over a polyhedron. Such problems can take many different forms, depending on how the polyhedral feasible region is represented. For instance, in Chapter 1 we used the form

$$\begin{array}{ll} \text{minimize} & f(x) = c^{\mathrm{T}}x + \frac{1}{2}x^{\mathrm{T}}Qx \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array} \tag{1}$$

where $Q \in R^{n\times n}$ is symmetric, $c \in R^n$, $A \in R^{m\times n}$ and $b \in R^m$. For some purposes, it is more convenient to consider another form, namely

$$\begin{array}{ll} \text{minimize} & f(x) = c^{\mathrm{T}}x + \frac{1}{2}x^{\mathrm{T}}Qx \\ \text{subject to} & Ax \geq b. \end{array} \tag{2}$$

The first form is readily converted to the second by writing the constraints as

$$\begin{bmatrix} A \\ I \end{bmatrix} x \geq \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

Ordinarily, the purpose of a quadratic program with objective function f and feasible region X is to determine a global minimum, that is, a vector $\bar{x} \in X$ such that

$$f(\bar{x}) \leq f(x) \quad \text{for all } x \in X.$$

Sometimes, however, it is necessary to settle for a local minimum, that is, a vector $\bar{x} \in X$ such that

$$f(\bar{x}) \leq f(x) \quad \text{for all } x \in X \cap N(\bar{x})$$

where $N(\bar{x})$ denotes a neighborhood of $\bar{x}$.

The existence of a global minimum

Every quadratic program has a continuous function as its objective and a closed set as its feasible region. Accordingly, the Bolzano-Weierstrass theorem guarantees the existence of a global minimum (or maximum) if the feasible region is bounded, for then it is compact. Known as the Frank-Wolfe theorem, the following result is of interest for cases where the feasible region is unbounded. It gives (necessary and) sufficient conditions for the existence of a global minimum in a quadratic programming problem. Note that the Frank-Wolfe theorem is *not* necessarily valid for *arbitrary* optimization problems.

2.8.1 Theorem. If the quadratic function f is bounded below on the nonempty polyhedron X, then f attains its infimum on X. (That is, if there exists a real number γ such that $f(x) \geq \gamma$ for all $x \in X$, then there exists a vector $\bar{x} \in X$ such that $f(\bar{x}) \leq f(x)$ for all $x \in X$.) □

In quadratic programming, an a *priori* lower bound for the objective function is not always available, but when it is, Theorem **2.8.1** can be applied. For instance, the quadratic programming formulation of the LCP (q, M) given in (1.4.2) has zero as the lower bound of the objective function over the polyhedron FEA(q, M). This fact, alone, justifies our interest in the Frank-Wolfe theorem.

It is possible to give a characterization for a quadratic function to be bounded below on a polyhedron. Two special cases are considered in Exercise **2.10.25** and Proposition **3.7.14**. See also **3.13.14**.

First-order optimality conditions

In any theory of optimization, it is essential to have a way to identify optimal solutions. In nonlinear programming with differentiable functions, necessary conditions of local optimality are given by the Karush-Kuhn-Tucker theorem. Here we shall specialize this famous result for the quadratic programming problem. To this end, it is convenient to start with problems of the form (2).

Let $X = \{x : Ax \geq b\}$ denote the feasible region of the problem. For $\bar{x} \in X$, let

$$\alpha(\bar{x}) = \{i : A_{i\cdot}\bar{x} = b_i\}.$$

The constraints for which $i \in \alpha(\bar{x})$ are said to be *active* or *binding* at $\bar{x}$. When there are no active constraints at $\bar{x}$, the index set $\alpha(\bar{x})$ is empty, and $\bar{x}$ is an interior point of X. Under such circumstances, the standard first-order optimality condition for $\bar{x}$ to be a local minimum is simply that the gradient of the objective function vanishes there; that is,

$$\nabla f(\bar{x}) = c + Q\bar{x} = 0.$$

Generalizing this simple consideration, we have the following result which provides a necessary and sufficient condition for a vector to be a stationary point of the quadratic program (2).

2.8.2 Theorem. A vector $\bar{x} \in X$ is a stationary point for the quadratic program (2) if and only if there exists a vector $\bar{y}$ such that

$$c + Q\bar{x} - A^{\mathrm{T}}\bar{y} = 0, \quad \bar{y} \geq 0, \quad \bar{y}^{\mathrm{T}}(A\bar{x} - b) = 0. \quad \square \tag{3}$$

In Exercise **2.10.27**, the reader is asked to give a direct proof of the above theorem by using the duality theory of linear programming. This theorem implies that when $\bar{x}$ is a stationary point for (2), there is a vector $\bar{y}$ such that $\bar{x}$ is a stationary point of the function

$$L(x, \bar{y}) = c^{\mathrm{T}}x + \tfrac{1}{2}x^{\mathrm{T}}Qx - \bar{y}^{\mathrm{T}}(Ax - b).$$

In general,

$$L(x, y) = c^{\mathrm{T}}x + \tfrac{1}{2}x^{\mathrm{T}}Qx - y^{\mathrm{T}}(Ax - b)$$

is called the *Lagrangian function* for (2) and the components of $\bar{y}$ are called *Lagrange multipliers*. Indeed, the Lagrange multipliers are in one-to-one correspondence with the constraints of the problem. The nonnegativity of the Lagrange multipliers stems from the particular inequality form of the constraints. It is important to observe that a Lagrange multiplier will be positive only when the corresponding constraint is active.

The first-order optimality conditions for problems of the form (1) involving nonnegative variables are easily derived from Theorem **2.8.2**. A version of these conditions particularly suited to LCP formulation is stated as (1.2.2).

Second-order optimality conditions

The first-order optimality conditions given by Theorem **2.8.2** provide a characterization for stationary points; hence, they are merely *necessary* conditions of optimality. Satisfaction of (3) by a feasible vector $\bar{x}$ does not by itself guarantee the global or even local optimality of $\bar{x}$. As noted in Section 1.2, the conditions (3) are sufficient for the the global optimality of $\bar{x}$ when the objective function $f(x) = c^{\mathrm{T}}x + \frac{1}{2}x^{\mathrm{T}}Qx$ is convex, i.e., when Q is positive semi-definite. (For a discussion of weaker conditions under which a feasible solution of (3) is a global minimum, see **2.11.16**.) The convexity assumption on f stated above can be viewed as a second-order optimality condition, but it is not the sort of thing we have in mind. We are interested in necessary and sufficient conditions for feasible vectors to be local minima. It is true that global minima are more desirable than local minima, but we are after a nontrivial criterion (i.e., necessary and sufficient conditions) for local optimality. Such a thing is not generally available for global optimality. To set the stage for the result we have in mind, we need to introduce the following definition.

2.8.3 Definition. For a vector $\bar{x} \in X = \{x : Ax \geq b\}$, let $\alpha(\bar{x})$ denote the index set of the active constraints at $\bar{x}$. The nonzero solutions of the homogeneous system

$$A_{i\boldsymbol{.}}v \geq 0 \quad \text{for all } i \in \alpha(\bar{x}) \tag{4}$$

are called *feasible directions* at $\bar{x}$. Collectively, the solutions of (4) form a (polyhedral) cone $\mathcal{F}$. If $\alpha(\bar{x}) = \emptyset$, then $\mathcal{F} = R^n$.

The cone $\mathcal{F}$ is called the *cone of feasible directions* at $\bar{x}$ (even though, strictly speaking, only its nonzero members are genuine directions). We are now in a position to state the desired second-order criterion for local optimality.

2.8.4 Theorem. A feasible solution $\bar{x}$ of (2) is a local minimum if and only if

$$\nabla f(\bar{x})^{\mathrm{T}}v \geq 0 \quad \text{for all } v \in \mathcal{F}, \tag{5}$$

and

$$v^{\mathrm{T}}Qv \geq 0 \quad \text{for all } v \in \mathcal{F} \cap (\nabla f(\bar{x}))^{\perp}. \quad \square \tag{6}$$

The set $(\nabla f(\bar{x}))^{\perp}$ consists of all vectors in R^n that are perpendicular to $\nabla f(\bar{x})$. In the event that $\nabla f(\bar{x}) = 0$, we have $(\nabla f(\bar{x}))^{\perp} = R^n$.

2.8.5 Remark. The first-order condition expressed in (5) can be interpreted as a version of the Karush-Kuhn-Tucker conditions; these alone are necessary conditions of optimality. The second-order conditions given in (6) ask that the quadratic form associated with the Hessian matrix Q of the objective function f be nonnegative on the intersection of the cone of feasible directions and the orthogonal complement of the space spanned by the gradient vector at the point $\bar{x}$. This intersection is still a cone, and the condition that $v^{\mathrm{T}}Qv$ be nonnegative there is a *copositivity* restriction with respect to this cone. It is obviously satisfied when Q is positive semi-definite.

Duality in quadratic programming

In the special instance of *convex* quadratic programming, i.e., when the objective function is convex, there is a duality theory that completely generalizes its linear programming counterpart. The results are most easily expressed for quadratic programs of the form (1) or extensions thereof. Indeed, let us consider the problem

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x + \frac{1}{2}x^{\mathrm{T}}Qx + \frac{1}{2}y^{\mathrm{T}}Py \\ \text{subject to} & Ax + Py \geq b \\ & x \geq 0, \end{array} \tag{7}$$

In this case, the matrices Q and P are assumed to be symmetric and positive semi-definite. Notice that when P equals the positive semi-definite matrix 0, we recover our original problem (1), and when both Q and P are zero, the problem is just the linear programming primal problem (2.7.14). Accordingly, (7) will be called the primal problem. The corresponding dual problem is

$$\begin{array}{ll} \text{maximize} & b^{\mathrm{T}}y - \frac{1}{2}x^{\mathrm{T}}Qx - \frac{1}{2}y^{\mathrm{T}}Py \\ \text{subject to} & -Qx + A^{\mathrm{T}}y \leq c \\ & y \geq 0. \end{array} \tag{8}$$

As in linear programming, there is a weak duality theorem for these quadratic programs. Indeed, if $(\bar{x}, \bar{y})$ is a feasible solution of the primal problem (7) and $(\hat{x}, \hat{y})$ is a feasible solution of the dual problem (8), then

$$b^{\mathrm{T}}\hat{y} - \tfrac{1}{2}\hat{x}^{\mathrm{T}}Q\hat{x} - \tfrac{1}{2}\hat{y}^{\mathrm{T}}P\hat{y} \leq c^{\mathrm{T}}\bar{x} + \tfrac{1}{2}\bar{x}^{\mathrm{T}}Q\bar{x} + \tfrac{1}{2}\bar{y}^{\mathrm{T}}P\bar{y}.$$

If equality holds in this inequality, the solutions are optimal. This fact is useful in proving the following existence result which, incidentally, generalizes **2.7.13**.

2.8.6 Theorem. If the primal problem (7) and the dual problem (8) are both feasible, then there exist vectors $\bar{x}$ and $\bar{y}$ such that $(\bar{x}, \bar{y})$ is feasible for both programs and

$$b^{\mathrm{T}}\bar{y} - \tfrac{1}{2}\bar{x}^{\mathrm{T}}Q\bar{x} - \tfrac{1}{2}\bar{y}^{\mathrm{T}}P\bar{y} = c^{\mathrm{T}}\bar{x} + \tfrac{1}{2}\bar{x}^{\mathrm{T}}Q\bar{x} + \tfrac{1}{2}\bar{y}^{\mathrm{T}}P\bar{y}. \quad \square$$

Note that in this theorem, the solution $(\bar{x}, \bar{y})$ is optimal for each program of the dual pair. Such a solution is said to be *jointly optimal.* The approach to proving the theorem is analogous to the one mentioned for the linear programming case.

The *strong duality theorem* for quadratic programming is stated as follows.

2.8.7 Theorem. If the primal problem (7) has an optimal solution $(\bar{x}, \bar{y})$, then the dual problem (8) has an optimal solution $(\hat{x}, \hat{y})$ and

$$b^{\mathrm{T}}\hat{y} - \tfrac{1}{2}\hat{x}^{\mathrm{T}}Q\hat{x} - \tfrac{1}{2}\hat{y}^{\mathrm{T}}P\hat{y} = c^{\mathrm{T}}\bar{x} + \tfrac{1}{2}\bar{x}^{\mathrm{T}}Q\bar{x} + \tfrac{1}{2}\bar{y}^{\mathrm{T}}P\bar{y}. \quad \square$$

Of course, the assertion of this theorem—that both problems have optimal solutions when the primal has an optimal solution—also means (by **2.8.6**) that there exists a jointly optimal solution when the primal has an optimal solution. The analogous statement can be made when it is known that the dual has an optimal solution.

Just as in the linear programming case, we relegate the statements of other dual pairs and other theorems to the reader.

2.9 Degree and Dimension

As seen in Chapter 1 and, for that matter, in this entire book, the linear complementarity problem appears in many contexts and can be studied from many viewpoints. One particular viewpoint which we will adopt in Chapter 6, is the geometric one. We have already introduced the basic tools needed for this in Section 1.3. In that section we introduced complementary matrices, complementary cones, and the complementary range. In Section 1.4 we introduced the idea of a nondegenerate solution to an LCP (Definition **1.4.3**) along with the mapping (1.4.8) and its connection to the LCP via Proposition **1.4.4**. As we will see in Chapter 6, all these will play a part in the geometric view of the LCP.

A useful concept with which to study the LCP is that of the degree of a mapping. Another concept incorporated in the geometric view of the LCP is that of the dimension of a set. Both degree theory and dimension theory are extensive areas of topology, and we will only use some of the more basic results from these fields. The purpose of the present section is to go over this basic material. A more complete discussion of degree theory and dimension theory can be found in several of the references cited in **2.11.18** and **2.11.19**.

Homogeneous functions

We start by defining homogeneous functions.

2.9.1 Definition. Let $D \subseteq R^m$ be a cone. A function $f : D \to R^n$ is said to be *(positive) homogeneous of degree* k, where k is an integer, if for all (positive) real numbers t, and all $x \in D$, we have $f(tx) = t^k f(x)$.

In this book we will frequently be dealing with positive homogenous functions of degree 1. As a convenience, we will define the unmodified term *homogeneous function* to mean a positive homogeneous function of degree 1. In addition, we will say that the homogeneous function f is *nondegenerate* if $f(x) = 0$ implies $x = 0$. Otherwise, we will say that f is *degenerate*.

2.9.2 Example. A linear function $f(x) = c^{\mathrm{T}}x$ is clearly homogeneous of degree 1; a quadratic function of the form $f(x) = x^{\mathrm{T}}Mx$ is homogeneous of degree 2. We may consider the determinant function, which maps the matrix $M \in R^{n\times n}$ into $\det M$, as a function from $R^{n\times n}$ into R. From basic linear algebra, we see that this function is homogeneous of degree n.

Let

$$S^{n-1} = \{x \in R^n : \|x\|_2 = 1\}$$

denote the $(n-1)$-dimensional unit sphere in R^n. Suppose $D \subseteq R^n$ is a cone, and $f : D \to R^n$ is a nondegenerate homogeneous function. We can learn a fair amount about f by studying the function $f_S : S^{n-1} \cap D \to S^{n-1}$ given by $f_S(x) = f(x)/\|f(x)\|_2$. As f is nondegenerate, it follows that $\|f(x)\|_2 \neq 0$ for all $x \in S^{n-1}$. Thus, $f_S(x)$ is well-defined.

For those readers already familiar with degree theory, we give a brief discussion connecting the results stated in the next subsection with the results one is more likely to see in other works. Since this discussion will not be used elsewhere in the book, the reader may safely skip to the next subsection.

The usual development of degree theory would deal with the function f_S, from S^{n-1} to S^{n-1}, rather than with the nondegenerate homogeneous function f. The key entity in the usual development would be the index of f_S at x, which is defined if f_S is continuously differentiable at x and the Jacobian is nonsingular. The value of the index depends on whether $\nabla f_S(x)$ maps the tangent space at x into the tangent space at $f_S(x)$ in a manner that preserves orientation (a positive index) or that reverses orientation (a negative index).

Degree theory can also be developed for maps $f : D \to R^n$ with D a compact domain in R^n, or with D a cone in R^n and f nondegenerate homogeneous. As in the case of $f_S : S^{n-1} \to S^{n-1}$, we define the index at a point x where $f(x)$ is continuously differentiable and the Jacobian nonsingular by whether or not the orientation of the tangent space is preserved. If D is a cone and f is nondegenerate homogeneous, and if $f_S(z) = f(x)/\|f(x)\|$, where $z = x/\|x\|$, then one can obtain a close relationship between the index for f at x and the index for f_S at z.

Degree theory

We will now develop the notion of degree to the extent that will be needed for this book. The following can be generalized in several ways and the interested reader should consult the references. As usual, we represent the closure, boundary, and relative interior of the set D by, respectively, $\mathrm{cl}\, D$, $\mathrm{bd}\, D$, and $\mathrm{ri}\, D$.

2.9.3 Definition. Let $D \subseteq R^n$ be an open cone. Let $f : \mathrm{cl}\, D \to R^n$ be a continuous nondegenerate homogeneous function. If, for some point $x \in D$, the function f is continuously differentiable in an open set containing x and, further, if the Jacobian matrix $\nabla f(x)$ is nonsingular, then we define the *index* of f at x to be $\mathrm{sgn}(\det(\nabla f(x)))$. We will denote this by $\mathrm{ind}_f(x)$ or simply by $\mathrm{ind}(x)$ when it is clear which f is meant. Notice that the index is always either $+1$ or -1.

2.9.4 Definition. Let $D \subseteq R^n$ be an open cone. Let $f : \mathrm{cl}\, D \to R^n$ be a continuous nondegenerate homogeneous function. If, for some point $y \in R^n$, the set $f^{-1}(y)$ consists of finitely many points and, further, if for each $x \in f^{-1}(y)$ the index of f at x is well-defined (using **2.9.3**), then we define the *local degree* of f at y to be

$$\sum_{x \in f^{-1}(y)} \mathrm{ind}_f(x) .$$

We will denote this by $\deg_f(y)$ or simply by $\deg(y)$ when it is clear which f is meant.

2.9.5 Remark. Note that index is defined for points in the domain of f while local degree is defined for points in the range of f. Also, note that if $f^{-1}(y) = \emptyset$, then the local degree of f at y is well defined and equal to zero (by convention, the empty sum is defined to equal zero). However, if $y \in f(\mathrm{bd}\, D)$, then the local degree of f at y is not well-defined.

The key property of degree is stated in the next theorem.

2.9.6 Theorem. Let $D \subseteq R^n$ be an open cone. Let $f : \mathrm{cl}\, D \to R^n$ be a continuous nondegenerate homogeneous function. Suppose that both $\deg_f(y)$ and $\deg_f(y')$ are well defined for $y, y' \in R^n$ (using **2.9.4**). If y and y' are in the same connected component of $R^n \setminus f(\mathrm{bd}\, D)$, then $\deg_f(y) = \deg_f(y')$. □

2.9.7 Corollary. Let $D \subseteq R^n$ be an open cone. Let $f : \mathrm{cl}\, D \to R^n$ be a continuous nondegenerate homogeneous function. If $R^n \setminus f(\mathrm{bd}\, D)$ has only one connected component, then the value of $\deg_f(y)$ is the same for all $y \in R^n$ which have a well-defined local degree. □

The common value of the local degrees given in Corollary **2.9.7** is called the *degree* of f, denoted $\deg f$. Notice, if $D = R^n$, then $\operatorname{bd} D$ is empty and $R^n \setminus f(\operatorname{bd} D)$ has exactly one connected component. Hence, if f is a continuous nondegenerate homogeneous function defined on the whole space R^n, then $\deg f$ is well defined.

It seems reasonable to expect that if a function is, in some way, constructed from simpler component functions, then the degree of the more complicated function can be calculated from the degrees of the component functions. To some extent this is true, as can be seen in the next two theorems.

2.9.8 Theorem. Let $f, g : R^n \to R^n$ be continuous nondegenerate homogeneous functions. For all $x \in R^n$, define $h(x) = f(g(x))$. Then $h : R^n \to R^n$ is a continuous nondegenerate homogeneous function and $\deg h = (\deg f)(\deg g)$. □

2.9.9 Theorem. Let $f : R^n \to R^n$ and $g : R^m \to R^m$ be continuous nondegenerate homogeneous functions. For all $(x, y) \in R^{n+m}$, define $h(x, y) = (f(x), g(y))$. Then $h : R^{n+m} \to R^{n+m}$ is a continuous nondegenerate homogeneous function and $\deg h = (\deg f)(\deg g)$. □

We end this subsection with some theorems concerning the invariance of degree under homotopic transformations. To this end, we first state what homotopy means in the current context.

2.9.10 Definition. Let $D \subseteq R^n$ be an open cone. Let $f, g : \operatorname{cl} D \to R^n$ be continuous nondegenerate homogeneous functions. We say that f and g are *homotopic* if there exists a continuous function $h : \operatorname{cl} D \times [0, 1] \to R^n$ such that:

(a) For each $t \in [0, 1]$, the function $h_t(x) : \operatorname{cl} D \to R^n$, defined by $h_t(x) = h(x, t)$ for all $x \in \operatorname{cl} D$, is a nondegenerate homogeneous function.

(b) For each $x \in \operatorname{cl} D$, $f(x) = h(x, 0)$ and $g(x) = h(x, 1)$.

If it exists, the function h is called a *homotopy* between f and g.

2.9.11 Theorem. Let $D \subseteq R^n$ be an open cone, and $f, g : \operatorname{cl} D \to R^n$ be continuous nondegenerate homogeneous functions. Suppose that both

$\deg_f(y)$ and $\deg_g(y)$ are well defined for $y \in R^n$ (using **2.9.4**). If the function $h : \operatorname{cl} D \times [0,1] \to R^n$ is a homotopy between f and g such that $y \notin h(\operatorname{bd} D \times [0,1])$, then $\deg_f(y) = \deg_g(y)$. □

2.9.12 Theorem. Let $f, g : R^n \to R^n$ be continuous nondegenerate homogeneous functions. The functions f and g are homotopic if and only if $\deg f = \deg g$. □

2.9.13 Remark. The degree results presented here are, for the most part, special cases of quite general results. The exception to this is the "if" part of Theorem **2.9.12**. This result is special to spheres and, hence, to nondegenerate homogeneous functions.

Dimension theory

We will only need a few results from dimension theory. For a set X in R^n, we will denote the dimension of X as $\dim X$. By definition, $\dim \emptyset = -1$. We will, in fact, not need a precise definition of dimension. The following result will suffice.

2.9.14 Proposition. Let X be a subset of R^n. Suppose the affine hull of X can be expressed as $\{q + Mz : z \in R^m\}$, where q is in X and the rank of $M \in R^{n \times m}$ is equal to m. If $\operatorname{ri} X \neq \emptyset$, then $\dim X = m$. □

The next two propositions contain some basic facts about dimension which will be useful.

2.9.15 Proposition. Let X and Y be subsets of R^n.

(a) If $X \subseteq Y$, then $\dim X \leq \dim Y$.

(b) Suppose that Y is closed and $X = \operatorname{ri} X$. If $\dim X > \dim Y$, then $\dim(X \setminus Y) = \dim X$.

(c) Suppose that Y is closed and $X = \operatorname{ri} X$. If X is path connected and if $\dim X - \dim Y \geq 2$, then $X \setminus Y$ is path connected. □

2.9.16 Proposition. In R^n, the countable union of closed sets of dimension less than or equal to m is a set of dimension less than or equal to m. □

The final result we will state is more from measure theory than dimension theory. However, it seems most appropriate to mention it here.

2.9.17 Proposition. If $X \subset R^n$ is the union of finitely many sets each of whose affine hulls has dimension $n-1$ or less (see **2.9.14**), then $R^n \setminus X$ is dense in R^n. □

The reader familiar with measure theory will recognize that the given set X, in Proposition **2.9.17**, will have zero measure in R^n and, as the proposition states, the complement of such a set will be dense in R^n.

2.10 Exercises

2.10.1 Let $f : \mathcal{D} \to R^m$ be Lipschitz continuous on the open set $\mathcal{D} \subseteq R^n$. Show that if f is directionally differentiable at $x \in \mathcal{D}$, then the directional derivative $f'(x,d)$ is a Lipschitz continuous function in the direction $d \in R^n$ with the same Lipschitz modulus as f.

2.10.2 Prove Theorem **2.1.10**. Show by examples that none of the conditions in the theorem can be dispensed with.

2.10.3 Let $f : \mathcal{D} \subseteq R^n \to R^n$ be a local homeomorphism on $\mathcal{D}$. Show that if $\mathcal{D}$ is an open set, then so is the image $f(\mathcal{D})$.

2.10.4 Prove the formula (2.2.1). Deduce from it the following expansion of the characteristic polynomial

$$\det(\lambda I - A) = \lambda^n - E_1(A)\lambda^{n-1} + E_2(A)\lambda^{n-2} + \cdots + (-1)^n E_n(A)$$

where $E_k(A)$ is the sum of all the principal minors of A of order k. What are $E_1(A)$ and $E_n(A)$?

2.10.5 Let $A \in R^{n\times n}$ be a symmetric positive definite matrix.

(a) Show that the function $\|x\|_A$ given by (2.1.1) defines a vector norm. (Hint: use Proposition **2.2.16**(e).)

(b) Give an example of a matrix A to show that the norm $\|x\|_A$ is not necessarily monotone.

2.10.6 Let $\|\cdot\|$ and $\|\cdot\|'$ be two vector norms on R^n. Show that the two statements are equivalent:

(a) there exist constants $c_1, c_2 > 0$ such that for all $x \in R^n$,

$$c_1\|x\| \leq \|x\|' \leq c_2\|x\|;$$

(b) for every $\{x^k\} \subseteq R^n$,

$$\lim_{k\to\infty} \|x^k\| = 0 \quad \Leftrightarrow \quad \lim_{k\to\infty} \|x^k\|' = 0.$$

2.10.7 Prove Propositions **2.2.16**, **2.2.17**, **2.2.20** and Corollary **2.2.22**.

2.10.8 This question concerns the continuity and differentiability properties of a vector norm as a function of its argument.

(a) Let $\|\cdot\|$ be a vector norm on R^n. Show that $\|x\|$ is Lipschitz continuous in x with a modulus equal to unity. Conclude from this fact that the unit ball $\mathcal{B}$ and the unit sphere $\mathcal{S}$ associated with any norm on R^n are compact sets.

(b) Show that the three norms in Example **2.1.2** and an elliptic norm are directionally differentiable functions in x. Exhibit the directional derivatives for each of these norms. Are these norms F-differentiable functions in x? At what points are they not F-differentiable?

2.10.9 The spectral radius can be considered a function $\rho : C^{n\times n} \to R_+$ defined on the set of $n \times n$ complex matrices.

(a) Is the spectral radius function a matrix norm (on $C^{n\times n}$) in the sense of **2.2.7**? Prove your answer if it is in the affirmative; give a counterexample to the violated axiom if your answer is in the negative.

(b) Show that the spectral radius $\rho(A)$ is a continuous function in the entries of the matrix $A \in C^{n\times n}$.

2.10.10 Let $A \in R^{n\times n}$ be a symmetric matrix. Let $\lambda_n(A)$ denote the largest eigenvalue of A.

(a) Use Proposition **2.2.10** to show that

$$\lambda_n(A) = \max_{x\in\mathcal{S}} x^{\mathrm{T}}Ax$$

where $\mathcal{S}$ is the unit sphere in R^n associated with the Euclidean norm.

(b) Deduce that $\lambda_n(A)$ is a Lipschitz continuous and convex function of A.

(c) What can be said about the smallest eigenvalue $\lambda_1(A)$ as a function of A?

2.10.11 Let $A \in R^{n\times n}$ be a positive semi-definite matrix.

(a) Show that if A is symmetric, then there exists a positive scalar σ such that for all $x \in R^n$,

$$x^{\mathrm{T}}Ax \geq \sigma\|Ax\|_2^2.$$

Give an interpretation of σ in terms of the eigenvalues of A.

(b) Suppose that A, in addition to being positive semi-definite, satisfies the property:

$$x^{\mathrm{T}}Ax = 0 \quad \Rightarrow \quad Ax = 0 \tag{1}$$

which clearly holds if A is symmetric. Show that the conclusion of part (a) remains valid under this generalized condition.

(c) Show that any matrix A of the form $P^{\mathrm{T}}QP$ where Q is positive definite satisfies the property (1). Are there positive semi-definite matrices satisfying (1) that are not of this particular form?

2.10.12 Let $f : D \subseteq R^n \to R^n$ be a nondegenerate homogeneous function, with D being a cone. If f is continuously differentiable at x, show that $\nabla f(x)x = f(x)$. Let $g(x_1, x_2) = (x_1, x_2 + 2\sqrt{x_1^2 + x_2^2})$. Show that g is nondegenerate homogeneous on R^2. At which points in R^2 does g have an index, and what is the index at each of these points? How do these indices relate to the behavior of $g_S : S^1 \to S^1$?

2.10.13 Suppose that x^* is a zero of the mapping $f : R^n \to R^n$ and that f is continuously differentiable in a neighborhood of x^*. Assume $\nabla f(x^*)$ is nonsingular. Let g be the mapping defined by (2.5.9). Show that g is well defined in a neighborhood of x^* and that g has a strong F-derivative at x^* which is equal to zero.

2.10.14 Prove Theorems **2.5.8** and **2.5.9**.

2.10.15 Let $\theta : R^n \to R$ be a continuously differentiable function. Let σ and ρ be scalars in $(0, 1)$.

(a) Show that if $\theta'(x, d) < 0$, then there exists a positive scalar $\bar{\tau}$ such that the inequality (2.5.13) holds for all $\tau \in [0, \bar{\tau}]$. Interpret this geometrically.

(b) Let $\{x^\nu\}$ be an infinite sequence of vectors generated as follows: for each $\nu = 0, 1, 2, \ldots$,

$$x^{\nu+1} = x^\nu + \tau_\nu d^\nu$$

where d^ν is a descent direction for θ at x^ν, and $\tau_\nu = \rho^{m_\nu}$ with m_ν being the smallest nonnegative integer m for which the stepsize $\tau = \rho^m$ satisfies the inequality (2.5.13) associated with the pair (x^ν, d^ν). Let the sequence $\{d^\nu\}$ have the property that whenever $\{x^\nu : \nu \in \kappa\}$ is a convergent subsequence for which

$$\lim_{\nu \in \kappa, \nu \to \infty} \nabla\theta(x^\nu) \neq 0,$$

the corresponding subsequence of directions $\{d^\nu : \nu \in \kappa\}$ is bounded and satisfies

$$\liminf_{\nu \in \kappa, \nu \to \infty} |\nabla\theta(x^\nu)^{\mathrm{T}} d^\nu| > 0.$$

Show that every accumulation point of $\{x^\nu\}$ is an unconstrained stationary point of the function θ.

(c) Suppose the function θ satisfies the property:

$$\lim_{\|x\| \to \infty} \theta(x) = \infty. \tag{2}$$

Show that the sequence $\{x^\nu\}$ described in part (b) is bounded.

(d) Suppose $\theta(x) = \frac{1}{2} f(x)^{\mathrm{T}} f(x)$ for some continuously differentiable function $f : R^n \to R^n$. Show that if $\bar{x}$ is an unconstrained stationary point of θ and if the Jacobian matrix $\nabla f(\bar{x})$ is nonsingular, then $\bar{x}$ is a zero of the mapping f.

2.10.16 Let $\{x^1, \ldots, x^r\}$ and $\{y^1, \ldots, y^s\}$ denote two finite subsets of R^n. Show that if P is the convex hull of the first set and C is the conical hull of the second set, then $X = P + C$ is a polyhedron.

2.10.17 Show that if X is a nonempty polyhedron given by the linear inequality system $Ax \geq b$, $x \geq 0$ then X is bounded if and only if the system $u^{\mathrm{T}} A < 0$, $u \geq 0$ has a solution.

2.10.18 Let S be a nonempty subset of R^n. A vector $y \in R^n$ is said to be a *recession direction* of S if there exists a vector $x \in S$ such that the ray $\{x + ty : t \in R_+\}$ is contained in S. Let 0^+S denote the set of all recession directions of S.

(a) Show that 0^+S is a cone that is not necessarily convex.

(b) Suppose S is closed and convex. Show that if y is a recession direction of S, then $z + ty \in S$ for all $z \in S$ and all $t \in R_+$. Deduce from this that the cone 0^+S is convex.

(c) Suppose S is closed and convex. Show that S is bounded if and only if 0^+S consists of the zero vector alone.

(d) Consider the polyhedron X given in Theorem **2.6.23**. Show that the cone of recession directions of X coincides with the cone C in this theorem.

2.10.19 Follow the argument outlined below to establish Theorem **2.7.1**.

(a) A function $\theta : R^n \to R$ is said to be *coercive* if the limit condition (2) holds. Show that if θ is a continuous and coercive function defined on R^n and X is a closed subset of R^n, then there exists a global minimum of the mathematical program

$$\begin{array}{ll} \text{minimize} & \theta(x) \\ \text{subject to} & x \in X. \end{array}$$

(b) Let $\theta : R^n \to R$ be a given function. Show that if θ is convex on R^n, then it is continuous. Show also that if θ is strongly convex, then it is coercive. Deduce that if θ is a strongly convex function, then the above nonlinear program has a unique solution.

(c) Use the result of part (b) to establish the existence and uniqueness assertion of the projection point in Theorem **2.7.1**.

(d) Finally, apply the minimum principle to deduce the variational characterization of the projection vector.

2.10.20 Use Theorem **2.7.1** to prove **2.7.2**.

2.10.21 In the setting of Theorem **2.7.1**, define the projection mapping $\Pi_C : R^n \to C$ where $\Pi_C(x)$ is the projection vector of x onto the closed convex set C under the l_2-norm.

(a) Show that Π_C is nonexpansive (and hence Lipschitz continuous).

(b) Give an example to show that Π_C is not necessarily F-differentiable. Show, however, that the squared least-distance function defined by

$$d_C(x) = \tfrac{1}{2}\|x - \Pi_C(x)\|_2^2$$

is F-differentiable with the gradient vector given by

$$\nabla d_C(x) = x - \Pi_C(x).$$

Is d_C twice F-differentiable?

2.10.22 Consider the problem of finding a vector in a given polyhedron that is closest to a given vector $a \in R^n$ under the l_∞-norm. Let the polyhedron be represented by

$$P = \{x \in R^n : Ax = b, x \geq 0\}$$

and assume that P is nonempty.

(a) Formulate this nearest-point problem as a linear program.

(b) Prove, using only results from linear programming, that this linear program must have an optimal solution, which we denote $\Pi_P(a)$.

(c) Show that there exists a constant $\lambda > 0$, dependent on A and b only, such that for all vectors $a \in R^n$,

$$\|a - \Pi_P(a)\|_\infty \leq \lambda\|(a^-, Aa - b)\|_\infty.$$

2.10.23 The theorems of the alternative, the duality theorems of linear programming and the separation theorems of convex polyhedra are all equivalent in the sense that by using any one of these, all the others can be derived. We have illustrated this with the derivation of Theorem **2.7.6** by Corollary **2.7.3**.

(a) Deduce Theorem **2.7.13** from **2.7.9**.

(b) Use **2.7.9** and **2.7.13** to show that if the linear program (2.7.14) is feasible and its objective function is bounded below on the feasible set, then an optimal solution exists.

(c) Deduce Theorem **2.7.9** from the result in part (b) and the strong duality theorem of linear programming (a version of which is given by Theorem **2.7.14**).

(d) Pick any two theorems of the alternative given in Section 2.7 and show that they can be deduced from each other.

2.10.24 Prove the following properties of the dual cone.

(a) The dual cone of a convex polyhedron is polyhedral.

(b) The dual cone of a simplicial cone is simplicial.

(c) Suppose C is a simplicial cone in R^n that is self-dual0 (i.e., $C = C^*$). Show that $C = \operatorname{pos} A$ for some orthogonal matrix $A \in R^{n \times n}$.

2.10.25 This problem is concerned with some special properties of a quadratic function.

(a) Show that a quadratic function is equal to a constant on the set of its (unconstrained) stationary points.

(b) Let $\theta(x) = b^{\mathrm{T}}x + \frac{1}{2}x^{\mathrm{T}}Ax$ where $A \in R^{n \times n}$ is symmetric. Show that θ is bounded below on the whole space R^n if and only if A is positive semi-definite and b is in the column space of A.

2.10.26 Prove Theorem **2.6.24**. Deduce from this result that if S_1 and S_2 are two polyhedra in R^n, then so is their sum $S_1 + S_2$. Hence, the sum of two polyhedra is a closed set. In general, is the sum of two closed convex sets closed?

2.10.27 Use linear programming duality to prove Theorem **2.8.2**.

2.10.28 Let $C \subseteq R^n$ be a convex cone and let L be the lineality space of C. Show that L is a subspace and, in addition, any subspace contained in C is contained in L.

2.10.29 Let $C \subseteq R^n$ be a finite cone.

(a) Show that the lineality space of C is the subspace in R^n orthogonal to the affine hull of C^*.

(b) Show that if $C \neq R^n$, then there exists an $(n-1)$-dimensional hyperplane H such that $H \cap C$ equals the lineality space of C.

2.11 Notes and References

2.11.1 Fixed-point theorems are fundamental results that are useful for various solution concepts in mathematical game theory and for the demonstration of the existence of equilibria in economics. There are many such theorems; the one stated as **2.1.24** is due to Brouwer (1912). Traditionally, the proof of this important theorem was based on a *nonconstructive* argument that did not lend itself to the actual computation of a fixed point. It was not until Lemke and Howson published their renowned algorithm for computing a bimatrix game equilibrium point in 1964 that the subject of computing fixed points started to blossom (see Section 4.4 for the description of this algorithm and Section 4.12 for further notes and references concerning this path-breaking method). Another seminal work on this subject is the paper by Scarf (1967) who developed the first computational scheme for approximating fixed points of a continuous mapping. Scarf's method was greatly influenced by the ingenious Lemke-Howson algorithm. Since then, a tremendous growth has followed. Indeed, this subject has now developed into a very fruitful discipline and has a huge body of literature of its own. The following are some books and special volumes on these *fixed-point methods*: Scarf (1973), Todd (1976d), Karamardian and Garcia (1977), Robinson (1979), Garcia and Zangwill (1981), Eaves, Gould, Peitgen, and Todd (1983), Talman and van der Laan (1987).

2.11.2 The term "multivalued mapping" has been called many different names in various disciplines. Some commonly used synonyms are: "point-to-set map", "set-valued mapping", "multifunction", and "correspondence". Chapter 6 of the book by Berge (1963) contains a good summary of the basic continuity properties of these mappings. One notable distinction between Berge's definitions and those used by many authors (including us) is that the former require the images to be compact sets. Another good source for discussion of this topic is Chapter 11 of Border (1985) whose terminology differs slightly from ours.

2.11.3 Theorem **2.1.10** is a special case of a more general result due to Ostrowski (1966) who proved that under the first two assumptions of the theorem, the set of accumulation points of $\{x^\nu\}$ is closed and *connected*.

2.11.4 T.D. Parsons used the name "pivotal algebra" as the title of a set of seminar notes taken from lectures on Combinatorial Algebra given by A.W. Tucker at Princeton University in 1964-65. We list these notes as Tucker (1967). Other related references are Tucker (1960, 1963), Cottle (1964a, 1968a), Cottle and Dantzig (1968), and Keller (1969). Section 4.1 of this book contains several more results on pivotal algebra.

2.11.5 Formula (2.3.13) was noted by Cottle (1990). We do not know of an earlier reference.

2.11.6 The eponym of the "Schur complement" is I. Schur whose formula (2.3.14) has already been mentioned. Our presentation is based largely on a paper of Cottle which in turn builds on several earlier studies, notably by E. Haynsworth who coined the name. See Cottle (1974a) and the references therein. Many other publications on the subject have appeared since 1974.

2.11.7 The origins of lexicographic ordering are obscure. We do know that the concept was used by Dantzig, Orden and Wolfe (1955) as a way to avoid cycling in the simplex method for linear programming. See Dantzig (1963) and Murty (1976, 1988). There is a close connection between "lexicography" and "epsilon perturbation" of the constant (right-hand side) vector. See Dantzig (1963, Section 10-2). Eaves (1971a) used lexicographic ordering in a major way in his treatment of Lemke's method.

2.11.8 Matrix factorization is a basic numerical tool useful for solving systems of linear equations and a host of other linear algebraic problems. The text by Golub and Van Loan (1989) is an excellent reference on this subject and contains an extensive bibliography. The study of updates of matrix factorizations was, to a large extent, motivated by the need to develop some numerically stable implementation procedure for the simplex method of linear programming in order to handle very large, sparse problems. Among the early papers in this area are Bartels, Golub and Saunders (1970), and Gill, Golub, Murray and Saunders (1974). These updating schemes are now a central part in many of the computer codes written for solving linear and nonlinear programming problems, see Gill, Murray and Wright (1981, 1991).

2.11.9 The theory of solving systems of linear equations by iterative methods has undergone an extensive development. Chapter 7 in Berman and Plemmons (1979) presents a concise summary of the major convergence results. An important tool employed therein is the Perron-Frobenius theory of nonnegative matrices. The chapter ends with a set of notes giving a detailed historical account of these iterative methods.

2.11.10 The study of convex sets and functions, polyhedra and linear inequalities has a long history and quite an extensive literature. Systematic scholarly coverage of these topics will be found in Stoer and Witzgall (1970), Rockafellar (1970), and Grünbaum (1967). These references can be supplemented by consulting many textbooks on mathematical programming such as Dantzig (1963), Gale (1960), Simonnard (1966), Mangasarian (1969), and Murty (1983, 1988). Sources for the important named theorems cited in Section 2.6 are Minkowski (1896), Weyl (1935), and Goldman (1956). The latter paper appears in an influential volume, Kuhn and Tucker (1956), on linear inequalities and related systems.

2.11.11 Using separation theorems is not the only way to produce theorems of the alternative. Indeed, some authors prefer the constructive approach based on the simplex method of linear programming. For some this preference is simply a matter of pedagogy; for others it is rooted in the (Intuitionist) philosophy of mathematics. The latter outlook is well summarized by Hermann Weyl's often-quoted pronouncement, "Whenever you can settle a question by explicit construction, be not satisfied with purely existential arguments." A more pragmatic statement of the same idea was expressed by Ford and Fulkerson (1962) who in their seminal monograph *Flows in Networks* said "Other things being nearly equal, we prefer a constructive proof of a theorem to a non-constructive one, and a constructive proof that leads to an efficient computational scheme is, to our way of thinking, just that much better."

2.11.12 Motzkin (1936) favored the term "transposition theorem" for what we call "theorem of the alternative." That publication, Motzkin's doctoral dissertation, included many contributions to the theory of linear inequalities as well as a wealth of information on the preceding literature. Our development of theorems of the alternative is based on Theorem **2.7.6**

which is a version of the so-called Farkas' lemma. (Actually, in the work of Farkas (1902), the result is identified as a *Grundsatz*, not as a *Hilfsatz*.) The sources for the other named theorems of the alternative (**2.7.10**, **2.7.11**, and **2.7.12**) are Gordan (1873), Ville (1938), and Stiemke (1915), respectively.

2.11.13 The theory of duality is well covered in many textbooks on linear programming. See for example Gale (1960), Dantzig (1963), Murty (1983), and Schrijver (1986). The actual duality theorem of linear programming is ordinarily attributed to von Neumann (1947) and Gale, Kuhn and Tucker (1951).

2.11.14 Theorem **2.8.1** appeared in an appendix to the paper by Frank and Wolfe (1956). The result now known as the Frank-Wolfe theorem has been proved in various ways and has also been generalized to some extent. It is a very important existence result for quadratic programming, and hence for the linear complementarity problem.

2.11.15 The first-order optimality conditions in nonlinear programming were laid down by Karush (1939) and Kuhn and Tucker (1951). As remarked in **1.7.3**, the paper by Kuhn (1976) is of historical interest in this regard. Our presentation in **2.8.2** is restricted to the special case of quadratic programming. The fact that problems of the latter kind always have linear constraints implies that "constraint qualification" is automatically satisfied. For treatments of second-order necessary and sufficient conditions for local optimality, see Ritter (1965), Majthay (1971), and Contesse (1980).

2.11.16 We have noted in Section 2.8 that a feasible point satisfying the first-order optimality conditions of a convex quadratic program must be a global optimum. Actually, the same can be said if the objective function of the quadratic program is *pseudo-convex* on its feasible region. A differentiable function f is pseudo-convex on a convex set C if and only if for all x and $\bar{x}$ in C,

$$\nabla f(\bar{x})(x - \bar{x}) \geq 0 \quad \Rightarrow \quad f(x) \geq f(\bar{x}).$$

In the case where the feasible region is a solid subset of the nonnegative orthant and the linear term of the objective is nonzero, it is enough for the objective function to be *quasi-convex* on the nonnegative orthant. For a differentiable function f, the latter property is equivalent to the condition

$$f(x) \leq f(\bar{x}) \quad \Rightarrow \quad \nabla f(\bar{x})(x - \bar{x}) \leq 0$$

for all x and $\bar{x}$ in C. For a detailed treatment of these issues, see Cottle and Ferland (1971), but beware of an unfortunate typographical error in that paper: the first inequality in its display (1) should read $\geq$ and not $\leq$. Cottle and Ferland (1972) gives matrix-theoretic criteria for quasi-convexity and pseudo-convexity of quadratic functions on the nonnegative orthant.

2.11.17 Dennis (1959) and Dorn (1960a, 1960b, 1961) were the first to study duality in quadratic programming. The approach given in Section 2.8 is based on the *symmetric duality theory* for quadratic programming given by Cottle (1963).

2.11.18 The concepts of index and local degree are first seen in Kronecker (1869a, 1869b). The idea of global degree is credited to Brouwer (1912). As noted in **2.11.1**, Brouwer (1912) is also the paper in which **2.1.24** is first shown. Degree theory is a well studied subject in mathematics, and it has been treated by several different approaches. Section 2.9 adopts the analytic perspective which is the one used in Ortega and Rheinboldt (1970) and Lloyd (1978). The references of Milnor (1965), Guillemin and Pollack (1974), and Hirsch (1976) develop degree theory from the viewpoint of differential topology. A third approach to the subject is via combinatorial topology; details of this treatment can be found in the texts by Lefschetz (1930) and Cronin (1964). Several of these cited references also contain additional historical remarks and references concerning degree theory. The reader should consult Dieudonné (1989) for a more complete history of degree theory.

2.11.19 The notion of dimension is undoubtedly quite old and is linked to the intuitive feeling that R^m and R^n are "different" if $m \neq n$. However, with the rise of modern set theory and topology, it became clear that if the notion of dimension was to have any true meaning, then it must be shown that R^m and R^n are, indeed, not homeomorphic if $m \neq n$. This was finally shown by Brouwer (1911). It is interesting to note that this result was also shown by Lebesgue (1911) which, as it happens, is the very next paper appearing after Brouwer (1911) in volume 70 of *Mathematische Annalen*. The two papers use different approaches. It should be noted that the proof in Lebesgue (1911) is flawed and was later corrected in Lebesgue (1921).

A rigorous definition of topological dimension was first given by Brouwer (1913). This definition was laid out, in an intuitive manner, by Poincaré (1912). A different definition of topological dimension grew out of Lebesgue (1911). However, the two definitions are equivalent for separable pseudo-metric spaces such as R^n. The reader should consult Hurewicz and Wallman (1941), Nagata (1965) and Pears (1975) for additional historical remarks, details, and references regarding dimension theory.

2.11.20 Although we do not directly make use of the concept of measure, there are many places within this text in which this idea can be fruitfully incorporated. These junctures will be apparent to readers familiar with measure theory. The following texts are listed here for those interested in further pursuing the subject of measure: Halmos (1950), Royden (1968), Rudin (1974), and Dudley (1989).

Chapter 3

EXISTENCE AND MULTIPLICITY

In this chapter we present results pertaining to the existence and multiplicity of solutions to the linear complementarity problem. In essence, there are two general approaches to establishing the existence of a solution to the LCP. One is the *constructive approach,* in which one assumes appropriate conditions and actually produces a solution by means of an algorithm. The other is the *analytic approach,* in which one relies on an equivalent formulation of the LCP as a certain familiar mathematical programming problem (such as a quadratic program or a fixed-point problem) and then invokes an existence theorem (which is presumably proven by other means) for the latter problem. In this chapter we concentrate on the analytic approach.

"Multiplicity" refers to the number of solutions to the LCP, which can be finite or infinite. The special case of a unique solution is of particular interest. There are two types of uniqueness results: *global* and *local* uniqueness. A global uniqueness result asserts when a solvable LCP has only one solution, whereas a local uniqueness result provides conditions under which a given solution to an LCP is the only solution in one of its neighborhoods. Locally unique solutions are said to be *isolated.*

In connection with the multiplicity of solutions, it is also of interest to explore the structure of the solution set of an LCP. Such information often has important sensitivity and algorithmic implications for the problem.

3.1 Positive Definite and Semi-definite Matrices

As the source problems in Section 1.2 suggest, matrix classes play a strong role in the theory of the LCP. Two of the most fundamental classes are those consisting of the positive definite and positive semi-definite matrices. In addition to the fact that these matrices are the most commonly found in applications, they have nice properties which serve as a model for extension. In this section, we establish the existence and several properties of solutions to a linear complementarity problem with such a matrix M. As in linear complementarity theory generally, most of these results do not require that M be symmetric.

The cornerstone for the existence results to be presented is the quadratic programming formulation of the LCP (cf. (1.4.2)):

$$\begin{array}{ll} \text{minimize} & z^{\mathrm{T}}(q + Mz) \\ \text{subject to} & q + Mz \geq 0 \\ & z \geq 0. \end{array} \tag{1}$$

We start by stating a basic property of this program. It should be noticed that this result is valid for *arbitrary* real square matrices M.

3.1.1 Lemma. If the LCP (q, M) is feasible, then the quadratic program (1) has an optimal solution, z^*. Moreover, there exists a vector u^* of multipliers satisfying the conditions

$$q + (M + M^{\mathrm{T}})z^* - M^{\mathrm{T}}u^* \geq 0 \tag{2}$$

$$(z^*)^{\mathrm{T}}\big(q + (M + M^{\mathrm{T}})z^* - M^{\mathrm{T}}u^*\big) = 0 \tag{3}$$

$$u^* \geq 0 \tag{4}$$

$$(u^*)^{\mathrm{T}}(q + Mz^*) = 0. \tag{5}$$

Finally, the vectors z^* and u^* satisfy

$$(z^* - u^*)_i\big(M^{\mathrm{T}}(z^* - u^*)\big)_i \leq 0 \quad \text{for all } i = 1, \ldots, n. \tag{6}$$

Proof. Since (q, M) is feasible, so is the quadratic program (1). As the objective function of the quadratic program is bounded below on the feasible region, the Frank-Wolfe theorem implies that there exists an optimal solution to (1). Such an optimal solution z^* and a suitable vector u^* of multipliers will satisfy the Karush-Kuhn-Tucker conditions (2) – (5). To prove (6), we examine the inner product (3) at the componentwise level and deduce that for all $i = 1, \ldots, n$,

$$z_i^* \big(M^{\mathrm{T}}(z^* - u^*)\big)_i \leq 0 \tag{7}$$

using the fact that $z^* \in \mathrm{FEA}(q, M)$. Similarly, multiplying the i-th component in (2) by u_i^* and then invoking the complementarity condition $u_i^*(q + Mz^*)_i = 0$ which is implied by (4), (5), and the feasibility of z^*, we obtain

$$- u_i^* \big(M^{\mathrm{T}}(z^* - u^*)\big)_i \leq 0. \tag{8}$$

Now, (6) follows by adding (7) and (8). □

With Lemma **3.1.1**, we prove the following existence result for the LCP with a positive semi-definite matrix.

3.1.2 Theorem. Let M be a positive semi-definite matrix. If the LCP (q, M) is feasible, then it is solvable.

Proof. By **3.1.1**, there exist vectors z^* and u^* such that z^* is feasible in (q, M) and conditions (2) – (6) hold. Adding the n inequalities in (6), we obtain

$$(z^* - u^*)^{\mathrm{T}} M^{\mathrm{T}} (z^* - u^*) \leq 0.$$

Since M is positive semi-definite, this inequality must hold as an equation. Reviewing how (6) is derived, we deduce that (7) must hold as equality for all i. Consequently, by (3), we conclude

$$(z^*)^{\mathrm{T}}(q + Mz^*) = 0.$$

Thus, z^* is a solution of the LCP (q, M). □

If the matrix M is positive definite, a stronger conclusion about the LCP (q, M) can be drawn. We first state a lemma.

3.1.3 Lemma. If M is a positive definite matrix, then there exists a vector z such that

$$Mz > 0, \quad z > 0. \tag{9}$$

Proof. Indeed, if no such vector z exists, then by Ville's theorem of the alternative, it follows that there exists a *nonzero* vector $u \geq 0$ such that

$$M^{\mathrm{T}}u \leq 0.$$

Multiplying the above inequality by u yields $u^{\mathrm{T}}M^{\mathrm{T}}u \leq 0$, contradicting the positive definiteness of M. □

3.1.4 Definition. A square matrix M for which a vector z satisfying (9) exists is called an $\boldsymbol{S}$-matrix ($\boldsymbol{S}$ stands for Stiemke) . The class of $\boldsymbol{S}$-matrices is denoted by $\boldsymbol{S}$.

It should be noted that (9) is feasible if and only if

$$Mz > 0, \quad z \geq 0 \tag{10}$$

is feasible. Clearly, (10) is implied by (9). On the other hand, suppose a vector $z \geq 0$ is given such that $Mz > 0$. Since Mz is continuous in z, it follows that $M(z + \lambda e) > 0$ for all $\lambda > 0$ small enough. As $z + \lambda e > 0$, we have (9).

Lemma **3.1.3** shows that a positive definite matrix M must belong to the class $\boldsymbol{S}$. As a matter of fact, an arbitrary $\boldsymbol{S}$-matrix is related to the feasibility of the LCP in the following way.

3.1.5 Proposition. The matrix $M \in R^{n \times n}$ is an $\boldsymbol{S}$-matrix if and only if the LCP (q, M) is feasible for all $q \in R^n$.

Proof. Consider an arbitrary LCP (q, M) in which $M \in \boldsymbol{S}$. Let $\hat{z}$ be a solution to (9). We find

$$\lambda M\hat{z} = M(\lambda\hat{z}) \geq -q$$

for a suitably large positive scalar λ, and of course $\lambda\hat{z} > 0$, so that $\lambda\hat{z}$ is feasible for (q, M). Conversely, if (q, M) is feasible for every q, take any $q < 0$. Any feasible solution $\hat{z}$ of (q, M) will satisfy $M\hat{z} \geq -q > 0, \ \hat{z} \geq 0$. Thus, M is an $\boldsymbol{S}$-matrix. □

Combining **3.1.2**, **3.1.3**, and **3.1.5**, we obtain the existence part of the following result.

3.1.6 Theorem. If $M \in R^{n\times n}$ is positive definite, then the LCP (q, M) has a unique solution for all $q \in R^n$.

Proof. In light of what has been said above, it suffices to prove the uniqueness part of the assertion. Let $q \in R^n$ be given. Any solution to the LCP (q, M) must be an optimal solution to the quadratic program (1). If M is positive definite, the objective function is strictly convex. Hence (1) has a unique optimal solution. Consequently so does (q, M). □

In general, the LCP with a positive semi-definite matrix can have multiple solutions . For instance, the LCP with

$$q = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

has solutions

$$z^1 = (1,0), \quad z^2 = (0,1), \quad z^3 = (\tfrac{1}{2}, \tfrac{1}{2}).$$

Observe that $w = q + Mz$ is the same for all three solutions z^i $(i = 1, 2, 3)$.

The theorem below describes several properties of the solution set of an LCP of positive semi-definite type.

3.1.7 Theorem. Let $M \in R^{n\times n}$ be positive semi-definite, and let $q \in R^n$ be arbitrary. The following hold:

(a) If z^1 and z^2 are two solutions of (q, M), then

$$(z^1)^{\mathrm{T}}(q + Mz^2) = (z^2)^{\mathrm{T}}(q + Mz^1) = 0. \tag{11}$$

(b) If $z^* \in \mathrm{SOL}(q, M)$ has the property that (i) z^* is nondegenerate and (ii) $M_{\alpha\alpha}$ is nonsingular where

$$\alpha = \{i : z_i^* > 0\},$$

then z^* is the unique solution of (q, M).

(c) If (q, M) has a solution, then $\mathrm{SOL}(q, M)$ is polyhedral and equal to

$$P = \{z \in R^n_+ : q + Mz \geq 0,\ q^{\mathrm{T}}(z - \bar{z}) = 0,\ (M + M^{\mathrm{T}})(z - \bar{z}) = 0\}$$

where $\bar{z}$ is an arbitrary solution.

(d) If M is symmetric (as well as positive semi-definite), then $Mz^1 = Mz^2$ for any two solutions z^1 and z^2.

Proof. (a) Let $w^i = q + Mz^i$ for $i = 1, 2$. We have $w^1 - w^2 = M(z^1 - z^2)$. By the positive semi-definiteness of M and by the fact that z^1 and z^2 solve the LCP (q, M), we obtain

$$0 \leq (z^1 - z^2)^{\mathrm{T}} M (z^1 - z^2) = -(z^1)^{\mathrm{T}} w^2 - (z^2)^{\mathrm{T}} w^1 \leq 0. \tag{12}$$

Consequently, we must have $(z^1)^{\mathrm{T}} w^2 = (z^2)^{\mathrm{T}} w^1 = 0$, as desired.

(b) Let z' be any solution. By (11) we have

$$(q + Mz')_i = 0 \quad \text{for all } i \in \alpha. \tag{13}$$

If $i \notin \alpha$, then $(q + Mz^*)_i > 0$ by the nondegeneracy of z^*. By (11) again, we deduce that $z'_i = 0$ for $i \notin \alpha$. Thus, (13) becomes the square system of linear equations

$$q_\alpha + M_{\alpha\alpha} z'_\alpha = 0$$

whose solution must be unique by the nonsingularity assumption on $M_{\alpha\alpha}$.

(c) Let $\bar{z}$ be a given solution and z an arbitrary solution. By the proof of (12), we can show that $(z - \bar{z})^{\mathrm{T}} M (z - \bar{z}) = 0$. From this we obtain $(M + M^{\mathrm{T}})(z - \bar{z}) = 0$, for when a positive semi-definite quadratic form vanishes, so does its gradient. Thus, we have

$$z^{\mathrm{T}}(M + M^{\mathrm{T}})z = z^{\mathrm{T}}(M + M^{\mathrm{T}})\bar{z},$$

and

$$\bar{z}^{\mathrm{T}}(M + M^{\mathrm{T}})\bar{z} = \bar{z}^{\mathrm{T}}(M + M^{\mathrm{T}})z.$$

The last two equations imply that $z^{\mathrm{T}} M z = \bar{z}^{\mathrm{T}} M \bar{z}$. At the same time, we have

$$0 = \bar{z}^{\mathrm{T}}(q + M\bar{z}) = z^{\mathrm{T}}(q + Mz).$$

Consequently, $q^{\mathrm{T}}z = q^{\mathrm{T}}\bar{z}$ and $z \in P$. Conversely, suppose that $z \in P$. To show that z solves the LCP (q, M), it suffices to show $z^{\mathrm{T}}(q + Mz) = 0$. From $(M + M^{\mathrm{T}})(z - \bar{z}) = 0$, by the argument we have just used, it follows that $z^{\mathrm{T}}Mz = \bar{z}^{\mathrm{T}}M\bar{z}$. Thus, as $q^{\mathrm{T}}(z - \bar{z}) = 0$,

$$z^{\mathrm{T}}(q + Mz) = \bar{z}^{\mathrm{T}}(q + M\bar{z}) = 0$$

because $\bar{z}$ is a given solution of (q, M).

(d) The hypotheses of this part include those of (c). The desired conclusion now follows from the symmetry assumption on M and the condition $(M + M^{\mathrm{T}})(z - \bar{z}) = 0$ in the definition of the solution set P. □

From (d) we see that a linear complementarity problem of symmetric positive semi-definite type has the property that $w = q+Mz$ is constant for all solutions z. Accordingly, we say that the solutions of such a problem are *w-unique*. As an application of this idea, consider a linear complementarity problem in which

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_t \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} M_1 & & & \\ M_1 & M_2 & & \\ \vdots & \vdots & \ddots & \\ M_1 & M_2 & \cdots & M_t \end{bmatrix}$$

and $q_i \in R^n$. The M_i are assumed to be *symmetric* positive semi-definite matrices of order n. If $\bar{z} = [\,\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_t\,]$ solves (q, M), then $\bar{z}_1$ must solve (q_1, M_1). Moreover, $\bar{z}_2$ must solve $(q_2 + M_1\bar{z}_1, M_2)$, and so forth. This suggests an obvious sort of decomposition of a large problem of this special structure into a set of t smaller ones. But one condition requires checking. If the problem (q_1, M_1), for example, has multiple solutions, does the choice of one particular solution affect the feasiblility of the second-stage problem? That is, if z_1' and z_1'' both solve (q_1, M_1) could it happen that $(q_2 + M_1 z_1', M_2)$ is solvable while $(q_2 + M_1 z_1'', M_2)$ is not? By the symmetry and positive semi-definiteness of the M_i, part (d) of **3.1.7** tells us that the answer is in the negative. We can be sure that $M_i z_i' = M_i z_i''$ for any two solutions of the i-th stage problem, hence they lead to exactly the same problem at the next stage.

In general, the solution set of an arbitrary LCP (q, M) if nonempty is the union of a finite number of convex polyhedra

$$X_\alpha = \{\, x \in R^n_+ : (q + Mx)_\alpha = 0, \quad (q + Mx)_{\bar{\alpha}} \geq 0, \quad x_{\bar{\alpha}} = 0 \,\}$$

where α runs over all subsets of $\{1, \ldots, n\}$. (Some of the sets X_α may be empty.) Being such a union of polyhedra, the solution set of an arbitrary LCP is typically nonconvex. Indeed, it is convex if and only if it is polyhedral; see Theorem **3.1.8**.

Theorem **3.1.7** shows that the solution set of an LCP of the positive semi-definite type is a convex polyhedron. Moreover, if M is positive semi-definite, then (11) holds for any two solutions z^1 and z^2 of the LCP (q, M). As a matter of fact, condition (11) characterizes the convexity (and therefore the polyhedrality) of the solution set of an arbitrary LCP .

3.1.8 Theorem. Let $M \in R^{n\times n}$ and $q \in R^n$ be given. The following two statements are equivalent:

(a) The solution set of (q, M) is convex.

(b) For any two solutions z^1 and z^2 of (q, M), equation (11) holds.

Moreover, if $\text{SOL}(q, M)$ is convex, then it is equal to X_α where

$$\alpha = \{i : z_i > 0 \text{ for some } z \in \text{SOL}(q, M)\}. \tag{14}$$

Proof. (a) $\Rightarrow$ (b). Let z^1 and z^2 be any two solutions of (q, M). By the convexity assumption, the vector $z = \tau z^1 + (1 - \tau)z^2$ is also a solution for any $\tau \in (0, 1)$. By letting $w^i = q + Mz^i$ for $i = 1, 2$, we have

$$0 = (\tau w^1 + (1 - \tau)w^2)^{\mathrm{T}}(\tau z^1 + (1 - \tau)z^2) = \tau(1 - \tau)\big((w^1)^{\mathrm{T}} z^2 + (w^2)^{\mathrm{T}} z^1\big)$$

from which (11) holds.

(b) $\Rightarrow$ (a). This follows easily by reversing the argument used in the first part.

Finally, to prove that $\text{SOL}(q, M)$, if convex, is equal to X_α where α is as given in (14), it suffices to show $\text{SOL}(q, M) \subseteq X_\alpha$. But this is obvious in view of the equation (11) and the definition of the index set α. $\square$

It is interesting to compare part (c) of **3.1.7** with the last assertion of **3.1.8**. In the former result, the set $\mathrm{SOL}(q, M)$ is completely determined provided that *one* solution of (q, M) is known; nevertheless, this is not the case with the latter result which requires the knowledge of the special index set α as given in (14). This comparison points out one particular feature of a positive semi-definite LCP (see also Section 3.5).

3.2 The Classes Q and Q_0

In Section 3.1, we showed that if M is positive definite, then the LCP (q, M) has a unique solution for all vectors q. This prompts the following

Question 1. What is the class of matrices M for which the LCP (q, M) has a unique solution for all vectors q?

It turns out that a complete answer to this question is available. It will be discussed in the next section. One can ask a related question by dropping the uniqueness requirement.

Question 2. What is the class of matrices M for which the LCP (q, M) has a solution for all vectors q? This class is denoted $\boldsymbol{Q}$, and its elements are called $\boldsymbol{Q}$-matrices.

Intrinsically, Questions 1 and 2 involve a continuum of vectors q. An ideal answer for them would be to provide a set of necessary and sufficient conditions which could be used to check, in finite time, if any given matrix M belongs to the class. In the case of Question 2, some partial—albeit not entirely satisfactory—answers are available.

In terms of the complementary cones, Question 2 is asking for the class of $n \times n$ matrices M for which the complementary range of M equals R^n. For an arbitrary matrix M, the cone $K(M)$ is generally not even convex, much less equal to R^n. We can at least ask for the class of matrices M for which $K(M)$ is convex. This matter is closely related to

Question 3. What is the class of matrices M for which the LCP (q, M) is solvable whenever it is feasible? This class is denoted $\boldsymbol{Q}_0$, and its elements are called $\boldsymbol{Q}_0$-matrices.

According to Theorem **3.1.2**, every positive semi-definite matrix belongs to the class $\boldsymbol{Q}_0$. Likewise, Theorem **3.1.6** implies that every positive definite matrix belongs to the class $\boldsymbol{Q}$.

We now establish the equivalence between the class $\boldsymbol{Q}_0$ and the convexity of $K(M)$.

3.2.1 Proposition. Let $M \in R^{n\times n}$. The following are equivalent:

(a) $M \in \boldsymbol{Q}_0$.

(b) $K(M)$ is convex.

(c) $K(M) = \text{pos}\,(I, -M)$.

Proof. (a) $\Rightarrow$ (b). Let q^1 and q^2 be two vectors in $K(M)$. Thus, the LCP (q^i, M) is solvable for $i = 1, 2$. Obviously, the LCP (q^λ, M) is feasible for all $q^\lambda = \lambda q^1 + (1-\lambda)q^2$ with $\lambda \in [0,1]$. Thus, by (a), it follows that (q^λ, M) is solvable. Hence $q^\lambda \in K(M)$ and (b) follows.

(b) $\Rightarrow$ (c). This is clear because the convex hull of $K(M)$ is equal to $\text{pos}\,(I, -M)$.

(c) $\Rightarrow$ (a). The cone $\text{pos}\,(I, -M)$ consists of all vectors q for which the LCP (q, M) is feasible. Therefore, if (c) holds, (a) follows readily. □

From Proposition **3.1.5**, it is clear that the classes $\boldsymbol{Q}$ and $\boldsymbol{Q}_0$ are related through the equation

$$\boldsymbol{Q} = \boldsymbol{Q}_0 \cap \boldsymbol{S}. \tag{1}$$

From the definition of an $\boldsymbol{S}$-matrix, it follows that testing whether an arbitrary matrix M is in the class $\boldsymbol{S}$ is a matter of checking the feasibility of the inequality system (3.1.9). This, in turn, can be settled by linear programming. Therefore, the relation (1) implies that if one has a finite test for a $\boldsymbol{Q}_0$-matrix, then that test can easily be made into one for a $\boldsymbol{Q}$-matrix. At present, no efficient test exists for either matrix class.

3.3 $\boldsymbol{P}$-matrices and Global Uniqueness

In this section, we establish a characterization for the class of matrices M such that the LCP (q, M) has a unique solution for all vectors q. For this purpose, we introduce the class of $\boldsymbol{P}$-matrices.

3.3.1 Definition. A matrix $M \in R^{n \times n}$ is said to be a ***P**-matrix* if all its principal minors are positive. The class of such matrices is denoted $\boldsymbol{P}$.

Obviously, if M is a $\boldsymbol{P}$-matrix, then so are each of its principal submatrices and its transpose. It is well known that a *symmetric* matrix is positive definite if and only if it belongs to $\boldsymbol{P}$. As easily constructed examples show, this equivalence breaks down when the symmetry assumption is dropped. Nevertheless, it will follow from Theorem **3.3.7** below and from Theorem **3.1.6** that every positive definite matrix belongs to the class $\boldsymbol{P}$.

3.3.2 Example. Let

$$M = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}.$$

Clearly, M is a $\boldsymbol{P}$-matrix. However, letting $x = (1,1)$, we note that $x^{\mathrm{T}}Mx = -1 < 0$ which shows that M is not positive definite.

In order to state the first characterization theorem on $\boldsymbol{P}$-matrices, we introduce the important notion of sign reversing.

3.3.3 Definition. The matrix $M \in R^{n \times n}$ *reverses the sign* of the vector $z \in R^n$ if $z_i(Mz)_i \leq 0$ for all $i = 1, \ldots, n$.

3.3.4 Theorem. Let $M \in R^{n \times n}$. The following statements are equivalent:

(a) M is a $\boldsymbol{P}$-matrix.

(b) M reverses the sign of no nonzero vector, i.e.,

$$[\, z_i(Mz)_i \leq 0 \text{ for all } i \,] \quad \Rightarrow \quad [\, z = 0 \,].$$

(c) All real eigenvalues of M and its principal submatrices are positive.

Proof. (a) $\Rightarrow$ (b). This is clearly true for $n = 1$. Using induction, we will now assume this implication holds for the case $n-1$, where $n > 1$. Suppose that the $\boldsymbol{P}$-matrix $M \in R^{n \times n}$ reverses the sign of the nonzero vector $z \in R^n$. If $z_i = 0$, for some i, then the principal submatrix $M_{\bar{\imath}\bar{\imath}}$ is a $\boldsymbol{P}$-matrix which reverses the sign of the nonzero vector $z_{\bar{\imath}}$. This contradicts the induction hypothesis, so no component of z is zero. We may now write

$$(Mz)_i = d_i z_i \qquad \text{with} \qquad d_i = (Mz)_i / z_i \leq 0.$$

Letting $D = \text{diag}(d_1, \ldots, d_n)$, we obtain $(M - D)z = 0$. Using (2.2.1) we have

$$\det(M - D) = \sum_{\alpha} \det(-D_{\alpha\alpha}) \det M_{\bar{\alpha}\bar{\alpha}}$$

where α runs over the index subsets of $\{1, \ldots, n\}$. Since D is a nonpositive diagonal matrix and M is a $\boldsymbol{P}$-matrix, it follows that $M - D$ must be nonsingular. Thus, we obtain a contradiction, and (b) follows.

(b) $\Rightarrow$ (c). Since M reverses the sign of no nonzero vector, the same can be said of each principal submatrix of M. Hence it suffices to show that all real eigenvalues of M are positive. Let λ be one such eigenvalue and z an associated eigenvector. The vector z must be nonzero and, as λ is real, we may take z to be real. We have $Mz = \lambda z$. Since M does not reverse the sign of z, it follows that $\lambda > 0$.

(c) $\Rightarrow$ (a). Since the determinant of a matrix is equal to the product of all the (real as well as complex) eigenvalues, and since the complex eigenvalues always appear in conjugate pairs for real matrices, it follows that, if (c) holds, the determinant of M and all its principal submatrices must be positive. □

3.3.5 Corollary. Every $\boldsymbol{P}$-matrix is an $\boldsymbol{S}$-matrix.

Proof. We prove the contrapositive. If M is not an $\boldsymbol{S}$-matrix, then by Ville's theorem of the alternative, there exists a vector $u \neq 0$ such that $u \geq 0$ and $M^{\mathrm{T}}u \leq 0$. Thus, M^{T} reverses the sign of the nonzero vector u. By part (b) of Theorem **3.3.4**, M^{T} is not a $\boldsymbol{P}$-matrix. Therefore, M is not a $\boldsymbol{P}$-matrix. □

3.3.6 Example. The converse of Corollary **3.3.5** is false. Let

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

As $Me = (3, 3) > 0$, we see that M is an $\boldsymbol{S}$-matrix. However, M is not a $\boldsymbol{P}$-matrix as its determinant is negative.

Using Theorem **3.3.4**, we may prove

3.3.7 Theorem. A matrix $M \in R^{n \times n}$ is a $\boldsymbol{P}$-matrix if and only if the LCP (q, M) has a unique solution for all vectors $q \in R^n$.

Proof. Suppose M is a $\boldsymbol{P}$-matrix. From **3.3.5**, it follows that $M \in \boldsymbol{S}$. In particular, the LCP (q, M) is feasible for each $q \in R^n$. By Lemma **3.1.1**, the quadratic program (3.1.1) has an optimal solution z^*, and there exists a vector u^* such that $(z^*, \boldsymbol{u}^*)$ satisfies the conditions (3.1.2)–(3.1.6). Since M^{T} is a $\boldsymbol{P}$-matrix, it follows from (3.1.6) that $z^* = u^*$. Hence z^* solves the LCP (q, M) by (3.1.5).

To show that z^* is the unique solution, let z' be an alternate solution. Write $w^* = q + Mz^*$ and $w' = q + Mz'$. Subtracting, we deduce $w^* - w' = M(z^* - z')$. Thus, for all $i = 1, \ldots, n$,

$$0 \geq (z^* - z')_i(w^* - w')_i = (z^* - z')_i(M(z^* - z'))_i,$$

contradicting the fact that M reverses the sign of no nonzero vector.

Conversely, suppose M is not a $\boldsymbol{P}$-matrix. From **3.3.4**, there exists a vector $z \neq 0$ such that $z_i(Mz)_i \leq 0$ for all i. Let $z^+ = \max(0, z)$ and $z^- = \max(0, -z)$ be the positive and negative parts of z, respectively. As $z \neq 0$, we have $z^+ \neq z^-$. Similarly, let $u^+ = \max(0, Mz)$ and $\boldsymbol{u}^- = \max(0, -Mz)$. Noticing that $z = z^+ - z^-$ and $Mz = u^+ - \boldsymbol{u}^-$, we define

$$\bar{q} = \boldsymbol{u}^+ - Mz^+ = \boldsymbol{u}^- - Mz^-.$$

If $z_i > 0$, then $(Mz)_i \leq 0$. Thus, $z_i^+ \boldsymbol{u}_i^+ = 0$. Consequently, z^+ is a solution to $(\bar{q}, M)$. Similarly, one can show that the same is true for z^-. Therefore, $(\bar{q}, M)$ has two distinct solutions. This completes the proof. □

It follows from **3.3.7** that every $\boldsymbol{P}$-matrix belongs to the class $\boldsymbol{Q}$. Indeed, this theorem completely answers Question 1 in Section 3.2. Thus, given an arbitrary matrix $M \in R^{n \times n}$, by testing whether $M \in \boldsymbol{P}$, one can decide whether the LCP (q, M) has a unique solution for all $q \in R^n$. Notice from the definition, there is a finite, but not necessarily efficient, test which will determine if a matrix is in $\boldsymbol{P}$. At present, there is no efficient test to determine whether an arbitrary matrix is in $\boldsymbol{P}$.

The proof of **3.3.7** shows that if M is a $\boldsymbol{P}$-matrix, then the unique solution of the LCP (q, M) is also the unique solution of the quadratic program (3.1.1). If M is not positive semi-definite, the objective function in (3.1.1) is generally nonconvex.

As we have pointed out, the class of positive definite matrices is a subclass of $\boldsymbol{P}$. In the sequel, we introduce another subclass of $\boldsymbol{P}$-matrices which generalize the positive definite matrices. In general, if M is a $\boldsymbol{P}$-matrix, then so is any positively scaled matrix DME where D and E are arbitrary diagonal matrices with positive diagonal entries. Moreover, a vector z is a solution of (q, M) if and only if the vector $E^{-1}z$ is a solution of (Dq, DME). If the matrix DME is positive definite, then the LCP (Dq, DME) (and thus (q, M)) is equivalent to a *convex* quadratic program. Motivated by this equivalent formulation, one is led to ask the question: For a given $\boldsymbol{P}$-matrix M, is it always possible to find positive diagonal matrices D and E so that DME is positive definite? Presumably, an affirmative answer would allow one to convert an LCP with a $\boldsymbol{P}$-matrix into a convex quadratic program. This question leads to the following definition.

3.3.8 Definition. A matrix $M \in R^{n\times n}$ is said to be *(positive) stable* if there exists a symmetric positive definite matrix H such that HM is positive definite. The matrix M is said to be *diagonally (positive) stable* if there exists a positive-diagonal matrix D such that DM is positive definite.

Historically, the class of stable matrices was first discussed in the context of the stability analysis of a dynamical system. The general notion of a stable matrix does not seem to be too useful in the context of the LCP because scaling the matrix M by a nondiagonal matrix H would destroy the complementarity condition of the problem. Nevertheless, diagonally stable matrices do have some significance in the study of the LCP as we have pointed out.

Clearly, a diagonally stable matrix is in the class $\boldsymbol{P}$. The following result states that several ways of scaling a matrix to make it positive definite are equivalent.

3.3.9 Theorem. Let $M \in R^{n\times n}$. The following statements are equivalent:

(a) M is diagonally stable.

(b) There exist positive-diagonal matrices D and E such that DME is positive definite.

(c) There exists a positive-diagonal matrix F such that $F^{-1}MF$ is positive definite.

Moreover, if M is diagonally stable, then all eigenvalues of M have positive real parts.

Proof. (a) $\Rightarrow$ (b). This is obvious.

(b) $\Rightarrow$ (c). If DME is positive definite, then so is

$$(DE)^{-1/2}(DME)(DE)^{-1/2} = F^{-1}MF$$

where $F = E^{1/2}D^{-1/2}$.

(c) $\Rightarrow$ (a). If $F^{-1}MF$ is positive definite, then so is

$$F^{-1}(F^{-1}MF)F^{-1} = F^{-2}M.$$

Finally, let M be a diagonally stable matrix. By (c), we may assume with no loss of generality that M itself is positive definite. Let $\lambda = a + ib$ be an eigenvalue of M, and let $z = u + iv$ be a corresponding eigenvector. By equating the real and imaginary parts in the equation $Mz = \lambda z$, we obtain

$$Mu = au - bv \quad \text{and} \quad Mv = av + bu,$$

from which we deduce $0 < u^{\mathrm{T}}Mu + v^{\mathrm{T}}Mv = a(u^{\mathrm{T}}u + v^{\mathrm{T}}v)$. Thus, $a > 0$. Consequently, all eigenvalues of M have positive real parts. □

Using **3.3.9**, one can give an example of a $\boldsymbol{P}$-matrix which can not be scaled positive definite.

3.3.10 Example. Let

$$M = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -17 \\ 4 & 0 & 1 \end{bmatrix}.$$

The eigenvalues of M are 5 and $-1 \pm i\sqrt{13}$. Thus, M can not be scaled positive definite. It is easy to verify that $M \in \boldsymbol{P}$.

Another important subclass of $\boldsymbol{P}$ is a generalization of the diagonally dominant matrices.

3.3.11 Definition. A matrix $M \in R^{n \times n}$ is said to be an $\boldsymbol{H}$-matrix if there exists an n-vector $d > 0$ such that for all $i = 1, \ldots, n$,

$$|m_{ii}|d_i > \sum_{j \neq i} |m_{ij}|d_j.$$

The class of $\boldsymbol{H}$-matrices plays a particularly important role in several areas of the LCP. Among the $\boldsymbol{H}$-matrices, those with positive diagonal entries can be shown to belong to the class $\boldsymbol{P}$; indeed, we will now show that such $\boldsymbol{H}$-matrices are diagonally stable. For this purpose it is useful to introduce the following concept.

3.3.12 Definition. Let $M \in R^{n \times n}$. The matrix $\bar{M}$ defined as

$$\bar{m}_{ij} = \begin{cases} |m_{ij}| & \text{for } i = j, \\ -|m_{ij}| & \text{for } i \neq j, \end{cases}$$

is called the *comparison matrix* associated with M.

Observe that M is an $\boldsymbol{H}$-matrix if its comparison matrix $\bar{M}$ is in the class $\boldsymbol{S}$. In general, if a matrix $M \in R^{n \times n}$ has positive diagonal entries, then for an arbitrary vector $z \in R^n$

$$z^{\mathrm{T}} M z \geq |z|^{\mathrm{T}} \bar{M} |z| \,. \tag{1}$$

Thus, if $\bar{M}$ is positive definite (positive semi-definite), then so is M.

3.3.13 Lemma. Let $M \in R^{n \times n}$. If the comparison matrix $\bar{M}$ is in $\boldsymbol{S}$, then $\bar{M}$ is in $\boldsymbol{P}$.

Proof. As $\bar{M}$ is in $\boldsymbol{S}$, there is a $d > 0$ for which $\bar{M}d > 0$. Thus, taking $D = \mathrm{diag}(d)$, we have $\bar{M}D$ is strictly row diagonally dominant. Let $z \in R^n$ be an arbitrary nonzero vector. We may assume z_1 is the component with the largest absolute value. As $\bar{M}D$ is strictly row diagonally dominant, it is not hard to show that $z_1(\bar{M}Dz)_1 > 0$. Thus, $\bar{M}D$ reverses the sign of no nonzero vector. Therefore, by Theorem **3.3.4**, $\bar{M}D$ is in $\boldsymbol{P}$, hence, so is $\bar{M}$. □

3.3.14 Remark. The above lemma is a special case of a more general theorem from the theory of $\boldsymbol{Z}$-matrices. We will treat this more fully in Section 3.11. Notice the assumption that $\bar{M}$ is a comparison matrix is important. (See Example **3.3.6**.)

3.3.15 Theorem. Let $M \in R^{n \times n}$. If M is an $\boldsymbol{H}$-matrix with positive diagonal entries, then M is diagonally stable.

Proof. Let $\bar{M}$ be the comparison matrix of M. As M is an $\boldsymbol{H}$-matrix, then $\bar{M}$ is an $\boldsymbol{S}$-matrix. Thus, we can find a positive diagonal matrix D such that $\bar{M}D$ is strictly row diagonally dominant, i.e., $\bar{M}De > 0$. From Lemma **3.3.13** we know that $\bar{M}D$ is a $\boldsymbol{P}$-matrix. Thus, $D\bar{M}^{\mathrm{T}}$ is also a $\boldsymbol{P}$-matrix. It follows from Corollary **3.3.5** that $D\bar{M}^{\mathrm{T}}$ is also an $\boldsymbol{S}$-matrix. Again, this means that a positive diagonal matrix E exists such that $N = D\bar{M}^{\mathrm{T}}E$ is strictly row diagonally dominant. Notice that N is also strictly column diagonally dominant. Thus, $(N + N^{\mathrm{T}})$ is symmetric, equal to its own comparison matrix, and strictly row and column diagonally dominant. From Lemma **3.3.13**, $(N + N^{\mathrm{T}})$ is in $\boldsymbol{P}$ and, hence, is positive definite. Therefore, N is positive definite and from (1) we see that EMD is positive definite. Theorem **3.3.9** now implies that M is diagonally stable. □

3.4 P_0-matrices and w-Uniqueness

In Theorem **3.1.7(d)**, we showed that if M is a symmetric positive semi-definite matrix, then any two solutions z^1 and z^2 of the LCP (q, M) give rise to the same vector $w = q + Mz^i$ $(i = 1, 2)$. This property of w-uniqueness can be characterized by a certain condition on M related to the notion of sign reversing . The characterization is somewhat like that of z-uniqueness by means of the $\boldsymbol{P}$-property. Before establishing criteria for w-uniqueness, we define a generalization of the class $\boldsymbol{P}$.

3.4.1 Definition. A matrix $M \in R^{n \times n}$ is said to be a $\boldsymbol{P_0}$-matrix if all its principal minors are nonnegative. The class of such matrices is denoted $\boldsymbol{P_0}$.

Parallel to the fact that any positive definite matrix must belong to $\boldsymbol{P}$, it will follow that a positive semi-definite matrix must belong to $\boldsymbol{P_0}$. The following result gives some characterizations for a $\boldsymbol{P_0}$-matrix very much like those for a $\boldsymbol{P}$-matrix (cf. **3.3.4**).

3.4.2 Theorem. Let $M \in R^{n \times n}$. The following statements are equivalent:

(a) M is a $\boldsymbol{P_0}$-matrix.

(b) For each vector $z \neq 0$, there exists an index k such that $z_k \neq 0$ and $z_k(Mz)_k \geq 0$.

(c) All real eigenvalues of M and its principal submatrices are nonnegative.

(d) For each $\varepsilon > 0$, $M + \varepsilon I$ is a $\boldsymbol{P}$-matrix.

Proof. (a) $\Rightarrow$ (d). From formula (2.2.1), for any diagonal matrix D,

$$\det(M + D) = \sum_{\alpha} \det D_{\alpha\alpha} \det M_{\bar{\alpha}\bar{\alpha}} \tag{1}$$

where α runs over the index subsets of $\{1, \ldots, n\}$. Let $D = \varepsilon I$ in (1). As M is a $\boldsymbol{P_0}$-matrix, each term in the sum will be nonnegative and the term with $\alpha = \{1, \ldots, n\}$ will be $\varepsilon^n > 0$. Thus, $\det(M + \varepsilon I) > 0$. As each principal submatrix of M must also be in $\boldsymbol{P_0}$, this argument applies to each of these submatrices. Therefore, $M + \varepsilon I$ is a $\boldsymbol{P}$-matrix.

(d) $\Rightarrow$ (b). Let $z \neq 0$ be given. Since $M + \varepsilon I$ is a $\boldsymbol{P}$-matrix for $\varepsilon > 0$, there exists an index i (depending on ε) such that $z_i((M + \varepsilon I)z)_i > 0$. Let $\{\varepsilon_k\}$ be a sequence converging to zero. There must exist an index j such that $z_j((M + \varepsilon_k I)z)_j > 0$ for infinitely many ε_k. Clearly, $z_j \neq 0$. Also, as $k \to \infty$, we have $\varepsilon_k \to 0$, and we deduce that $z_j(Mz)_j \geq 0$, as desired.

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a). These are proved in much the same way as in Theorem **3.3.4**. □

3.4.3 Example. The condition that $z_k \neq 0$ in **3.4.2(b)** is essential. Consider the matrix

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which is clearly not in $\boldsymbol{P_0}$. Given any $z = (z_1, z_2) \neq 0$, we find $z_1(Mz)_1 = z_1^2 \geq 0$. Thus, M would satisfy **3.4.2(b)** if we dropped the condition that $z_k \neq 0$. It can be seen that the matrix M does not meet the full requirement of **3.4.2(b)** by letting $z = (0, 1)$.

The next result identifies a class of matrices M for which all solutions of the LCP (q, M) must be w-unique for all $q \in K(M)$.

3.4.4 Theorem. Let $M \in R^{n\times n}$. The following statements are equivalent:

(a) For all $q \in K(M)$, if z^1 and z^2 are any two solutions of (q, M), then $Mz^1 = Mz^2$.

(b) Every vector whose sign is reversed by M belongs to the nullspace of M, i.e.,

$$[\, z_i(Mz)_i \le 0 \text{ for all } i = 1, \ldots, n\,] \quad \Rightarrow \quad [\, Mz = 0\,]. \qquad (2)$$

(c) M is a $\boldsymbol{P_0}$-matrix and for each index set α with $\det M_{\alpha\alpha} = 0$, the columns of $M_{\bullet\alpha}$ are linearly dependent.

Proof. (a) $\Rightarrow$ (b). Suppose there exists a nonzero vector z such that $z_i(Mz)_i \le 0$ for all i and $Mz \ne 0$. Defining z^+, z^-, u^+, u^- and $\bar{q}$ as in the proof of **3.3.7**, we deduce that z^+ and z^- are solutions of $(\bar{q}, M)$ with $\bar{q} + Mz^+ = u^+ \ne u^- = \bar{q} + Mz^-$, which contradicts (a).

(b) $\Rightarrow$ (c). If the implication (2) holds, it follows from **3.4.2(b)** that M must be a $\boldsymbol{P_0}$-matrix. Suppose that there is an index set α for which $\det M_{\alpha\alpha} = 0$. Then there is a vector $z_\alpha \ne 0$ such that $M_{\alpha\alpha}z_\alpha = 0$. Define $z = (z_\alpha, 0)$. The nonzero vector z satisfies $[\, z_i(Mz)_i = 0$ for all $i\,]$. By (2), we must have $0 = Mz = M_{\bullet\alpha}z_\alpha$. Hence the columns of $M_{\bullet\alpha}$ are linearly dependent, and (c) follows.

(c) $\Rightarrow$ (b). Let $\hat{z}$ be a vector whose sign is reversed by M. Without loss of generality, we may assume that $\hat{z}$ is nonnegative. (Otherwise, we may apply the argument below to the matrix $\bar{M} = DMD$ where D is the diagonal matrix with diagonal entries $d_{ii} = 1$ if $\hat{z}_i \ge 0$ and $d_{ii} = -1$ if $\hat{z}_i < 0$. The matrix $\bar{M}$ is easily seen to satisfy the assumptions of (c), given that M does. Moreover, $\bar{z}_i(\bar{M}\bar{z})_i \le 0$ for all $i = 1, \ldots, n$ where $\bar{z} = D\hat{z} \ge 0$.) Let $\hat{w} = M\hat{z}$. Suppose that $\hat{w} \ne 0$. The system

$$\hat{w} = Mz, \;\; z \ge 0$$

has a solution, $\hat{z}$. Let α be the support of $\hat{z}$, thus $\alpha \ne \emptyset$. Notice, $\hat{w}_\alpha \le 0$ as $\hat{z}_\alpha > 0$. By linear programming theory, there is a (basic) feasible solution $\tilde{z} \ge 0$ with support $\beta \subseteq \alpha$ such that $M_{\bullet\beta}$ has linearly independent columns. Note, $\beta \ne \emptyset$ as $0 \ne \hat{w} = M\tilde{z}$. Obviously, $M_{\beta\beta}$ is a $\boldsymbol{P_0}$-matrix. As a matter of fact, $M_{\beta\beta}$ is a $\boldsymbol{P}$-matrix for if $\det M_{\gamma\gamma} = 0$ for some $\gamma \subseteq \beta$, then the columns of $M_{\bullet\gamma}$ are linearly dependent, contradicting the linear independence of $M_{\bullet\beta}$. As $\beta \subseteq \alpha$, we find that $M_{\beta\beta}$ reverses the sign of the positive vector $\tilde{z}_\beta$. This is a contradiction and, therefore, (b) follows.

(b) $\Rightarrow$ (a). The proof of this assertion is similar to that of the uniqueness part of Theorem **3.3.7**. □

3.4.5 Remark. It is clear that if M is a nonsingular matrix satisfying the condition (c) of **3.4.4**, then M must indeed be a $\boldsymbol{P}$-matrix.

3.4.6 Definition. A matrix $M \in \boldsymbol{P_0} \cap R^{n \times n}$ is said to be

(a) *column adequate* if for each $\alpha \subseteq \{1, \ldots, n\}$

$$[\det M_{\alpha\alpha} = 0] \quad \Rightarrow \quad [M_{\bullet\alpha} \text{ has linearly dependent columns}].$$

(b) *row adequate* if M^{T} is column adequate.

(c) *adequate* if M is both column and row adequate.

3.4.7 Remark. According to Theorem **3.4.4**, the column adequacy of M characterizes the uniqueness of the w-part of any solution of the LCP (q, M) for all vectors q.

Obviously, any $\boldsymbol{P}$-matrix is adequate, as is any symmetric positive semi-definite matrix. Not every (asymmetric) positive semi-definite matrix is adequate, however. Moreover, there are adequate matrices which are neither in the class $\boldsymbol{P}$ nor positive semi-definite. In the context of the LCP, an adequate matrix differs from a $\boldsymbol{P}$-matrix in two major ways. One is that adequate matrices need not belong to $\boldsymbol{S}$, hence if M is adequate, an arbitrary LCP (q, M) is not guaranteed to have even a feasible solution. (A result on the solvability of such linear complementarity problems will be derived in the next section.) The other difference is that even if (q, M) is solvable, there is no guarantee that the z-solution is unique, although the w-solution must be.

3.4.8 Examples. The matrix

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

is positive semi-definite, but is neither row nor column adequate. The matrix

$$M = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

is adequate but is neither a $\boldsymbol{P}$-matrix nor positive semi-definite.

3.5 Sufficient Matrices

In Sections 3.1 and 3.3, we relied on the quadratic program (3.1.1) to derive the existence of a solution to the LCP (q, M). The Karush-Kuhn-Tucker conditions (3.1.2) – (3.1.5) played a key role in these analytic arguments. In essence, the various assumptions on the matrix M (e.g., positive definiteness, the $\boldsymbol{P}$-property) were used to establish the existence of at least one Karush-Kuhn-Tucker pair (z, u) and that any such vector z must solve the LCP (q, M). In this section, we characterize the class of matrices M for which this type of analytic approach can be successfully applied. Interestingly enough, the transpose of a matrix belonging to this class is intimately related to the convexity of the solution set of the LCP.

3.5.1 Definition. A matrix $M \in R^{n \times n}$ is said to be *column sufficient* if it satisfies the implication:

$$[\, z_i(Mz)_i \leq 0 \text{ for all } i\,] \quad \Rightarrow \quad [\, z_i(Mz)_i = 0 \text{ for all } i\,] \tag{1}$$

The matrix M is called *row sufficient* if its transpose is column sufficient. If M is both column and row sufficient, then it is called *sufficient.*

3.5.2 Example. The matrix

$$M = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

is column sufficient, but not row sufficient.

It is clear that any column (row) adequate matrix must be column (row) sufficient; moreover, any positive semi-definite matrix is obviously sufficient. The next proposition collects together some simple, but useful, facts concerning sufficient matrices in general.

3.5.3 Proposition. Let $M \in R^{n\times n}$ be given. If M is row sufficient then:

(a) M is a $\boldsymbol{P_0}$-matrix.

(b) All principal submatrices of M are row sufficient.

(c) If $i, j \in \{1, \dots, n\}$ and $i \neq j$, then

$$[\, m_{ii} = 0,\ m_{ji} \geq 0\,] \quad \Rightarrow \quad [\, m_{ij} \leq 0\,].$$

Proof. (a) Using the characterization of a $\boldsymbol{P_0}$-matrix given by Theorem **3.4.2(b)**, it is easy to see that a row sufficient matrix must be $\boldsymbol{P_0}$.

(b) Take an index set $\alpha \subseteq \{1, \dots, n\}$, and suppose $M_{\alpha\alpha}$ is not row sufficient. There then exists a z_α such that $z_i(M_{\alpha\alpha}^{\mathrm{T}} z_\alpha)_i \leq 0$ for $i \in \alpha$, with strict inequality holding for some i. Letting $z_{\bar\alpha} = 0$, we see that $z_i(M^{\mathrm{T}}z)_i \leq 0$ for $i \in \{1, \dots, n\}$, with strict inequality holding for the same i as before. This contradicts the row sufficiency of M, thus $M_{\alpha\alpha}$ is row sufficient.

(c) Since, from part (a) above, M is a $\boldsymbol{P_0}$-matrix, then $m_{jj} \geq 0$. If in addition $m_{ii} = 0$, $m_{ji} \geq 0$, and $m_{ij} > 0$, then let the vector z be such that $z_i > m_{jj}/m_{ij}$ and $z_j = -1$, with all other components of z equal to zero. We find $z_k(M^{\mathrm{T}}z)_k$ is nonpositive for $k = i$, negative for $k = j$, and zero for all other k. This contradicts the row sufficiency of M, thus statement (c) follows. □

In the result below, we establish how a row sufficient matrix characterizes the relationship between a Karush-Kuhn-Tucker point of the program (3.1.1) and a solution to the LCP (q, M).

3.5.4 Theorem. Given $M \in R^{n\times n}$, the following two statements are equivalent:

(a) M is row sufficient.

(b) For each vector $q \in R^n$, if (z, u) is a Karush-Kuhn-Tucker pair of the quadratic program (3.1.1), then z solves the LCP (q, M).

Proof. (a) ⇒ (b). Let q be given, and let (z, u) be a Karush-Kuhn-Tucker pair of (3.1.1). By Lemma **3.1.1**, we deduce (cf. (3.1.6))

$$(z-u)_i\big(M^{\mathrm{T}}(z-u)\big)_i \leq 0 \quad \text{for all } i = 1, \dots, n. \tag{2}$$

Since M is row sufficient, equality holds in (2) for all i. Thus, it follows from the derivation of (2) that (cf. (3.1.7))

$$z_i\big(M^{\mathrm{T}}(z-u)\big)_i = 0 \quad \text{for all } i = 1, \ldots, n.$$

By the Karush-Kuhn-Tucker condition (3.1.3), we obtain

$$0 = z^{\mathrm{T}}\big(q + (M + M^{\mathrm{T}})z - M^{\mathrm{T}}u\big) = z^{\mathrm{T}}(q + Mz).$$

Thus z solves (q, M).

(b) $\Rightarrow$ (a). Suppose that M is not row sufficient. There then exists a vector x such that $x_i(M^{\mathrm{T}}x)_i \leq 0$ for all i with strict inequality holding for at least one i, say j. Without loss of generality, we may assume that $x_j > 0$. Proceeding similar to the proof of Theorem **3.3.7**, define $z = x^+$, $u = x^-$, and $q = -Mz + (M^{\mathrm{T}}x)^-$. It is then easy to show that (z, u) is a Karush-Kuhn-Tucker pair. On the other hand, we have $z_j > 0$ and $(q + Mz)_j > 0$, in contradiction to (b). This completes the proof. □

Notice that the preceding theorem does *not* assert that if M is row sufficient, then the LCP (q, M) is solvable for all q. This is because there is no guarantee that a Karush-Kuhn-Tucker point will exist. If such a point exists, then (q, M) will indeed be solvable, provided that M is row sufficient. In turn, if (q, M) is feasible, then the quadratic program (3.1.1) has an optimal solution which must be a Karush-Kuhn-Tucker point. Thus, we have proved

3.5.5 Corollary. Every row sufficient matrix is a $\boldsymbol{Q_0}$-matrix. □

An adequate matrix is both row and column adequate, so by combining Corollary **3.5.5** above and Theorem **3.4.4** we obtain

3.5.6 Corollary. Let $M \in R^{n \times n}$ be adequate and let $q \in R^n$ be arbitrary. If (q, M) is feasible, then there exist a unique vector w and a vector z satisfying

$$w = q + Mz \geq 0, \quad z \geq 0, \quad w^{\mathrm{T}}z = 0. \quad \square$$

3.5.7 Example. One should remember that these matrix classes are not contained in $\boldsymbol{Q}$. For example, the matrix

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

is symmetric and positive semi-definite. Therefore, it is also adequate and sufficient. However, it is not a $\boldsymbol{Q}$-matrix as (q, M) has no solution for $q = (-1, -1)$.

In order to motivate the following discussion, we recall that if M is a positive semi-definite matrix, then LCP (q, M) has a convex (possibly empty) solution set for all vectors q (see **3.1.7**). Moreover, for an arbitrary M, the convexity of the solution set of (q, M) is characterized by a certain condition (3.1.11) holding (see **3.1.8**). Using the latter characterization, we establish another result.

3.5.8 Theorem. Given $M \in R^{n\times n}$, the following two statements are equivalent:

(a) M is column sufficient.

(b) For each vector $q \in R^n$, the LCP (q, M) has a (possibly empty) convex solution set.

Proof. (a) $\Rightarrow$ (b). Let q be given. If (q, M) has less than two solutions, there is nothing to prove. Therefore, suppose that z^1 and z^2 are two solutions of (q, M). It suffices to show that condition (3.1.11) holds, i.e., that

$$(z^1)^{\mathrm{T}} w^2 = (z^2)^{\mathrm{T}} w^1 = 0 \tag{3}$$

where $w^k = q + Mz^k$ for $k = 1, 2$. For each $i = 1, \ldots, n$, we have

$$0 \geq (z^1 - z^2)_i (w^1 - w^2)_i = (z^1 - z^2)_i \big(M(z^1 - z^2)\big)_i . \tag{4}$$

Thus, by the column sufficiency of M, it follows that equality must hold throughout (4). The desired condition (3) now follows.

(b) $\Rightarrow$ (a). Suppose that M is not column sufficient. There then exists a vector x for which $x_i(Mx)_i \leq 0$ for all i with strict inequality holding for at least one index i, say j. Proceeding as in the proof of Theorem

3.3.7, let $z^1 = x^+$ and $z^2 = x^-$ be the positive and negative parts of x, respectively. Let u^+ and u^- be the positive and negative parts of Mx, respectively. Define the vector $q = u^+ - Mx^+ = u^- - Mx^-$. It is then easy to show that both z^1 and z^2 are solutions to (q, M). Nevertheless, we have either $z_j^1 w_j^2 > 0$ or $z_j^2 w_j^1 > 0$ depending on whether $x_j > 0$ or $x_j < 0$. This contradicts the convexity of the solution set of (q, M). □

The defining property (1) does not permit one to check the column sufficiency of an arbitrary matrix $M \in R^{n\times n}$ in finite time. In the next result, we establish a necessary and sufficient condition for column sufficiency which is stated in terms of a finite number of linear inequality systems. This characterization makes use of the sign-changing operation discussed in Section 2.3.

3.5.9 Proposition. Let $M \in R^{n\times n}$ be given. The following two statements are equivalent:

(a) M is column sufficient.

(b) For each pair of disjoint index sets α, $\beta \subseteq \{1, \ldots, n\}$, whose union is nonempty, the system

$$0 \neq \begin{bmatrix} M_{\alpha\alpha} & -M_{\alpha\beta} \\ -M_{\beta\alpha} & M_{\beta\beta} \end{bmatrix} \begin{bmatrix} x_\alpha \\ x_\beta \end{bmatrix} \leq 0$$
$$\begin{bmatrix} x_\alpha \\ x_\beta \end{bmatrix} > 0 \tag{5}$$

has no solution.

Proof. (a) $\Rightarrow$ (b). Suppose system (5) is consistent for some pair of index sets α and β as stated. Let

$$z_\alpha = -x_\alpha\,, \quad z_\beta = x_\beta\,, \quad \text{and} \quad z_\gamma = 0\,,$$

where $\gamma = \{1, \ldots, n\} \setminus (\alpha \cup \beta)$. The vector z violates the defining property (1) of column sufficiency.

(b) $\Rightarrow$ (a). Conversely, suppose z is a vector such that

$$z_i(Mz)_i \leq 0 \qquad \text{for all } i = 1, \ldots, n$$

with strict inequality holding for at least one i. Define

$$\alpha = \{i : z_i < 0\}, \quad \beta = \{i : z_i > 0\}, \quad \text{and} \quad \gamma = \{i : z_i = 0\}.$$

It follows that $(x_\alpha, x_\beta) = (-z_\alpha, z_\beta)$ satisfies system (5). □

Obviously, by applying condition (b) to M^T, we obtain a finite characterization of a row sufficient matrix. The resulting characterization is a generalization of the sign pattern established in part (c) of Proposition **3.5.3** which is a necessary condition for row sufficiency.

3.6 Nondegenerate Matrices and Local Uniqueness

There are many generalizations of the class $\boldsymbol{P}$. Three of these ($\boldsymbol{P_0}$, adequate and sufficient matrices) were discussed in Sections 3.4 and 3.5. In this section, we introduce another generalization and analyze how it is related to the LCP.

3.6.1 Definition. A matrix $M \in R^{n \times n}$ is called *nondegenerate* if all its principal minors are nonzero.

It is important to realize that the nondegeneracy of matrices is unrelated to the concept of nondegeneracy of basic solutions of equations, nor is nondegeneracy of matrices related to nondegeneracy of vectors as defined in **1.4.3**.

Like $\boldsymbol{P}$-matrices, the class of nondegenerate matrices characterizes a certain uniqueness property of solutions of the LCP. In what follows, we shall develop this and a related characterization.

3.6.2 Definition. A solution z^* of the LCP (q, M) is said to be *locally unique* (or, *isolated*) if there exists a neighborhood of z^* within which z^* is the only solution of (q, M).

3.6.3 Theorem. Let $M \in R^{n \times n}$. The following statements are equivalent:

(a) M is nondegenerate.

(b) For all vectors q, the LCP (q, M) has a finite number (possibly zero) of solutions.

(c) For all vectors q, any solution of the LCP (q, M), if it exists, must be locally unique.

Proof. (a) $\Rightarrow$ (b). Suppose that M is nondegenerate but that for a certain vector $q \in K(M)$, the LCP (q, M) has an infinite number of solutions. Since $K(M)$ is the union of finitely many complementary cones, there must exist one such cone in which q has infinitely many representations. If B is the associated complementary matrix, then the system $Bv = q$ has infinitely many solutions, and so B must be singular. Thus, if α is the (necessarily nonempty) set of indices k such that $-M_{\bullet k}$ appears in B, then the submatrix $M_{\alpha\alpha}$ must be singular. This contradicts the nondegeneracy of M.

(b) $\Rightarrow$ (c). This is obvious.

(c) $\Rightarrow$ (a). Suppose that for some nonempty index set α, the principal submatrix $M_{\alpha\alpha}$ is singular. Let u_α be a nonzero vector with $M_{\alpha\alpha}u_\alpha = 0$. Define $q_\alpha = -M_{\alpha\alpha}e_\alpha$. Let $q_{\bar{\alpha}}$ be such that

$$q_{\bar{\alpha}} + M_{\bar{\alpha}\alpha}(e_\alpha + \theta u_\alpha) \geq 0$$

for all $\theta \geq 0$ sufficiently small. For the vector $q = (q_\alpha, q_{\bar{\alpha}})$ so defined, each vector z defined by

$$z_\alpha = e_\alpha + \theta u_\alpha \quad \text{and} \quad z_{\bar{\alpha}} = 0$$

is a solution of the LCP (q, M), provided that $\theta \geq 0$ is small enough to make $z_\alpha \geq 0$. Since this contradicts (c), the proof is complete. □

3.6.4 Remark. It is generally not true that if a set S has the property that all its elements are locally unique (i.e., isolated), then S must be a finite set. (An example would be $S = \{1/k : k = 1, 2, \ldots, \}$.) Thus the implication (c) $\Rightarrow$ (b) in Theorem **3.6.3** above is a nontrivial assertion.

Theorem **3.6.3** characterizes the class of matrices M for which the LCP (q, M) has locally unique solutions for all vectors q. In the next result, we give a set of necessary and sufficient conditions under which a *given solution* to (q, M) for a *specific* q is locally unique.

3.6.5 Theorem. Let $z^* \in \text{SOL}(q, M)$. Define the index sets

$$\alpha = \{i : z_i^* > 0\} = \operatorname{supp} z^*$$

$$\beta = \{i : z_i^* = 0 \quad \text{and} \quad (q + Mz^*)_i = 0\}.$$

Then, z^* is a locally unique solution if and only if $(z_\alpha, z_\beta) = (0, 0)$ is the only solution of the system:

$$\begin{aligned} M_{\alpha\alpha} z_\alpha + M_{\alpha\beta} z_\beta &= 0 \\ w_\beta = M_{\beta\alpha} z_\alpha + M_{\beta\beta} z_\beta &\geq 0 \\ z_\beta &\geq 0 \\ (w_\beta)^{\mathrm{T}} z_\beta &= 0. \end{aligned} \tag{1}$$

In particular, if z^* is a nondegenerate solution, and if $M_{\alpha\alpha}$ is nonsingular, then z^* is locally unique.

Proof. Suppose that z^* is a locally unique solution of (q, M) and that the system (1) possesses a nonzero solution $(\tilde{z}_\alpha, \tilde{z}_\beta)$. Let γ be the complement of $\alpha \cup \beta$ and let $\tilde{z}_\gamma = 0$. Notice that $\gamma = \{i : (q + Mz^*)_i > 0\}$. It is then easy to verify that the vector $z^* + \theta\tilde{z}$ is a solution of (q, M) for all $\theta \geq 0$ sufficiently small. This contradicts the local uniqueness of z^*.

Conversely, suppose that z^* is not locally unique. There must be a sequence of vectors $\{z^k\}$ converging to z^* such that each z^k is a solution of (q, M). Let $w^* = q + Mz^*$, $w^k = q + Mz^k$, and $u^k = z^k - z^*$. Thus, $w^k - w^* = Mu^k$. Since $z^k \to z^*$, it follows that for all k sufficiently large $z^k_\alpha > 0$, $w^k_\gamma > 0$, and, by complementarity, $w^k_\alpha = 0$, and $z^k_\gamma = u^k_\gamma = 0$.

The normalized sequence $\{u^k \,/\, \|u^k\|\}$ is bounded, and thus has an accumulation point, say $u^* \neq 0$. Notice, as $u^k_\gamma = 0$ for all large k, we have $(u^*_\alpha, u^*_\beta) \neq 0$. Without loss of generality, we may assume that the entire sequence $\{u^k \,/\, \|u^k\|\}$ converges to u^*. For all large k, we have

$$0 = w^k_\alpha - w^*_\alpha = (Mu^k)_\alpha = M_{\alpha\alpha} u^k_\alpha + M_{\alpha\beta} u^k_\beta \,.$$

Thus, dividing through by $\|u^k\|$ and letting $k \to \infty$, we obtain

$$0 = M_{\alpha\alpha} u^*_\alpha + M_{\alpha\beta} u^*_\beta \,.$$

Since $w^*_\beta = z^*_\beta = 0$, it follows that

$$u^*_\beta = \lim_{k\to\infty} u^k_\beta \,/\, \|u^k\| = \lim_{k\to\infty} z^k_\beta \,/\, \|u^k\| \geq 0$$

and, for all large k,

$$0 \leq w^k_\beta = (Mu^k)_\beta = M_{\beta\alpha} u^k_\alpha + M_{\beta\beta} u^k_\beta \,.$$

Again, dividing by $\|u^k\|$ and letting $k \to \infty$, we obtain

$$0 \le M_{\beta\alpha} u^*_\alpha + M_{\beta\beta} u^*_\beta .$$

Finally, since $0 = (z^k_\beta)^{\mathrm{T}} w^k_\beta = (u^k_\beta)^{\mathrm{T}} (M u^k)_\beta$, we deduce

$$(u^*_\beta)^{\mathrm{T}} (M_{\beta\alpha} u^*_\alpha + M_{\beta\beta} u^*_\beta) = 0.$$

Thus, the nonzero vector (u^*_α, u^*_β) satisfies the system (1). This establishes the characterization of local uniqueness.

Finally, if z^* is a nondegenerate solution of (q, M), then $\beta = \emptyset$. If, in addition, $M_{\alpha\alpha}$ is nonsingular, then (1) holds only if $z_\alpha = 0$, and z^* is locally unique. This completes the proof of the theorem. □

The system (1) is an instance of a mixed LCP. If $M_{\alpha\alpha}$ is nonsingular, we can solve for $z_\alpha = -M^{-1}_{\alpha\alpha} M_{\alpha\beta} z_\beta$. Substituting the latter expression into the rest of (1), we obtain the LCP $(0, N)$ where

$$N = M_{\beta\beta} - M_{\beta\alpha} M^{-1}_{\alpha\alpha} M_{\alpha\beta}.$$

Thus, provided that $M_{\alpha\alpha}$ is nonsingular, the solution z^* of LCP (q, M) is locally unique if and only if the LCP $(0, N)$ has the zero vector as its only solution. A noteworthy point about the LCP $(0, N)$ is that it is a homogeneous problem and its size is equal to the cardinality of the index set β which consists of the indices corresponding to the degenerate variables associated with the solution z^*.

3.7 An Augmented LCP

A key theme of this chapter is to find conditions under which the LCP (q, M) is guaranteed to have a solution. In Sections 3.1 and 3.3 we developed such conditions by relying on the quadratic programming formulation (3.1.1) of the LCP (q, M).

In this section we will define an *augmented* LCP $(\tilde{q}, \tilde{M})$ which is obtained from (q, M) by adding one extra variable and one extra constraint. This augmented LCP $(\tilde{q}, \tilde{M})$ will always have a solution. In turn, the existence of a solution to the original LCP (q, M) can be inferred from the LCP $(\tilde{q}, \tilde{M})$ provided that certain conditions are satisfied.

Given an LCP (q, M), a scalar $\lambda \geq 0$, and a vector $d > 0$, consider the augmented LCP $(\tilde{q}, \tilde{M})$ where

$$\tilde{q} = \begin{bmatrix} q \\ \lambda \end{bmatrix} \quad \text{and} \quad \tilde{M} = \begin{bmatrix} M & d \\ -d^{\mathrm{T}} & 0 \end{bmatrix}. \tag{1}$$

We may ask whether this augmented LCP has a solution. That is, we may ask whether or not there exist a vector z and a scalar θ which satisfy the system

$$\begin{aligned} w = q + Mz + \theta d &\geq 0, \quad z \geq 0, \quad w^{\mathrm{T}}z = 0, \\ \sigma = \lambda - d^{\mathrm{T}}z &\geq 0, \quad \theta \geq 0, \quad \sigma \cdot \theta = 0. \end{aligned} \tag{2}$$

If (z^*, θ^*) is a solution to $(\tilde{q}, \tilde{M})$ with $\theta^* = 0$, then z^* solves (q, M). Thus, in order to solve (q, M), it suffices to seek a solution (z^*, θ^*) to $(\tilde{q}, \tilde{M})$ with $\theta^* = 0$. Notice that $\theta^* = 0$ will hold if $\lambda > d^{\mathrm{T}}z^*$.

It turns out that the augmented LCP $(\tilde{q}, \tilde{M})$ will always have a solution. There are several ways of proving this. If we assume that M is symmetric, we could borrow from the theory of quadratic programming as we did in Sections 3.1 and 3.3. Indeed in this case the augmented LCP $(\tilde{q}, \tilde{M})$ becomes the Karush-Kuhn-Tucker conditions of the quadratic program

$$\begin{array}{ll} \text{minimize} & z^{\mathrm{T}}q + \frac{1}{2}z^{\mathrm{T}}Mz \\ \text{subject to} & z \geq 0 \\ & d^{\mathrm{T}}z \leq \lambda \end{array} \tag{3}$$

as in (1.2.1) and (1.2.2). Since the feasible region of the QP (3) is compact, there is an optimal solution z^* to the QP (3) which together with a Lagrange multiplier θ^* must satisfy the conditions in (2).

Another proof is needed in the more general case where M is not assumed to be symmetric. One approach is to use pivoting arguments which we will develop in Section 4.4. A second approach is to use geometric methods which we will develop in Chapter 6. In fact, we will give both of these proofs in the appropriate chapters. For now, we give a proof based on the following fundamental existence result for a variational inequality problem defined over a compact convex region.

3.7.1 Theorem. Let K be a nonempty compact convex subset of R^n. Let the function $f : R^n \to R^n$ be continuous. There will then exist a vector $z^* \in K$ satisfying

$$(y - z^*)^{\mathrm{T}} f(z^*) \geq 0 \quad \text{for all} \quad y \in K. \tag{4}$$

Proof. Let $\Pi_K(\cdot)$ denote the projection operator onto the set K under the l_2-norm; i.e., for every vector $x \in R^n$, $\Pi_K(x)$ is the unique vector in the set K that is closest to x under the stated norm. According to Exercise **2.10.21**, $\Pi_K(\cdot)$ is a continuous mapping. Moreover, by Theorem **2.7.1**, we have for every $x \in R^n$,

$$(z - \Pi_K(x))^{\mathrm{T}}(\Pi_K(x) - x) \geq 0 \quad \text{for all } z \in K.$$

From this, it is easy to deduce (see Exercise **3.12.6**) that a vector z^* solves the problem VI(K, f) if and only if

$$z^* = \Pi_K(z^* - f(z^*)),$$

i.e., if z^* is a fixed point of the mapping $F : K \to K$ defined by

$$F(x) = \Pi_K(x - f(x)).$$

Since the mapping F is obviously continuous and K is compact and convex, the existence of a fixed point of F, and thus, a solution of VI(K, f), follows from Brouwer's fixed-point theorem, **2.1.24**. □

3.7.2 Remark. When K is the nonnegative orthant R^n_+, the mapping F in the above proof reduces to

$$F(x) = \max(0, x - f(x)) = x - \min(x, f(x))$$

which is precisely the mapping given in (1.4.4) in the case where $f(x) = q + Mx$.

Using Theorem **3.7.1**, we may now show that the augmented LCP $(\tilde{q}, \tilde{M})$ always has a solution.

3.7.3 Theorem. Let the LCP (q, M) be given. For any scalar $\lambda \geq 0$ and vector $d > 0$, there exist a vector z and a scalar θ satisfying the system (2).

Proof. Let

$$K = \{ z \in R^n_+ : d^{\mathrm{T}} z \leq \lambda \} \quad \text{and} \quad f(z) = q + Mz \, .$$

We observe that a vector $z^* \in K$ satisfies (4) if and only if there exists a θ^* such that (z^*, θ^*) satisfies (2). (Cf. Section 1.2.) In fact, given $z^* \in K$ satisfying (4), it is easy to show that (z^*, θ^*) satisfies (2) where $\theta^* = 0$ if $d^{\mathrm{T}} z^* < \lambda$ or if $f(z^*) \geq 0$, and otherwise

$$\theta^* \; = \; \max_i \{ -f(z^*)_i / d_i \} \, .$$

The theorem now follows from Theorem **3.7.1**. □

3.7.4 Remark. We do not need to use the vector d twice in defining the augmented LCP. Instead of $\sigma = \lambda - d^{\mathrm{T}} z \geq 0$, the last constraint of (2) could have been $\sigma = \lambda - \bar{d}^{\mathrm{T}} z \geq 0$ with any vector $\bar{d} > 0$. It will still be true that the augmented LCP will always have a solution. This slightly more general result follows from Theorem **3.7.3** by noticing that if an LCP $(\tilde{q}, \tilde{M})$ has a solution, then it will continue to have a solution even after the columns of $\tilde{M}$ have been positively rescaled.

From Theorem **3.7.3** we may deduce the following existence result for the LCP (q, M).

3.7.5 Corollary. Let $q \in R^n$ and $M \in R^{n \times n}$ be given. If there exists a scalar $\kappa > 0$ such that

$$[\, x \geq 0, \; e^{\mathrm{T}} x = \kappa \,] \quad \Rightarrow \quad [\, x^{\mathrm{T}}(q + Mx) \geq 0 \,] \, , \tag{5}$$

then the LCP (q, M) has a solution.

Proof. Let (z^*, θ^*) be a solution to the augmented LCP $(\tilde{q}, \tilde{M})$ with $\lambda = \kappa$ and $d = e$ (the vector of all ones). If $\theta^* = 0$, then z^* solves (q, M). On the other hand, if $\theta^* > 0$, then $d^{\mathrm{T}} z^* = \kappa$. Hence,

$$0 = (w^*)^{\mathrm{T}} z^* = (z^*)^{\mathrm{T}}(q + Mz^*) + \theta^* \kappa > 0$$

by (5), which is a contradiction. Consequently, $\theta^* = 0$ and (q, M) has a solution. □

As a matter of fact, we may derive from Theorem **3.7.3** a set of necessary and sufficient conditions for the solvability of LCP (q, M).

3.7.6 Theorem. Let $q \in R^n$ and $M \in R^{n\times n}$ be given. Let d be an arbitrary positive n-vector. The following statements are equivalent:

(a) The LCP (q, M) is solvable.

(b) For every unbounded sequence $\{\lambda_k\}$ of positive scalars, the augmented LCP $(\tilde{q}^k, \tilde{M})$, with $\tilde{M}$ as given in (1) and

$$\tilde{q}^k = \begin{bmatrix} q \\ \lambda_k \end{bmatrix}, \tag{6}$$

has a solution (z^k, θ_k) where zero is an accumulation point of $\{\theta_k\}$.

(c) There exists an unbounded sequence $\{\lambda_k\}$ of positive scalars such that the augmented LCP $(\tilde{q}^k, \tilde{M})$, with $\tilde{M}$ as given in (1) and $\tilde{q}^k$ as given in (6), has a sequence of solutions (z^k, θ_k) for which zero is an accumulation point of $\{\theta_k\}$.

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c). These are obvious.

(c) $\Rightarrow$ (a). For a fixed vector $d > 0$ and a sequence $\{\lambda_k\}$ of positive scalars with $\lambda_k \to \infty$, let $\{(z^k, \theta_k)\}$ be a sequence such that (i) each (z^k, θ_k) is a solution of the augmented LCP $(\tilde{q}^k, \tilde{M})$, and (ii) zero is an accumulation point of $\{\theta_k\}$. Without loss of generality, we may assume that the sequence $\{\theta_k\}$ converges to zero. There must exist an index set $\alpha \subseteq \{1, \ldots, n\}$ and a subsequence $\{z^{k_i}\}$ such that for each k_i, $z_j^{k_i} > 0$ if and only if $j \in \alpha$. By complementarity, $w_j^{k_i} = 0$ for $j \in \alpha$. We have for each k_i,

$$\begin{aligned} q_\alpha + \theta_{k_i} d_\alpha &= -M_{\alpha\alpha} z_\alpha^{k_i}, \\ q_{\bar{\alpha}} + \theta_{k_i} d_{\bar{\alpha}} &= -M_{\bar{\alpha}\alpha} z_\alpha^{k_i} + w_{\bar{\alpha}}^{k_i}. \end{aligned}$$

Thus, $q + \theta_{k_i} d \in \operatorname{pos} C_M(\alpha)$. Since $\theta_k \to 0$, by Theorem **2.6.24**, there exists a vector $(w_{\bar{\alpha}}^*, z_\alpha^*) \geq 0$ such that

$$\begin{aligned} q_\alpha &= -M_{\alpha\alpha} z_\alpha^*, \\ q_{\bar{\alpha}} &= -M_{\bar{\alpha}\alpha} z_\alpha^* + w_{\bar{\alpha}}^*. \end{aligned}$$

The vector $z^* = (z_\alpha^*, 0)$ is easily seen to be a solution of (q, M). □

3.7.7 Remark. By Theorem **3.7.3**, each augmented LCP $(\tilde{q}^k, \tilde{M})$ has a solution. What is essential in assertions (b) and (c) above is the requirement that zero be an accumulation point of $\{\theta_k\}$ for a certain sequence of solutions $\{(z^k, \theta_k)\}$.

3.7.8 Examples. Let

$$q = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The LCP (q, M) does not have a solution. However, with $\tilde{q}$ and $\tilde{M}$ as in (1), the augmented LCP $(\tilde{q}, \tilde{M})$ has the unique solution $(0, \lambda, 1 + \lambda)$. For this example, Theorem **3.7.6** implies that zero is not an accumulation point of θ as $\lambda \to \infty$. This is certainly the case as $\theta = 1 + \lambda$ in this example. However, we must be careful in drawing conclusions from this. Theorem **3.7.6** guarantees that, if the LCP (q, M) has a solution, then *at least one* sequence of solutions (as $\lambda \to \infty$) will exist to the augmented LCP $(\tilde{q}, \tilde{M})$ with zero as an accumulation point of θ. The theorem does not state that this will occur for all solution sequences. If we slightly modify the above example by letting

$$q = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{7}$$

then the LCP (q, M) has the solution $(1, 0)$. Thus, the augmented LCP $(\tilde{q}, \tilde{M})$ has the solution $(1, 0, 0)$ for $\lambda \geq 1$. Yet, for $\lambda \geq 1/2$, the augmented LCP also has the solution $(0, \lambda, \lambda)$. For this sequence of solutions θ does not have zero as an accumulation point as $\lambda \to \infty$.

Theorem **3.7.3** states that a solution to the augmented LCP (2) always exists for each scalar $\lambda \geq 0$ and vector $d > 0$. Theorem **3.7.6** goes on to state that if zero is an accumulation point of $\{\theta_k\}$ as λ_k goes to infinity, then there is a solution to the LCP (q, M). The simplest case would be when $\theta_k = 0$ for all large values of λ_k. In this case, the z^k would represent solutions to the LCP (q, M) for all large values of λ_k. Notice, if the z^k represent the same solution and, hence, are equal for all large values of λ_k, then $\sigma_k \to \infty$ as $\lambda_k \to \infty$.

The question arises as to what happens in the opposite case where there is an infinite subsequence $\{\lambda_{k_i}\}$ in which the corresponding θ_{k_i} are positive and the corresponding σ_{k_i} equal zero. The following theorem provides us with some information concerning this situation. Notice, we can use (7) in Example **3.7.8** to show that it is possible to have $\liminf \theta_k = 0$ and $\limsup \theta_k = \infty$.

3.7.9 Theorem. Let the LCP (q, M) be given along with a vector $d > 0$ and an unbounded sequence of positive scalars $\{\lambda_k\}$. Let $\{(z^k, \theta_k)\}$ be a corresponding sequence of solutions to the augmented LCP (2) such that in each solution $\sigma_k = 0$. There will then exist a subsequence $\{\lambda_{k_i}\}$ along with (possibly different) corresponding solutions $\{(\bar{z}^{k_i}, \bar{\theta}_{k_i})\}$ to the augmented LCP (2), and two vectors u and v with $d^{\mathrm{T}}u = 0$, $v \geq 0$ and $d^{\mathrm{T}}v = 1$ such that for all k_i,

$$\bar{z}^{k_i} = u + \lambda_{k_i} v.$$

Proof. Theorem **1.3.4** implies that, for each $\lambda = \lambda_k$, the augmented LCP (2) has an extreme point (basic) solution $(\bar{z}^k, \bar{\theta}_k)$ in which $\sigma_k = 0$. Since any such solution corresponds to a certain basis of the matrix $(I, -\tilde{M})$, and since there are only finitely many such bases, there is a subsequence λ_{k_i} for which the corresponding basic solutions all use the same basis. The conclusion now follows. □

As we have pointed out, if M is symmetric, the augmented LCP $(\tilde{q}, \tilde{M})$ is related to the QP (3). In this case, we can prove the following specialization of Theorem **3.7.9**.

3.7.10 Corollary. If in Theorem **3.7.9** the matrix M is symmetric and each z^k is an optimal solution to the QP (3) with $\lambda = \lambda_k$, then we may take each $\bar{z}^{k_i}$ obtained by Theorem **3.7.9** to be an optimal solution to the QP (3) with $\lambda = \lambda_{k_i}$.

Proof. In Theorem **3.7.9** we used Theorem **1.3.4** to generate a sequence $\{(\bar{z}^k, \bar{\theta}_k)\}$ with certain desired properties. With the additional assumptions given in the corollary, we may use Theorem **1.3.5** to generate a sequence $\{(\bar{z}^k, \bar{\theta}_k)\}$ with the same desired properties and, in addition, with the property that each $\bar{z}^k$ is an optimal solution to the QP (3). The corollary now follows from proof of Theorem **3.7.9**. □

3.7.11 Example. Notice that Theorem **3.7.9** does not guarantee that the solutions in the original sequence $\{(z^k, \theta_k)\}$ will lie on the ray $u + \lambda v$. Let $\lambda_k = k$ and let

$$q = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For each λ_k, the augmented LCP (2) has the solution $z^k = (1/k, (k^2-1)/k)$ and $\theta_k = 1$. Notice that σ_k always equals zero. Yet, since the sequence $\{z_1^k\}$ is decreasing, there can be no ray $u + \lambda v$, with $v \geq 0$ as described in Theorem **3.7.9**, which contains more than one of these solutions. Theorem **3.7.9** only guarantees that for some sequence of solutions there will be a subsequence which lies on some ray. For example, if we consider the sequence of solutions $z^k = (0, k)$ and $\theta_k = 1$ to the augmented LCP (2), then each solution of the sequence lies along the ray λv where $v = (0, 1)$.

We will now apply Theorem **3.7.6** to get a sufficient condition for the existence of a solution to the LCP (q, M) which is somewhat related to condition (5) in **3.7.5**.

3.7.12 Corollary. Let $q \in R^n$ and $M \in R^{n\times n}$ be given. If the quadratic function $f(z) = z^{\mathrm{T}}(q + Mz)$ is bounded below for $z \geq 0$, then the LCP (q, M) has a solution.

Proof. Let $\{\lambda_k\}$ be an unbounded sequence of positive scalars, and let $\{(z^k, \theta_k)\}$ be a corresponding sequence of solutions to an augmented LCP (2). If $\inf\{\theta_k\} > 0$, then we must have $d^{\mathrm{T}}z^k = \lambda_k$ for all k; moreover,

$$f(z^k) = -\theta_k\lambda_k$$

which must diverge to $-\infty$ as $k \to \infty$. This contradicts the assumption that $f(z)$ is bounded below. Consequently, the existence of a solution to the LCP (q, M) follows from **3.7.6**. □

3.7.13 Remark. The assumptions in Corollaries **3.7.5** and **3.7.12** are different and neither one implies the other. Indeed, the pair

$$q = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

satisfies condition (5) with $\kappa = 1$, but the corresponding quadratic function $f(z) = z^{\mathrm{T}}(q + Mz)$ is unbounded below for $z \geq 0$; on the other hand, the pair

$$q = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

fails condition (5) for any positive κ, but the quadratic function $f(z)$ is bounded below on R^2.

We will now apply Theorem **3.7.9** to establish a necessary and sufficient condition for the quadratic function $f(z) = z^{\mathrm{T}}(q+Mz)$ to be bounded below on the nonnegative orthant.

3.7.14 Proposition. Let $q \in R^n$ and $M \in R^{n\times n}$ be given. The following statements are equivalent:

(a) The two implications hold

$$z \geq 0 \;\Rightarrow\; z^{\mathrm{T}}Mz \geq 0\,, \tag{8}$$

$$[\,z \geq 0,\; z^{\mathrm{T}}Mz = 0\,] \;\Rightarrow\; [\,q^{\mathrm{T}}z \geq 0\,]\,; \tag{9}$$

(b) For all $\sigma > 0$, the quadratic function $f_\sigma(z) = z^{\mathrm{T}}(q+\sigma Mz)$ is bounded below for $z \geq 0$;

(c) The quadratic function $f(z) = z^{\mathrm{T}}(q+Mz)$ is bounded below for $z \geq 0$.

Proof. (a) $\Rightarrow$ (b). Clearly, if the two implications (8) and (9) hold, then they hold if M is replaced by σM for any $\sigma > 0$. Thus, it suffices to prove (b) for $\sigma = \frac{1}{2}$. Suppose the contrary. With no loss of generality, we may assume that M is symmetric. Given $d > 0$ and λ a positive scalar we consider the QP (3) which has $f_{\frac{1}{2}}(z)$ for its objective function. As previously mentioned, this quadratic program will always have an optimal solution z^* and there will always exist a scalar θ^* such that (z^*, θ^*) satisfies the augmented LCP (2). Since the quadratic function $f_{\frac{1}{2}}(z)$ is unbounded below for $z \geq 0$, we deduce that there is an unbounded sequence of positive scalars $\{\lambda_k\}$ and a corresponding sequence $\{(z^k, \theta_k)\}$ of solutions to (2) such that z^k solves the QP (3) with $\lambda = \lambda_k$ and $d^{\mathrm{T}}z^k = \lambda_k$. Clearly, $\{f_{\frac{1}{2}}(z^k)\} \to -\infty$. By Corollary **3.7.10**, there exist a subsequence $\{\lambda_{k_i}\}$, a corresponding subsequence $\{(\bar{z}^{k_i}, \bar{\theta}_{k_i})\}$ of (possibly different) solutions to the augmented LCP (2) with each $\bar{z}^{k_i}$ being an optimal solution to the QP (3), and two vectors u and v, with $0 \neq v \geq 0$, such that $\bar{z}^{k_i} = u + \lambda_{k_i}v$ for all k_i. Since

$$f_{\frac{1}{2}}(\bar{z}^{k_i}) = f_{\frac{1}{2}}(u) + \lambda_{k_i}v^{\mathrm{T}}(q + Mu) + \tfrac{1}{2}\lambda_{k_i}^2 v^{\mathrm{T}}Mv,$$

and since $\{f_{\frac{1}{2}}(\bar{z}^{k_i})\} \to -\infty$, it follows from condition (8) that $v^{\mathrm{T}}Mv = 0$ and $v^{\mathrm{T}}(q + Mu) < 0$. Thus, by condition (9), we deduce $v^{\mathrm{T}}Mu < 0$. By condition (8) again, we have for all k_i

$$0 \leq (u + \lambda_{k_i}v)^{\mathrm{T}}M(u + \lambda_{k_i}v) = u^{\mathrm{T}}Mu + 2\lambda_{k_i}v^{\mathrm{T}}Mu$$

and the right-hand term tends to minus infinity as $k_i \to \infty$. This is a contradiction. Thus, statement (b) is established.

(b) $\Rightarrow$ (c). This is obvious.

(c) $\Rightarrow$ (a). It is clear that if z is any vector violating either condition (8) or (9), then $f(\lambda z) \to -\infty$ as $\lambda \to \infty$. This completes the proof. $\Box$.

3.7.15 Remark. If the quadratic function $f_\sigma(z)$ is unbounded below on R^n_+, then Proposition **3.7.14** implies that some $v \geq 0$ violates either (8) or (9). In either case, this v shows that $f_\sigma(z)$ is unbounded below on a ray emanating from the origin.

The augmented LCP $(\tilde{q}, \tilde{M})$ in (1) provides a useful tool for the study of a given LCP (q, M). In what follows, we introduce another augmented problem and discuss its application. To provide the motivation, we note that the examples in **3.7.8** illustrate the possibility that the LCP (q, M) has a solution and yet for some unbounded sequence $\{\lambda_k\}$ of nonnegative scalars the augmented LCP $(\tilde{q}^k, \tilde{M})$, with $\tilde{q}^k$ as given in (6), can have a sequence of solutions $\{(z^k, \theta_k)\}$ with $\inf_k \theta_k > 0$. It turns out that if M is column sufficient, this situation can never occur with the alternate augmented LCP introduced below. The benefit of this implication is that if one can solve the augmented problem (by any method), then by testing the solution obtained, one can successfully determine whether or not the LCP (q, M) is solvable (and trivially obtain a solution if it exists).

Given the LCP (q, M) of order n and an n-vector $a > 0$, consider the LCP (q', M') of order $2n$ where

$$q' = \begin{bmatrix} q \\ a \end{bmatrix} \quad \text{and} \quad M' = \begin{bmatrix} M & I \\ -I & 0 \end{bmatrix}. \tag{10}$$

(Similar to **3.7.4**, we may replace the two identity matrices in M' by two positive diagonal matrices; such replacement will not affect the validity of the results below.)

3.7.16 Theorem. Given $M \in R^{n\times n}$, $q \in R^n$, $a \in R^n$, and $a \geq 0$, the augmented LCP (q', M') as given in (10) has a solution.

Proof. It suffices to apply Theorem **3.7.1** with

$$K = \{z \in R^n_+ : z \leq a\} \quad \text{and} \quad f(z) = q + Mz\,.$$

Similar to the proof of Theorem **3.7.3**, one now notes that there exists a vector $z^* \in K$ satisfying (4) if and only if there exists a y^* such that (z^*, y^*) solves the augmented LCP (q', M'). In fact, given $z^* \in K$ satisfying (4), it is easy to show that (z^*, y^*) solves the augmented LCP (q', M') where $y^* = (Mz^* + q)^-$. □

Obviously, if $(z, y) \in R^{2n}$ is a solution of (q', M') with $y = 0$, then z solves (q, M); conversely, if z solves (q, M) and $a \geq z$, then $(z, 0)$ solves (q', M'). It turns out that if M is column sufficient and if (q, M) is solvable, then the only solutions (z, y) of (q', M') are those which have $y = 0$, provided that the components of the vector a are sufficiently large.

3.7.17 Theorem. Let $M \in R^{n \times n}$ be column sufficient and $q \in R^n$ be arbitrary. The LCP (q, M) is solvable if and only if for all scalars $\lambda > 0$ sufficiently large, *any* solution (z, y) of the LCP (q', M') given in (10), with $a = \lambda e$, must have $y = 0$.

Proof. It suffices to prove the "only if" part. For this purpose, let $\tilde{z}$ be a solution to the LCP (q, M) and write $\tilde{w} = q + M\tilde{z}$. Choose λ such that $\lambda e > \tilde{z}$. Let (z, y) be an arbitrary solution of (q', M') and write

$$w = q + Mz + y\,.$$

We have for all $i = 1, \ldots, n$,

$$0 \geq (z - \tilde{z})_i (w - \tilde{w})_i = (z - \tilde{z})_i \big(M(z - \tilde{z})\big)_i + (z - \tilde{z})_i y_i.$$

If $y_i > 0$, then $z_i = \lambda$ by complementarity; by the definition of λ, $z_i > \tilde{z}_i$. Consequently, it follows that for all i,

$$0 \geq (z - \tilde{z})_i \big(M(z - \tilde{z})\big)_i$$

which implies by the column sufficiency of M that

$$0 = (z - \tilde{z})_i \big(M(z - \tilde{z})\big)_i$$

for all i. Hence $y = 0$ as desired. □

Theorem **3.7.17** provides a necessary and sufficient condition for a column sufficient LCP to have a solution. We shall return to discuss the computational implication of this result in Section 5.9.

3.8 Copositive Matrices

In Section 3.1, we have presented results pertaining to the existence of a solution for the LCP (q, M) when M is either a positive definite or a positive semi-definite matrix. In the context of the LCP, such matrix definiteness properties seem a bit too strong, for after all, the LCP is defined relative to the nonnegative orthant. In this section, we shall establish several existence results for the LCP under the weaker notion of "definiteness over the nonnegative orthant." We first define this notion in a more formal way.

3.8.1 Definition. A matrix $M \in R^{n \times n}$ is said to be

(a) *copositive* if $x^{\mathrm{T}} M x \geq 0$ for all $x \in R^n_+$;

(b) *strictly copositive* if $x^{\mathrm{T}} M x > 0$ for all nonzero $x \in R^n_+$;

(c) *copositive-plus* if M is copositive and the following implication holds:

$$[\, x^{\mathrm{T}} M x = 0, \ x \geq 0 \,] \quad \Rightarrow \quad [\, (M + M^{\mathrm{T}}) x = 0 \,].$$

(d) *copositive-star* if M is copositive and the following implication holds:

$$[\, x^{\mathrm{T}} M x = 0, \ M x \geq 0, \ x \geq 0 \,] \quad \Rightarrow \quad [\, M^{\mathrm{T}} x \leq 0 \,].$$

The inclusion relationships among the above matrix classes is quite clear. The class of copositive matrices contains the copositive-star matrices which, in turn, contain the copositive-plus matrices which, in turn, contain the strictly copositive matrices. It is easy to see that a positive definite matrix must be strictly copositive. It is also straightforward to show (see Exercise **3.12.1**) that a positive semi-definite matrix must be copositive-plus. Obviously, a nonnegative matrix is copositive and a nonnegative matrix with positive diagonal entries is strictly copositive. The following examples serve to show that these various inclusion relationships do not hold with equality.

3.8.2 Examples. Consider the matrices

$$M_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad M_3 = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}.$$

M_1 is nonnegative with a positive diagonal, so it is strictly copositive. However, as $x^{\mathrm{T}}M_1x = -2 < 0$ for $x = (1, -1)$, we see that M_1 is not positive definite or positive semi-definite. M_2 is nonnegative, so it is copositive. However, M_2 is not copositive-star, for if $x = (1, 0)$, then $M_2x \geq 0$ and $x^{\mathrm{T}}M_2x = 0$, yet $M_2^{\mathrm{T}}x$ is not nonpositive. M_3 is copositive-star, but not copositive-plus. To see this, we let $x = (x_1, x_2)$ to find that $x^{\mathrm{T}}M_3x = x_2(x_1 + x_2)$. Clearly, if $x \geq 0$, then $x^{\mathrm{T}}M_3x \geq 0$, so M_3 is copositive. Also, if $x \geq 0$ and $x^{\mathrm{T}}M_3x = 0$ then $x_2 = 0$, which implies $M_3^{\mathrm{T}}x \leq 0$. Thus, M_3 is copositive-star. However, if $x = (1, 0)$, then $x^{\mathrm{T}}M_3x = 0$ and $(M_3 + M_3^{\mathrm{T}})x = (0, 1) \neq 0$, which means M_3 is not copositive-plus. Finally, the zero matrix is trivially seen to be copositive-plus but not strictly copositive.

Notice that all of the previous examples use symmetric matrices except for M_3. It can be shown (see Exercise **3.12.2**) that a symmetric matrix is copositive-star if and only if it is copositive-plus. In fact, assuming symmetry, we have the following characterization of copositivity.

3.8.3 Theorem. Let $M \in R^{n \times n}$ be symmetric. The following statements are equivalent:

(a) M is copositive.

(b) For every index set $\alpha \subseteq \{1, \ldots, n\}$, the system

$$M_{\alpha\alpha}x_\alpha \geq 0, \quad x_\alpha \geq 0 \tag{1}$$

has a nonzero solution.

Proof. (a) $\Rightarrow$ (b). Suppose (1) has only zero as a solution for some α. By Ville's theorem of the alternative there exists a $y \in R^n$ such that $y_\alpha > 0$ and $y_\alpha^{\mathrm{T}}M_{\alpha\alpha} < 0$. Clearly, setting $y_{\bar{\alpha}} = 0$ will give $y \geq 0$ and $y^{\mathrm{T}}My < 0$ which contradicts the copositivity of M. (Notice that this part of the proof does not use the symmetry of M.)

(b) $\Rightarrow$ (a). If $n = 1$, then (1) implies that M consists of a single nonnegative element. Hence, M is copositive and thus $[\,$(b) $\Rightarrow$ (a)$\,]$ for $n = 1$. Using induction, assume the implication holds for symmetric matrices of order less than n. Given that (b) holds for M, we see that (b) holds for all principal submatrices of M and, by induction, all proper principal submatrices of M are copositive. To complete the induction, we must show M

to be copositive. Let $\tilde{x}$ be a nonzero solution to (1) with $\alpha = \{1, \ldots, n\}$. Clearly, for any $x \geq 0$, there is some $\lambda \geq 0$ such that $x - \lambda\tilde{x}$ is nonnegative but not strictly positive. We have

$$\begin{aligned} x^{\mathrm{T}}Mx &= (x - \lambda\tilde{x})^{\mathrm{T}}M(x - \lambda\tilde{x}) + 2\lambda(x - \lambda\tilde{x})^{\mathrm{T}}M\tilde{x} + \lambda^2\tilde{x}^{\mathrm{T}}M\tilde{x} \\ &\geq (x - \lambda\tilde{x})^{\mathrm{T}}M(x - \lambda\tilde{x}) \\ &\geq 0 \end{aligned}$$

where the initial equality follows from the symmetry of M, the first inequality follows as $M\tilde{x} \geq 0$, and the second inequality follows as the proper principal submatrices of M are copositive. Therefore M is copositive. □

3.8.4 Example. In the previous theorem, the symmetry assumption is important in showing that [(b) ⇒ (a)]. Consider the matrix

$$M = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Since $x^{\mathrm{T}}Mx = -1 < 0$ for $x = (1, 1)$, it follows that M is not copositive. Yet, it is easily seen that M satisfies statement (b) in Theorem **3.8.3**.

The classes of copositive and strictly copositive matrices are closely related to the quadratic function $f(z) = z^{\mathrm{T}}(q + Mz)$. Indeed, according to Proposition **3.7.14**, the matrix $M \in R^{n \times n}$ is copositive if and only if, for all nonnegative q, this quadratic function $f(z)$ is bounded below on the nonnegative orthant; it is strictly copositive if and only if $f(z)$ is bounded below on the nonnegative orthant for all q.

The following result shows that a strictly copositive matrix belongs to the class $\boldsymbol{Q}$.

3.8.5 Theorem. If $M \in R^{n \times n}$ is strictly copositive, then for each $q \in R^n$ the LCP (q, M) has a solution.

Proof. This follows easily from Corollary **3.7.12** and the aforementioned remark that for all vectors q, the quadratic function $f(z) = z^{\mathrm{T}}(q + Mz)$ must be bounded below for $z \geq 0$ if M is strictly copositive. □

It is easy to see that the LCP (q, M) need not have a solution if M is only a copositive matrix. For example, if $M = 0$, then M is copositive but the LCP (q, M) has a solution if and only if $q \geq 0$. The next result provides a sufficient condition for (q, M) to be solvable.

3.8.6 Theorem. Let $M \in R^{n\times n}$ be copositive and let $q \in R^n$ be given. If the implication

$$[\, v \geq 0,\ Mv \geq 0,\ v^{\mathrm{T}}Mv = 0 \,] \quad \Rightarrow \quad [\, v^{\mathrm{T}}q \geq 0 \,] \tag{2}$$

is valid, then (q, M) has a solution.

Proof. It suffices to verify that condition (c) of Theorem **3.7.6** is satisfied. Let $d > 0$ be a given vector and let $\{\lambda_k\}$ be a sequence of positive scalars with $\lambda_k \to \infty$. For each k, let (z^k, θ_k) be a solution to the augmented LCP $(\tilde{q}^k, \tilde{M})$ where $\tilde{M}$ is given by (3.7.1) and $\tilde{q}^k$ by (3.7.6). Without loss of generality, we may assume that none of the θ_k is equal to zero. Thus, by complementarity, $d^{\mathrm{T}}z^k = \lambda_k$ for each k. The sequence $\{z^k\}$ is therefore unbounded. Let $y^k = z^k/\lambda_k$. The sequence $\{y^k\}$ is nonnegative and satisfies $d^{\mathrm{T}}y^k = 1$ for all k. Hence $\{y^k\}$ has an accumulation point, say $\hat{v}$. Obviously, $\hat{v} \geq 0$ and $d^{\mathrm{T}}\hat{v} = 1$. Without loss of generality, we may assume that the entire sequence $\{y^k\}$ converges to $\hat{v}$. Now, we have

$$0 = (z^k)^{\mathrm{T}}w^k = (z^k)^{\mathrm{T}}q + (z^k)^{\mathrm{T}}Mz^k + \theta_k\lambda_k \geq (z^k)^{\mathrm{T}}q + (z^k)^{\mathrm{T}}Mz^k. \tag{3}$$

Dividing by λ_k^2 and letting $k \to \infty$, we deduce $\hat{v}^{\mathrm{T}}M\hat{v} \leq 0$ which, by the copositivity assumption, implies $\hat{v}^{\mathrm{T}}M\hat{v} = 0$. Furthermore, we have

$$0 = (z^k)^{\mathrm{T}}q + (z^k)^{\mathrm{T}}Mz^k + \theta_k\lambda_k \geq (z^k)^{\mathrm{T}}q + \theta_k\lambda_k \tag{4}$$

by copositivity. Dividing by λ_k^2 and letting $k \to \infty$, we deduce $\theta_k/\lambda_k \to 0$. Since

$$q + Mz^k + \theta_k d \geq 0,$$

we can divide through by λ_k, then let $k \to \infty$ and conclude that $M\hat{v} \geq 0$. Consequently, by the implication (2), it follows that $\hat{v}^{\mathrm{T}}q \geq 0$. At the same time, from (3),

$$0 \geq (z^k)^{\mathrm{T}}q + (z^k)^{\mathrm{T}}Mz^k \geq (z^k)^{\mathrm{T}}q\,.$$

Therefore, $\hat{v}^{\mathrm{T}}q = 0$. From (4), we have

$$0 \geq (y^k)^{\mathrm{T}}q + \theta_k\,.$$

Letting $k \to \infty$, we deduce that $\theta_k \to 0$, verifying condition (c) of Theorem **3.7.6**. □

The converse of Theorem **3.8.6** is false. For example, the LCP (q, M) where

$$q = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

has a solution $z = (1, 1)$. Nevertheless, the implication (2) fails to hold with $v = (0, 1)$.

The implication (2) has an interesting geometric interpretation. Consider the solution set of the homogeneous LCP $(0, M)$. Clearly $\mathrm{SOL}(0, M)$ is a (possibly non-convex) cone. Thus, (2) holds if and only if the vector q belongs to the dual cone $(\mathrm{SOL}(0, M))^*$. Consequently, Theorem **3.8.6** says that if M is copositive, then $(\mathrm{SOL}(0, M))^* \subseteq K(M)$.

Another interpretation of (2) is based on the quadratic function $f(z) = z^{\mathrm{T}}(q + Mz)$. Indeed, by following the same line of proof as in Proposition **3.7.14**, it is not difficult to establish that if M is copositive, then (2) holds if and only if $f(z)$ is bounded below on the set $\{z \in R^n_+ : Mz \geq 0\}$.

Theorem **3.8.6** has a number of applications. We discuss several of these in the remainder of this section. The reader can find more in Exercise **3.12.4**. To start, we note that **3.8.6** generalizes **3.8.5**. Indeed, if M is strictly copositive, then the implication (2) holds trivially, hence $\mathrm{SOL}(q, M) \neq \emptyset$ for all q. More generally, when the zero vector is the only solution of the homogeneous LCP $(0, M)$, then (2) will hold trivially. Matrices with this property are significant enough to warrant their own notation.

3.8.7 Definition. Let $M \in R^{n \times n}$. Then M is called an $\boldsymbol{R_0}$-*matrix* if $\mathrm{SOL}(0, M) = \{0\}$. The class of such matrices is denoted $\boldsymbol{R_0}$.

We have already used the class $\boldsymbol{R_0}$ at the end of Section 3.6 in our discussion of local uniqueness. It is easy to see that if M is a nondegenerate matrix, then $M \in \boldsymbol{R_0}$. More will be said concerning the class $\boldsymbol{R_0}$ in Section 3.9 and in later chapters. For the moment, we will use it in the next corollary, which follows easily from Theorem **3.8.6**.

3.8.8 Corollary. If M is a copositive $\boldsymbol{R_0}$-matrix, then $M \in \boldsymbol{Q}$. □

3.8.9 Remark. The reader can easily verify that the matrix

$$M = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

is positive semi-definite (hence, copositive) and belongs to the class $\boldsymbol{P}$ (hence, to $\boldsymbol{R_0}$). But M is not strictly copositive.

We now turn our attention to the copositive-plus matrices. As seen in Definition **3.8.1**, a copositive matrix is copositive-plus if and only if every constrained minimum point of the quadratic form $f(z) = z^{\mathrm{T}}Mz$ on the nonnegative orthant is an unconstrained stationary point of f (i.e., where $\nabla f(z) = 0$). For copositive-plus matrices, it turns out that the implication (2) is equivalent to the feasibility of the LCP (q, M).

3.8.10 Corollary. Let $M \in R^{n\times n}$ be copositive-plus and let $q \in R^n$ be arbitrary. Implication (2) holds if and only if the LCP (q, M) is feasible. If (q, M) is feasible, then it is solvable.

Proof. Suppose (2) holds. If (q, M) is infeasible, then by Farkas' lemma there exists a vector $v \geq 0$ such that $v^{\mathrm{T}}M \leq 0$ and $v^{\mathrm{T}}q < 0$. Since M is copositive, $v^{\mathrm{T}}Mv = 0$, and since M is copositive-plus, we have $Mv = -M^{\mathrm{T}}v \geq 0$. This means that v violates (2). Hence (q, M) must be feasible.

Conversely, if (q, M) is feasible, and if $v \geq 0$ is a vector satisfying $Mv \geq 0$ and $v^{\mathrm{T}}Mv = 0$, then $M^{\mathrm{T}}v = -Mv \leq 0$ because M is copositive plus. This in turn implies $v^{\mathrm{T}}q \geq 0$ since (q, M) is feasible. In view of the equivalence between the feasibility of (q, M) and the implication (2), the last assertion is an immediate consequence of Theorem **3.8.6**. □

3.8.11 Remark. It follows from Corollary **3.8.10** that every copositive-plus matrix belongs to $\boldsymbol{Q_0}$.

The following is a consequence of Corollary **3.8.10**. It shows that the symmetric copositive-plus matrices provide a matrix class for which the converse of Corollary **3.7.12** holds. The reader is asked to supply the proof in Exercise **3.12.13**.

3.8.12 Corollary. Let $M \in R^{n \times n}$ be symmetric and copositive-plus. For any given $q \in R^n$, the following three statements are equivalent:

(a) The LCP (q, M) is feasible.

(b) The LCP (q, M) is solvable.

(c) The quadratic function $f(z) = z^{\mathrm{T}}(q+Mz)$ is bounded below for $z \geq 0$.

□

An equivalent way of stating Corollary **3.8.10** is that if M is copositive-plus, then

$$K(M) = \mathrm{pos}\,(I, -M) = (\mathrm{SOL}(0, M))^*. \tag{5}$$

The equality of $K(M)$ and $(\mathrm{SOL}(0, M))^*$, which holds for a copositive-plus matrix M, is analogous to the well-known fact in elementary linear algebra that the range of a matrix equals the orthogonal complement of the null space of its transpose.

It turns out that a complete characterization of the validity of (5) can be established within the class of copositive matrices. This is the main content of the next result.

3.8.13 Theorem. Let $M \in R^{n \times n}$ be copositive. Then, the set

$$T = \{x \in R^n_+ : M^{\mathrm{T}}x \leq 0\}$$

is a subset of $\mathrm{SOL}(0, M)$. In addition, the following three statements are equivalent:

(a) $T = \mathrm{SOL}(0, M)$.

(b) M is copositive-star.

(c) $(\mathrm{SOL}(0, M))^* = K(M) = \mathrm{pos}\,(I, -M)$.

Proof. Let $S = \mathrm{SOL}(0, M)$. We first show $T \subseteq S$. Suppose $x \in T$. We certainly know $x \geq 0$. In addition, as $x^{\mathrm{T}}M \leq 0$, we have $x^{\mathrm{T}}Mx \leq 0$. On the other hand, M is copositive so this implies $x^{\mathrm{T}}Mx = 0$. The last thing we must verify, to show $x \in S$, is that $Mx \geq 0$. Suppose $(Mx)_i < 0$ for some $i = 1, \ldots, n$. As M is copositive and $x^{\mathrm{T}}Mx = 0$, we have for $\theta > 0$

$$\begin{aligned} 0 &\leq (e_i + \theta x)^{\mathrm{T}} M (e_i + \theta x) \\ &= e_i^{\mathrm{T}} M e_i + \big((Mx)_i + (M^{\mathrm{T}}x)_i\big)\theta \\ &\leq e_i^{\mathrm{T}} M e_i + (Mx)_i \theta \end{aligned}$$

where the last inequality follows because $x \in T$. This will lead to a contradiction for large $\theta > 0$. Thus, $T \subseteq S$.

(a) $\Leftrightarrow$ (b). By definition, a copositive matrix is copositive-star if and only if $S \subseteq T$. Since we have just shown that $T \subseteq S$ for copositive matrices, the equivalence follows.

(a) $\Rightarrow$ (c). The inclusions

$$S^* \subseteq K(M) \subseteq \text{pos}\,(I, -M)$$

hold for any copositive M. Thus, it suffices to show that $\text{pos}\,(I, -M) \subseteq S^*$. Let $q \in \text{pos}\,(I, -M)$ and $v \in S$. There exist nonnegative vectors z and w such that $w = q + Mz$. Hence,

$$0 \leq v^{\mathrm{T}}w = v^{\mathrm{T}}q + v^{\mathrm{T}}Mz \leq v^{\mathrm{T}}q$$

where the last inequality follows as $v \in S = T$. This establishes (c).

(c) $\Rightarrow$ (a). As $T \subseteq S$ for all copositive matrices, we need only show that $S \subseteq T$. Suppose $v \in S$ and $(v^{\mathrm{T}}M)_i > 0$ for some $i = 1, \ldots, n$. We then have $-v^{\mathrm{T}}Me_i < 0$. As $-Me_i$ is in $\text{pos}\,(I, -M)$, this contradicts $S^* = \text{pos}\,(I, -M)$. Therefore, $v^{\mathrm{T}}M \leq 0$, hence $S \subseteq T$. □

If M is copositive, then both **3.8.8** and **3.8.13** give sufficient conditions for the existence of a solution for the LCP (q, M). Corollary **3.8.8** relates to the class $\boldsymbol{Q}$ and thus applies to all vectors q, whereas Theorem **3.8.13** concerns the set of "solvable" vectors q. By bringing in the matrix class $\boldsymbol{S}$ (which, as Proposition **3.1.5** shows, characterizes the class of matrices M for which the LCP (q, M) is feasible for all q), we obtain the following result.

3.8.14 Corollary. Let $M \in R^{n \times n}$ be copositive-star. The following statements are equivalent:

(a) $M \in \boldsymbol{S}$.

(b) $M \in \boldsymbol{R_0}$.

(c) $M \in \boldsymbol{Q}$.

Proof. Since [(b) $\Rightarrow$ (c)] follows from Corollary **3.8.8** and [(c) $\Rightarrow$ (a)] is obvious, it suffices to prove [(a) $\Rightarrow$ (b)]. Suppose that M is not an

$\boldsymbol{R}_0$-matrix. There is then a nonzero solution, say z, of $(0, M)$. Since M is assumed to be copositive-star, we obtain $M^{\mathrm{T}}z \leq 0$. Consequently, by Ville's theorem of the alternative, there cannot exist a vector $x > 0$ such that $Mx > 0$. Yet when $M \in \boldsymbol{S}$, there must exist such an x. This contradiction establishes the corollary. □

An important subclass of the class of copositive matrices is that consisting of the nonnegative matrices. If M is such a matrix, the above results provide sufficient conditions for the LCP (q, M) to have a solution for a specific vector q. In particular, it follows from Theorem **3.8.5** that any nonnegative matrix with positive diagonal elements must be a $\boldsymbol{Q}$-matrix. The following theorem shows that the property of positive diagonal elements actually gives a characterization for when a nonnegative matrix belongs to $\boldsymbol{Q}$.

3.8.15 Theorem. If $M \in R^{n \times n}$ is a nonnegative matrix, then $M \in \boldsymbol{Q}$ if and only if all diagonal entries of M are positive.

Proof. It suffices to show that if at least one diagonal entry of M is zero, then $M \notin \boldsymbol{Q}$. Without losing generality, we may assume $m_{11} = 0$. Let q be any vector satisfying $q_1 < 0 < q_i$ $(i = 2, \ldots, n)$. It is then easy to see that the LCP (q, M) cannot have a solution. □

3.9 Semimonotone and Regular Matrices

The two classes of matrices whose study we take up in this section are linked through similar properties relating to the uniqueness of solutions. Despite this likeness, these classes are distinct.

3.9.1 Definition. A matrix $M \in R^{n \times n}$ is said to be *semimonotone* if

$$[\,0 \neq x \geq 0\,] \quad \Rightarrow \quad [\,x_k > 0 \text{ and } (Mx)_k \geq 0 \text{ for some } k\,] \tag{1}$$

The class of such matrices is denoted $\boldsymbol{E}_0$, and its elements are called $\boldsymbol{E}_0$-matrices.

A few useful observations follow readily from this definition. First, every principal submatrix of a semimonotone matrix is again semimonotone. In particular, the diagonal entries of a semimonotone matrix must be nonnegative. Second, every $\boldsymbol{P_0}$-matrix is an $\boldsymbol{E_0}$-matrix. (See the characterization of $\boldsymbol{P_0}$-matrices given in Theorem **3.4.2**.) Third, all copositive matrices are semimonotone. These observations imply that $\boldsymbol{E_0}$ is a rather large class.

3.9.2 Example. The class of semimonotone matrices is larger than the union of the $\boldsymbol{P_0}$-matrices and the copositive matrices. Consider

$$M = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

M is not in $\boldsymbol{P_0}$ as $\det M = -1$. If $x = (1,1,0)$, then $x^{\mathrm{T}}Mx = -1$ and so M is not copositive. However, for $x = (x_1, x_2, x_3)$ with $0 \neq x \geq 0$, we see that either $x_2 > 0$ and $(Mx)_2 \geq 0$, or $x_2 = 0$ and $Mx \geq 0$. It follows that M is semimonotone.

The class $\boldsymbol{E_0}$ can be characterized in several ways, one of which pertains to the issue of uniqueness of solution to the LCP.

3.9.3 Theorem. Let $M \in R^{n \times n}$. The following statements are equivalent:

(a) M is semimonotone.

(b) The LCP (q, M) has a unique solution for every $q > 0$.

(c) For every index set $\alpha \subseteq \{1, \ldots, n\}$, the system

$$M_{\alpha\alpha} x_\alpha < 0, \quad x_\alpha \geq 0 \tag{2}$$

has no solution.

Proof. (a) $\Rightarrow$ (b). Let $q > 0$ be an arbitrary n-vector. Clearly, $z = 0$ solves the LCP (q, M). Suppose there exists a positive vector $\tilde{q}$ for which $(\tilde{q}, M)$ has a *nonzero* solution, $\tilde{z}$. Since M is semimonotone, there exists an index k such that $\tilde{z}_k > 0$ and $(M\tilde{z})_k \geq 0$. Consequently,

$$\tilde{q}_k + (M\tilde{z})_k > 0\,.$$

This contradicts the assumption that $\tilde{z}$ solves $(\tilde{q}, M)$.

(b) $\Rightarrow$ (c). Suppose there exists an index set α such that (2) has a solution, x_α. Define

$$\begin{aligned} x_{\bar\alpha} &= 0, \\ \tilde q_\alpha &= -M_{\alpha\alpha}x_\alpha, \\ \tilde q_{\bar\alpha} &> \max\{0, -M_{\bar\alpha\alpha}x_\alpha\}. \end{aligned}$$

Thus, $\tilde q > 0$ and $\tilde z = x$ is a nonzero solution of $(\tilde q, M)$. This contradicts (b).

(c) $\Rightarrow$ (a). Suppose that for every index set α the system (2) has no solution. Thus, the support of every nonzero nonnegative vector x contains an element k such that $(Mx)_k \geq 0$; hence M is semimonotone. □

Although semimonotonicity is equivalent to the uniqueness of solutions to the trivial linear complementarity problems (q, M) having $q > 0$, there is no guarantee that nontrivial, but solvable, problems have unique solutions. For instance, with M as in Example **3.9.2**, any vector of the form $(\lambda, 0, 0)$ or $(0, 0, \lambda)$, with $\lambda > 0$, is a solution of the LCP $(0, M)$. Notice, this implies that M is not an $\boldsymbol{R_0}$-matrix.

3.9.4 Definition. A matrix $M \in R^{n\times n}$ is said to be an $\boldsymbol{S_0}$-matrix if the system

$$Mz \geq 0, \quad 0 \neq z \geq 0 \tag{3}$$

has a solution.

Recall that for each member of the class $\boldsymbol{S}$ (defined in **3.1.4**), the homogeneous linear inequality system (3.1.9) has a solution. From this, it follows immediately that $\boldsymbol{S} \subseteq \boldsymbol{S_0}$.

Unlike $\boldsymbol{E_0}$, the classes $\boldsymbol{S}$ and $\boldsymbol{S_0}$ do not possess the inheritance property. That is, the principal submatrices of their members need not all belong to the same class. The study of many classes of matrices suggests that the inheritance property is an interesting one. For this reason, we introduce a bit of general language that we can put to use at once.

3.9.5 Definition. Let $\boldsymbol{Y}$ denote a (fixed) class of square matrices. The square matrix M will be called *completely-$\boldsymbol{Y}$* if M and all its principal submatrices belong to $\boldsymbol{Y}$. The class of completely-$\boldsymbol{Y}$ matrices is denoted $\overline{\boldsymbol{Y}}$. If $\boldsymbol{Y} = \overline{\boldsymbol{Y}}$, then the matrix class $\boldsymbol{Y}$ is said to be *complete*.

As a case in point, we now consider a result about the class $\bar{\boldsymbol{S}}_0$.

3.9.6 Lemma. If $M \in R^{n\times n}$, then $M \in \bar{\boldsymbol{S}}_0$ if and only if $M^{\mathrm{T}} \in \bar{\boldsymbol{S}}_0$.

Proof. Since $(M^{\mathrm{T}})^{\mathrm{T}} = M$, it suffices to verify the "only if" direction. The proof is by induction on n. The assertion is obvious for $n = 1$. Assume now that $n > 1$ and that the transpose of every $\bar{\boldsymbol{S}}_0$-matrix of order less than or equal to $n - 1$ belongs to $\bar{\boldsymbol{S}}_0$. Thus, if $M \in R^{n\times n}$, $M \in \bar{\boldsymbol{S}}_0$, and $M^{\mathrm{T}} \notin \bar{\boldsymbol{S}}_0$, it must be that M^{T} itself is not an $\boldsymbol{S}_0$-matrix. By Ville's theorem of the alternative, there then exists a vector x such that

$$Mx < 0 \quad \text{and} \quad x > 0\,. \tag{4}$$

Now since $M \in \boldsymbol{S}_0$, there exists a vector z satisfying (3). Clearly, there is some $\lambda > 0$ such that $x - \lambda z$ is nonnegative but not strictly positive. By (3), (4), and the positivity of λ we have $M(x - \lambda z) < 0$. Obviously, $x - \lambda z$ is nonzero. Thus, if α is the support of $x - \lambda z$, then α is a nonempty and proper subset of $\{1, \ldots, n\}$. We have

$$M_{\alpha\alpha}(x - \lambda z)_\alpha < 0 \quad \text{and} \quad (x - \lambda z)_\alpha > 0$$

which, by Ville's theorem of the alternative, implies that $M^{\mathrm{T}}_{\alpha\alpha}$ is not in $\boldsymbol{S}_0$. However, by induction we know that every proper principal submatrix of M^{T} is an $\boldsymbol{S}_0$-matrix. We have a contradiction; hence the lemma follows. □

In Definition **3.1.4** we pointed out that (3.1.9) and (3.1.10) are equivalent conditions. From this and Ville's theorem of the alternative we conclude that assertion (c) of Theorem **3.9.3** is just the statement that M^{T} is completely-$\boldsymbol{S}_0$. Lemma **3.9.6** now immediately enables us to prove

3.9.7 Corollary. Let $M \in R^{n\times n}$. The following statements are equivalent:

(a) M is semimonotone.

(b) M is completely-$\boldsymbol{S}_0$.

(c) M^{T} is completely-$\boldsymbol{S}_0$.

(d) M^{T} is semimonotone.

□

The proof of Lemma **3.9.6** is somewhat similar to the proof of Theorem **3.8.3**. In fact, we see that assertion (b) of Theorem **3.8.3** just says that $M \in \bar{\boldsymbol{S}}_0$. From this and Corollary **3.9.7** we conclude

3.9.8 Proposition. If $M \in R^{n\times n}$ is symmetric, then M is copositive if and only if M is semimonotone. □

This explains why we could not have used a symmetric matrix in Example **3.9.2**.

There are many matrix classes of which we are now aware. As we review these classes, there are certain patterns which seem to emerge. One such pattern is that some of the matrix classes are defined by certain inequalities, and natural subclasses are produced by requiring the inequalities to hold strictly. For example, $\boldsymbol{P}_0$-matrices, which require that the principal minors be nonnegative, have as a subclass the $\boldsymbol{P}$-matrices which require positive principal minors. Other examples would include the positive semi-definite matrices with the subclass of positive definite matrices, the copositive matrices with the subclass of strictly copositive matrices, and the $\boldsymbol{S}_0$-matrices with the subclass of $\boldsymbol{S}$-matrices. Following this pattern we introduce a very natural subclass of $\boldsymbol{E}_0$.

3.9.9 Definition. A matrix $M \in R^{n\times n}$ is said to be *strictly semimonotone* if

$$[\, 0 \neq x \geq 0 \,] \quad \Rightarrow \quad [\, x_k > 0 \text{ and } (Mx)_k > 0 \text{ for some } k \,] \tag{5}$$

The class of such matrices is denoted $\boldsymbol{E}$, and its elements are called $\boldsymbol{E}$-matrices.

3.9.10 Example. It is clear that a strictly copositive matrix is strictly semimonotone. Also, Theorem **3.3.4** shows that all $\boldsymbol{P}$-matrices are strictly semimonotone. However, this does not capture the entire class of $\boldsymbol{E}$-matrices. Consider

$$M = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}.$$

The situation is similar to that in Example **3.9.2**. One can show that M is strictly semimonotone but is neither a $\boldsymbol{P}$-matrix nor strictly copositive.

Condition (5) is just a stronger version of (1). This accounts for the inclusion $\boldsymbol{E} \subseteq \boldsymbol{E}_0$ as well as the following results.

3.9.11 Theorem. Let $M \in R^{n \times n}$. The following statements are equivalent:

(a) M is strictly semimonotone.

(b) The LCP (q, M) has a unique solution for every $q \geq 0$.

(c) For every index set $\alpha \subseteq \{1, \ldots, n\}$, the system

$$M_{\alpha\alpha} x_\alpha \leq 0, \quad 0 \neq x_\alpha \geq 0 \tag{6}$$

has no solution.

Proof. This is Exercise **3.12.18.** □

Now let $\bar{\boldsymbol{S}}$ denote the class of completely-$\boldsymbol{S}$ matrices. (See **3.9.5** and **3.1.4.**)

3.9.12 Lemma. If $M \in R^{n \times n}$, then $M \in \bar{\boldsymbol{S}}$ if and only if $M^{\mathrm{T}} \in \bar{\boldsymbol{S}}$.

Proof. This is Exercise **3.12.19.** □

This leads to the following two results.

3.9.13 Corollary. Let $M \in R^{n \times n}$. The following statements are equivalent:

(a) M is strictly semimonotone.

(b) M is completely-$\boldsymbol{S}$.

(c) M^{T} is completely-$\boldsymbol{S}$.

(d) M^{T} is strictly semimonotone.

□

3.9.14 Proposition. If $M \in R^{n \times n}$ is symmetric, then M is strictly copositive if and only if M is strictly semimonotone.

Proof. This is Exercise **3.12.20.** □

In order to discuss the existence of solutions for the linear complementarity problem with a semimonotone matrix, we introduce three fundamental index sets associated with an arbitrary solution of an LCP.

3.9.15 Notation. Let $z \in \mathrm{SOL}(q, M)$. Define the index sets

$$\alpha(z) = \{i : z_i > 0 = (q + Mz)_i\}$$

$$\beta(z) = \{i : z_i = 0 = (q + Mz)_i\}$$

$$\gamma(z) = \{i : z_i = 0 < (q + Mz)_i\}.$$

In terms of these index sets, a solution z of the LCP (q, M) is nondegenerate if $\beta(z) = \emptyset$. Note also that $\alpha(z) = \operatorname{supp} z$ and $\gamma(z) = \operatorname{supp} w$ where $w = q + Mz$. Indices in $\beta(z)$ correspond to the *degenerate variables* of the solution z.

The result below is similar to Theorem **3.8.6** in both content and proof. Its proof is based on the augmented LCP $(\tilde{q}, \tilde{M})$ defined in Section 3.7.

3.9.16 Theorem. Let $q \in R^n$ and $M \in R^{n \times n}$ be given. Let d be an arbitrary positive n-vector. If for every nonzero solution z (if any exists) of the LCP $(\tau d, M)$, with arbitrary $\tau \geq 0$, there exists a nonzero vector $y_\alpha \geq 0$ such that

$$y_\alpha^{\mathrm{T}} M_{\alpha\alpha} \geq 0, \quad y_\alpha^{\mathrm{T}} M_{\alpha\beta} \geq 0, \quad y_\alpha^{\mathrm{T}} q_\alpha \geq 0 \tag{7}$$

where $\alpha = \alpha(z)$ and $\beta = \beta(z)$ are as defined in **3.9.15**, then the LCP (q, M) has a solution.

Proof. Suppose the contrary. Consider the augmented LCP $(\tilde{q}^k, \tilde{M})$ defined in (3.7.1), (3.7.2), and (3.7.6). By Theorem **3.7.6**, there exist an unbounded sequence of positive scalars $\{\lambda_k\}$ and a corresponding sequence of solutions $\{(z^k, \theta_k)\}$ to $(\tilde{q}^k, \tilde{M})$ such that $\inf \theta_k > 0$. It follows that $d^{\mathrm{T}} z^k = \lambda_k$, that is $\sigma_k = 0$, for all k. By Theorem **3.7.9**, there exist two vectors u and v with $d^{\mathrm{T}} u = 0$, $v \geq 0$ and $d^{\mathrm{T}} v = 1$ and a subsequence $\{k_i\}$ such that for all k_i,

$$\bar{z}^{k_i} = u + \lambda_{k_i} v. \tag{8}$$

where $(\bar{z}^{k_i}, \bar{\theta}_{k_i})$ is a, possibly different, solution to the augmented LCP $(\tilde{q}^{k_i}, \tilde{M})$. Without loss of generality, we may assume $(\bar{z}^{k_i}, \bar{\theta}_{k_i}) = (z^{k_i}, \theta_{k_i})$ and the expression (8) holds for the entire sequence $\{z^k\}$. Similar to the proof of Theorem **3.8.6**, we note

$$0 = (z^k)^{\mathrm{T}} w^k = (z^k)^{\mathrm{T}} q + (z^k)^{\mathrm{T}} M z^k + \theta_k \lambda_k .$$

If we divide through by λ_k^2, it becomes apparent, since $d^T(z^k/\lambda_k) = 1$ and $d > 0$, that the sequence $\{\theta_k/\lambda_k\}$ is bounded. Thus, $\{\theta_k/\lambda_k\}$ has some accumulation point $\tau \geq 0$. Again, without loss of generality, we may assume that τ is the limit of the latter sequence. It is straightforward to show that v solves the LCP $(\tau d, M)$. Let $\alpha = \alpha(v)$, $\beta = \beta(v)$, and $\gamma = \gamma(v)$. If the LCP $(\tau d, M)$ has no nonzero solution, then a contradiction is derived, and the theorem is proved. Otherwise, by assumption, there exists a nonzero vector $y_\alpha \geq 0$ such that (7) holds. Thus, by the definition of the set α, we have

$$0 \leq y_\alpha^T M_{\alpha\alpha} v_\alpha = -\tau y_\alpha^T d_\alpha \leq 0$$

which implies $y_\alpha^T M_{\alpha\alpha} = 0$ in view of the fact that $v_\alpha > 0$.

Since $v_{\beta\cup\gamma} = 0$, it follows that $u_{\beta\cup\gamma} = z^k_{\beta\cup\gamma} \geq 0$. Moreover, by complementarity, we deduce,

$$u_\gamma^T(q + Mu + \lambda_k Mv + \theta_k d)_\gamma = 0 \,.$$

Dividing by λ_k and passing to the limit as $k \to \infty$, we deduce $u_\gamma = 0$ because $(\tau d + Mv)_\gamma > 0$. Since $v_\alpha > 0$, it follows that $z^k_\alpha > 0$ for all k large enough; thus, by complementarity,

$$0 = (q + Mu + \lambda_k Mv + \theta_k d)_\alpha$$

which implies

$$\begin{aligned} 0 &= y_\alpha^T(q + Mu + \lambda_k Mv + \theta_k d)_\alpha \\ &= y_\alpha^T(q_\alpha + M_{\alpha\beta} u_\beta + \theta_k d_\alpha) \end{aligned}$$

because $u_\gamma = y_\alpha^T M_{\alpha\alpha} = v_{\beta\cup\gamma} = 0$. Consequently, from $0 \neq y_\alpha \geq 0$, $d > 0$, and condition (7), it follows that $\theta_k = 0$ for all k sufficiently large. This contradiction establishes the theorem. □

Specializing Theorem **3.9.16** to the case where M is semimonotone, we derive the following existence result.

3.9.17 Corollary. Let $q \in R^n$ and let $M \in R^{n\times n}$ be given. Suppose that M is semimonotone and that for every $\alpha \subseteq \{1, \ldots, n\}$ for which there exists a vector $0 \neq z \geq 0$ such that $\alpha = \operatorname{supp} z$, and

$$M_{\alpha\alpha} z_\alpha = 0, \quad M_{\bar\alpha\alpha} z_\alpha \geq 0 \,,$$

there exists a nonzero vector $y_\alpha \geq 0$ such that (7) holds where

$$\beta = \{i \in \bar{\alpha} : M_{i\alpha} z_\alpha = 0\}.$$

It then follows that the LCP (q, M) has a solution.

Proof. This follows immediately from Theorem **3.9.16** by noting that if M is semimonotone, then the LCP $(\tau d, M)$ can not have a nonzero solution for $\tau > 0$ (see **3.9.3**). □

The above corollary provides a sufficient condition for the LCP (q, M) to have a solution when $M \in \boldsymbol{E_0}$. In what follows, we introduce a subclass of $\boldsymbol{E_0}$ which is contained in $\boldsymbol{Q_0}$.

3.9.18 Definition. Let $M \in R^{n \times n}$. Then $M \in \boldsymbol{E_1}$ if and only if for every nonzero vector $z \in \text{SOL}(0, M)$, there exist nonnegative diagonal matrices D_1 and D_2 such that $D_2 z \neq 0$ and $(D_1 M + M^{\text{T}} D_2) z = 0$. Let $\boldsymbol{L}$ be the intersection of $\boldsymbol{E_0}$ and $\boldsymbol{E_1}$.

Observe that class $\boldsymbol{E_1}$ contains $\boldsymbol{E}$ (by default) and all the copositive-plus matrices (by taking $D_1 = D_2 = I$). Moreover, if $M \in \boldsymbol{E_1}$, then for every nonzero vector $z \in \text{SOL}(0, M)$, the vector $y = D_2 z$ where D_2 is the diagonal matrix as given in Definition **3.9.18** must belong to $\text{SOL}(0, -M^{\text{T}})$ and the two solutions y and z are related in a special way through the equation $D_1 M z + M^{\text{T}} y = 0$. Hence, one way to interpret the class $\boldsymbol{E_1}$ is that it is comprised of those matrices M for which a certain "duality" relation exists between the two homogeneous linear complementarity problems $(0, M)$ and $(0, -M^{\text{T}})$.

3.9.19 Corollary. The class $\boldsymbol{L}$ is contained in $\boldsymbol{Q_0}$.

Proof. Let $q \in \text{pos}(I, -M)$. It suffices to verify the assumption of Corollary **3.9.17**. For this purpose, let z, α and β be as given in **3.9.17**. Then $z \in \text{SOL}(0, M)$. Since $M \in \boldsymbol{E_1}$, it follows that there exist nonnegative diagonal matrices D_1 and D_2 such that $D_2 z \neq 0$ and $M^{\text{T}} D_2 z = -D_1 M z$. Let $y = D_2 z$. Clearly, $y_{\bar{\alpha}} = 0$; hence, $y^{\text{T}} M = y_\alpha^{\text{T}} M_{\alpha \cdot}$. We have

$$y_\alpha^{\text{T}} M_{\alpha\alpha} = (y^{\text{T}} M)_\alpha = -((Mz)^{\text{T}} D_1)_\alpha.$$

Since D_1 is a diagonal matrix and $(Mz)_\alpha = 0$, it follows that $y_\alpha^{\mathrm{T}} M_{\alpha\alpha} = 0$. Similarly, $y_\alpha^{\mathrm{T}} M_{\alpha\beta} = 0$. Finally, it remains to be shown that $y_\alpha^{\mathrm{T}} q_\alpha \geq 0$. For this purpose, let $x \in \mathrm{FEA}(q, M)$. Then,

$$0 \leq y_\alpha^{\mathrm{T}}(q + Mx)_\alpha = y_\alpha^{\mathrm{T}} q_\alpha + y^{\mathrm{T}} M x \leq y_\alpha^{\mathrm{T}} q_\alpha$$

because $M^{\mathrm{T}} y \leq 0$ and $x \geq 0$. □

In **3.8.7** we defined the class $\boldsymbol{R_0}$. It follows from Corollary **3.9.17** that $\boldsymbol{E_0} \cap \boldsymbol{R_0} \subseteq \boldsymbol{Q}$. More generally, if for some vector $d > 0$ and for all $\tau \geq 0$ the LCP $(\tau d, M)$ has only one solution, namely the zero vector, then $M \in \boldsymbol{Q}$. This suggests the following.

3.9.20 Definition. Given $M \in R^{n \times n}$ and $d \in R^n_{++}$, we say that M is *d-regular* if for all $\tau \geq 0$ the LCP $(\tau d, M)$ has only one solution $(z = 0)$. A matrix M is called *regular*, if it is d-regular for some $d > 0$. The class of regular matrices is denoted by $\boldsymbol{R}$.

3.9.21 Remark. We have deviated slightly in Definition **3.9.20** in that a regular matrix is usually defined to be a matrix which is e-regular. (Here, as always, e is the vector of all ones.)

Note that the defining property for the class $\boldsymbol{R_0}$ corresponds to that of $\boldsymbol{R}$ when $\tau = 0$. It is therefore clear that $\boldsymbol{R} \subseteq \boldsymbol{R_0}$, and this explains why $\boldsymbol{R_0}$-matrices are sometimes called *pseudo-regular*. We now see that Corollary **3.9.17** implies

$$\boldsymbol{E_0} \cap \boldsymbol{R_0} \subseteq \boldsymbol{R} \subseteq \boldsymbol{Q}\,. \tag{9}$$

Incidentally, the term "regular" should not be construed as a synonym for "nonsingular." For example, the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is regular as well as singular.

It is clear that strictly semimonotone matrices are regular in the sense defined above, for when $\tau \geq 0$ is a nonnegative scalar, the vectors τd are all nonnegative (see **3.9.11**). However, the two classes are not the same.

Strictly semimonotone matrices must have positive diagonal elements; regular matrices can have negative diagonal elements as illustrated by the following e-regular matrix

$$M = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

It is clear from the above remarks that strictly semimonotone matrices belong to the class $\boldsymbol{Q}$. As a matter of fact, we have $\boldsymbol{E} \subset \boldsymbol{R} \subset \boldsymbol{Q}$.

At the beginning of this section we mentioned that a $\boldsymbol{P_0}$-matrix must be semimonotone. Thus, $\boldsymbol{P_0} \cap \boldsymbol{R_0} \subseteq \boldsymbol{R} \subseteq \boldsymbol{Q}$. In fact, the following characterization of $\boldsymbol{Q}$-matrices holds within the class of $\boldsymbol{P_0}$-matrices.

3.9.22 Theorem. Let $M \in R^{n\times n}$ be a $\boldsymbol{P_0}$-matrix. The following are equivalent:

(a) $M \in \boldsymbol{R_0}$;

(b) $M \in \boldsymbol{R}$;

(c) $M \in \boldsymbol{Q}$.

Proof. In view of the preceding discussions, it remains to verify the implication [(c) $\Rightarrow$ (a)]. Suppose that M is in $\boldsymbol{Q}$ and not in $\boldsymbol{R_0}$. Let z be a nonzero solution of the homogeneous LCP $(0, M)$. Let α denote the support of z. Let q be an arbitrary vector with $q_\alpha < 0$ and $q_{\bar{\alpha}} > 0$. Let $\tilde{z}$ be a solution of the problem (q, M). Consider $z - \tau\tilde{z}$, with $\tau > 0$. Select τ small enough so that $(z - \tau\tilde{z})_i > 0$ for all $i \in \alpha$.

Suppose $(z-\tau\tilde{z})_i \neq 0$. We claim that $(z-\tau\tilde{z})_i\big(M(z-\tau\tilde{z})\big)_i < 0$. To see this, first assume $i \notin \alpha$. Thus, $(z - \tau\tilde{z})_i = -\tau\tilde{z}_i < 0$. By complementarity $(M\tilde{z} + q)_i = 0$. Hence, $\big(M(z - \tau\tilde{z})\big)_i \geq \tau q_i > 0$, and the claim is true. If we assume $i \in \alpha$, then $(Mz)_i = 0$ and $\big(M(z - \tau\tilde{z})\big)_i \leq \tau q_i < 0$. As $(z-\tau\tilde{z})_i > 0$, the claim is true. Since z is nonzero, then α is nonempty and $z - \tau\tilde{z}$ is nonzero. Therefore, using assertion (b) of Theorem **3.4.2**, we see that the claim implies the contradiction $M \notin \boldsymbol{P_0}$, from which the theorem follows. □

As the above results show, the class $\boldsymbol{R_0}$ plays an important role in the existence theory for the LCP. It also has an interesting property related to the boundedness of the solution sets of LCPs as well as to the boundedness of the level sets of the associated quadratic program (3.1.1). As we shall see in Chapter 5, the boundedness of such level sets is in turn related to the convergence of iterative methods for solving the LCP (q, M).

3.9.23 Proposition. Let $M \in R^{n \times n}$. The following three statements are equivalent:

(a) $M \in \boldsymbol{R_0}$.

(b) For every $q \in R^n$ and every $\sigma, \tau \in R$ with $\sigma > 0$, the level set

$$L(\sigma, \tau) \;=\; \{ z \geq 0 \;:\; q + Mz \geq 0,\; z^{\mathrm{T}} q + \sigma z^{\mathrm{T}} M z \leq \tau \}$$

is bounded.

(c) For every $q \in R^n$, the solution set of (q, M) is bounded.

Proof. (a) $\Rightarrow$ (b). Suppose $M \in \boldsymbol{R_0}$ but the set $L(\sigma, \tau)$ is unbounded for some $q \in R^n$, some $\sigma > 0$, and some τ. Let $\{z^k\}$ be an unbounded sequence of vectors in $L(\sigma, \tau)$. It is then easy to show that any accumulation point of the normalized sequence $\{z^k / \|z^k\|\}$ (which is bounded) is a nonzero solution of $(0, M)$. This is a contradiction.

(b) $\Rightarrow$ (c). It suffices to note that the solution set of (q, M) is equal to the level set $L(1, 0)$.

(c) $\Rightarrow$ (a). If z is a nonzero solution of $(0, M)$, then so are all nonnegative multiples of z. $\square$

3.10 Completely-Q Matrices

When a matrix M belongs to the class $\boldsymbol{Q}$, there is no guarantee that all its principal submatrices enjoy this property. For example, the $\boldsymbol{Q}$-matrix

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

has a principal submatrix (namely the diagonal entry 0) which is not in $\boldsymbol{Q}$. Yet sometimes all the principal submatrices of a $\boldsymbol{Q}$-matrix do belong to $\boldsymbol{Q}$, as in the case of

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

It turns out that one can characterize the class of all real square matrices having the latter property, and this characterization is the main subject of this section. In line with **3.9.5**, we introduce the following terminology.

3.10.1 Definition. A matrix $M \in R^{n\times n}$ is said to be *completely-$\boldsymbol{Q}$* if M and all its principal submatrices belong to $\boldsymbol{Q}$. The class of such matrices is denoted $\bar{\boldsymbol{Q}}$.

As we shall see, this terminology and notation is superfluous, for $\bar{\boldsymbol{Q}}$ is in fact one of the classes we have already considered.

3.10.2 Lemma. Every strictly semimonotone matrix belongs to $\bar{\boldsymbol{Q}}$.

Proof. This is immediate from the inclusions $\boldsymbol{E} \subseteq \boldsymbol{R} \subseteq \boldsymbol{Q}$ and the fact that $\boldsymbol{E}$ is a complete class (see **3.9.13**). □

The converse is just as easy to prove.

3.10.3 Lemma. Every completely-$\boldsymbol{Q}$ matrix is strictly semimonotone.

Proof. We have already noted that $\boldsymbol{Q} \subseteq \boldsymbol{S}$. The assertion now follows from Corollary **3.9.13**. □

3.10.4 Remark. These two lemmas combine to give the fact that $\bar{\boldsymbol{Q}} = \boldsymbol{E}$. Identifying the class of $\boldsymbol{Q}$-matrices whose principal submatrices all belong to $\boldsymbol{Q}$ is of interest in its own right. There is also a "practical" side to the matter which will become evident when we study the so-called variable-dimension algorithms for the LCP. (See Section 4.6.) The class of matrices we consider next arose in just such a context.

3.10.5 Definition. A matrix $M \in R^{n\times n}$ is said to be a $\boldsymbol{V}$-matrix if for every index set $\alpha \subseteq \{1,\ldots,n\}$ the principal submatrix $M_{\alpha\alpha}$ has the property that there exists no positive vector $z_\alpha > 0$ such that the last component of $M_{\alpha\alpha}z_\alpha$ is nonpositive and the remaining components are zero. The class of such matrices is denoted simply as $\boldsymbol{V}$.

It is clear that $\boldsymbol{V}$ is a complete class. We now show that the elements of this new class are strictly semimonotone matrices and vice versa.

3.10.6 Lemma. A matrix $M \in R^{n \times n}$ is strictly semimonotone if and only if $M \in \boldsymbol{V}$.

Proof. Clearly $\boldsymbol{E}$ is a subset of $\boldsymbol{V}$. To establish the reverse inclusion, suppose there exists a matrix M in $\boldsymbol{V} \setminus \boldsymbol{E}$. There will then exist an index set $\alpha \subseteq \{1, \ldots, n\}$ of minimum cardinality such that the system

$$M_{\alpha\alpha} z_\alpha \leq 0, \quad 0 \neq z_\alpha \geq 0$$

has a solution. The minimality of the cardinality of α implies that z_α must be positive, rather than merely nonnegative. In addition, it implies that $M_{\alpha\alpha}$ must be nonsingular. To see this latter implication, assume that some $y_\alpha \neq 0$ exists for which $M_{\alpha\alpha} y_\alpha = 0$. There cannot be a scalar λ such that $z_\alpha = \lambda y_\alpha$ as then $M_{\alpha\alpha} z_\alpha$ would be zero, which implies $M \notin \boldsymbol{V}$. Thus, for some scalar λ, the vector $z_\alpha - \lambda y_\alpha$ will be nonnegative, nonzero, and not strictly positive. As $M_{\alpha\alpha}(z_\alpha - \lambda y_\alpha) = M_{\alpha\alpha} z_\alpha \leq 0$, this violates the minimality of the cardinality of α, hence $M_{\alpha\alpha}$ is nonsingular.

Now define

$$c_\alpha = (0, \ldots, 0, -1) \quad \text{and} \quad x_\alpha = M_{\alpha\alpha}^{-1} c_\alpha.$$

The vector x_α cannot be nonnegative. Indeed, x_α cannot equal zero, since c_α is not zero. If x_α were positive, then M could not be a member of $\boldsymbol{V}$. If x_α were nonnegative but neither zero nor positive, then the minimality of the cardinality of α would be violated. Now for all $\lambda \geq 0$ the vector $z_\alpha + \lambda x_\alpha \neq 0$. To see this, note that the case $\lambda = 0$ follows from the positivity of z_α. If $z_\alpha + \lambda x_\alpha = 0$ for $\lambda > 0$, then we have

$$M_{\alpha\alpha} x_\alpha \leq 0 \quad \Rightarrow \quad M_{\alpha\alpha}(\lambda x_\alpha) \leq 0 \quad \Rightarrow \quad M_{\alpha\alpha}(-z_\alpha) \leq 0.$$

Thus, $M_{\alpha\alpha} z_\alpha \geq 0$, while at the same time $M_{\alpha\alpha} z_\alpha \leq 0$. This implies that $M_{\alpha\alpha} z_\alpha = 0$ which is impossible since $M_{\alpha\alpha}$ is nonsingular and $z_\alpha \neq 0$. Now, for some $\lambda > 0$, the vector $z_\alpha + \lambda x_\alpha$ will be nonnegative, nonzero and not strictly positive. Nevertheless, $M_{\alpha\alpha}(z_\alpha + \lambda x_\alpha) \leq 0$. This contradicts the minimality of the cardinality of α. □

The three preceding lemmas and some results in Section 3.9 combine to give one grand characterization of the class of completely-$\boldsymbol{Q}$ matrices.

3.10.7 Theorem. Let $M \in R^{n\times n}$. The following statements are equivalent:

(a) $M \in \bar{\boldsymbol{Q}}$.

(b) $M \in \boldsymbol{E}$.

(c) $M \in \bar{\boldsymbol{S}}$.

(d) $M \in \boldsymbol{V}$.

(e) The LCP (q, M) has a unique solution for all $q \geq 0$.

(f) For every index set $\alpha \subseteq \{1, \ldots, n\}$, the system

$$M_{\alpha\alpha}x_\alpha \leq 0, \quad 0 \neq x_\alpha \geq 0$$

has no solution.

(g) $M^{\mathrm{T}} \in \bar{\boldsymbol{Q}}$.

(h) $M^{\mathrm{T}} \in \boldsymbol{E}$.

(i) $M^{\mathrm{T}} \in \bar{\boldsymbol{S}}$.

(j) $M^{\mathrm{T}} \in \boldsymbol{V}$.

(k) The LCP (q, M^{T}) has a unique solution for all $q \geq 0$.

(l) For every index set $\alpha \subseteq \{1, \ldots, n\}$, the system

$$M^{\mathrm{T}}_{\alpha\alpha}x_\alpha \leq 0, \quad 0 \neq x_\alpha \geq 0$$

has no solution. □

3.11 $\boldsymbol{Z}$-matrices and Least-Element Theory

In the previous sections, two approaches were used to derive results on the existence of a solution to the linear complementarity problem. Each approach relies on different properties of the matrix defining the problem. In this section, a third approach is used to obtain further existence results. We begin by introducing the class of $\boldsymbol{Z}$-matrices which plays the central role in the following development.

3.11.1 Definition. A square matrix is called a *$\boldsymbol{Z}$-matrix* if its off-diagonal entries are all non-positive. A $\boldsymbol{Z}$-matrix which is also a $\boldsymbol{P}$-matrix is called a *$\boldsymbol{K}$-matrix*. The classes of $\boldsymbol{Z}$-matrices and $\boldsymbol{K}$-matrices are denoted by $\boldsymbol{Z}$ and $\boldsymbol{K}$, respectively.

We have already seen examples of these matrices. Indeed, any comparison matrix is a $\boldsymbol{Z}$-matrix (see **3.3.12**). Moreover, the optimal stopping problem and the convex hull problem (see Section 1.2) give rise to LCPs with matrices in $\boldsymbol{Z}$ and $\boldsymbol{K}$, respectively.

Clearly, $\boldsymbol{Z}$ is a complete class of matrices (see **3.9.5**). Since the class $\boldsymbol{P}$ is also complete, so is $\boldsymbol{K}$. Moreover, the classes $\boldsymbol{Z}$ and $\boldsymbol{K}$ are invariant under transposition.

The feasible region of the LCP (q, M) where M is a $\boldsymbol{Z}$-matrix possesses a special property which we define below.

3.11.2 Definition. A subset S of R^n is called a *meet semi-sublattice* (under the componentwise ordering of R^n) if for any two vectors x and y in S, their *meet*, which is defined as the vector $z = \min(x, y)$, also belongs to S.

In essence, one could also define the *join* of two vectors and a *join semi-sublattice* by replacing the "min" operator in the above definition with the "max" operator. For our purpose here, we consider only the "meet" case.

3.11.3 Proposition. If M is a $\boldsymbol{Z}$-matrix and q is an arbitrary vector, then the feasible region of the LCP (q, M) is a meet semi-sublattice.

Proof. Let $S = \text{FEA}(q, M)$. Let x and y be two feasible vectors in S, and let z denote their meet. Obviously $z \geq 0$. Consider an arbitrary component i. Without loss of generality, we may assume that $z_i = x_i$. By the $\boldsymbol{Z}$-property of M and the fact that x is feasible, we have

$$(q + Mz)_i = q_i + m_{ii}x_i + \sum_{j \neq i} m_{ij}z_j \geq (q + Mx)_i \geq 0.$$

Similarly, one shows that $(q + Mz)_i \geq 0$ if $z_i = y_i$. This establishes the feasibility of the vector z and the meet semi-sublattice property of S. □

If M is a $\boldsymbol{Z}$-matrix, then the feasible region of the LCP (q, M), if nonempty, must possess a certain "least element". This concept is defined below.

3.11.4 Definition. A subset S of R^n is *bounded below* (relative to the componentwise ordering $\geq$) if there exists a vector $u \in R^n$ such that $x \geq u$

for all vectors $x \in S$. If such a vector u happens to belong to S, then u is called a *least element* of S.

Obviously, if a least element of a set exists, then it must be unique. The following result provides a sufficient condition for the existence of such an element.

3.11.5 Theorem. If S is a nonempty meet semi-sublattice that is closed and bounded below, then S has a least element.

Proof. Consider the mathematical program

$$\begin{array}{ll} \text{minimize} & p^{\mathrm{T}}x \\ \text{subject to} & x \in S \end{array} \tag{1}$$

where p is an arbitrary positive vector. This program has an optimal solution $\tilde{x}$. To see this, note that as S is bounded below there is a vector u such that $u \leq x$ for all $x \in S$, and as S is nonempty we can take some fixed vector x' to be in S. Thus (1) is equivalent to

$$\begin{array}{ll} \text{minimize} & p^{\mathrm{T}}x \\ \text{subject to} & x \in S \\ & x \geq u \\ & p^{\mathrm{T}}x \leq p^{\mathrm{T}}x' \end{array}$$

However, as p is positive and S is closed, this latter program has a compact feasible region and, hence, has an optimal solution.

We claim that $\tilde{x}$ is the least element of S. Indeed, let x be any vector in S. The vector $z = \min(x, \tilde{x})$ will then belong to S. By the definition of $\tilde{x}$, we have $p^{\mathrm{T}}\tilde{x} \leq p^{\mathrm{T}}z$. As p is positive, it follows that $\tilde{x} = z \leq x$, establishing that $\tilde{x}$ is the least element of S. □

The above theorem not only provides a sufficient condition for the existence of a least element, its proof actually suggests a constructive way to compute that element. Indeed, the least element of S is the unique optimal solution of the mathematical program (1). In the case where S is polyhedral (e.g., when S is the feasible region of an LCP), (1) is a linear program.

Applying **3.11.5** to the feasible region of the LCP (q, M), we derive the following result.

3.11.6 Theorem. Let M be a $\boldsymbol{Z}$-matrix and q an arbitrary vector. If the LCP (q, M) is feasible, then FEA(q, M) contains a least element u. Moreover, u solves the LCP (q, M).

Proof. Let $S = \text{FEA}(q, M)$. By Proposition **3.11.3**, S is a meet semi-sublattice. It is obviously bounded below (by zero), and is nonempty if the LCP (q, M) is feasible. The existence of the least element u therefore follows from Theorem **3.11.5**. It remains to show that u solves the LCP (q, M). Indeed, suppose that for some component i, both u_i and $(q+Mu)_i$ are positive. Consider the vector $z = u-\delta e_i$ where δ is some positive scalar. We claim that for $\delta > 0$ small enough, the vector z is feasible. Obviously, for such a δ, z is nonnegative and $(q+Mz)_i \geq 0$. Consider an index $j \neq i$. By the $\boldsymbol{Z}$-property of M, we may derive $(q + Mz)_j \geq (q + Mu)_j \geq 0$. Consequently, z is feasible provided that $\delta > 0$ is small enough. But this contradicts the least-element property of u. This establishes the theorem. □

To illustrate Theorem **3.11.6**, the feasible region of an LCP (q, M) with $M \in \boldsymbol{Z}$, along with the least-element solution u, is shown in Figure 3.1.

According to the proof of Theorem **3.11.5**, the least-element solution u in Theorem **3.11.6** can be computed by solving the linear program

$$\begin{array}{ll} \text{minimize} & p^{\mathrm{T}}z \\ \text{subject to} & q + Mz \geq 0 \\ & z \geq 0 \end{array} \tag{2}$$

for any positive vector p. This conclusion is closely related to Theorem **1.3.4** which states that any solvable LCP must possess an extreme point solution. Such a solution could be obtained by solving a linear program with a suitably chosen objective function. For an arbitrary LCP (q, M), the appropriate linear form to be used is typically not known in advance. However, the above discussion shows that when M is a $\boldsymbol{Z}$-matrix, such an objective function is easily available. This latter result will later be generalized to a larger class of matrices.

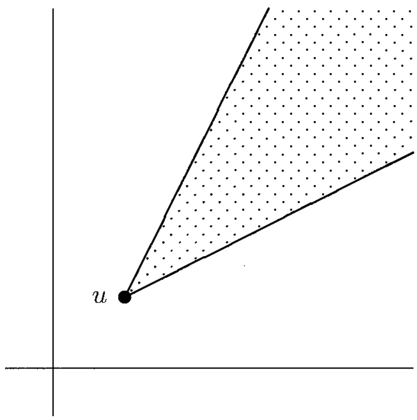

Figure 3.1

Theorem **3.11.6** shows that the $\boldsymbol{Z}$-property of the matrix M provides a sufficient condition for the existence of a least-element solution to all feasible LCP (q, M). It turns out that this property of M is also necessary as we establish the following characterization of a $\boldsymbol{Z}$-matrix.

3.11.7 Theorem. $M \in R^{n \times n}$ is a $\boldsymbol{Z}$-matrix if and only if for all vectors $q \in \text{pos}(I, -M)$, the feasible region of the LCP (q, M) contains a least element which is a solution of the LCP.

Proof. It suffices to show that the $\boldsymbol{Z}$-property is necessary. Suppose, on the contrary, that $m_{ij} > 0$ for some $i \neq j$. Let

$$q = e_j - M_{\cdot j}.$$

Obviously, e_j is a feasible vector for the LCP (q, M) with this chosen q. Thus, by assumption, (q, M) has a solution x satisfying $0 \leq x \leq e_j$, which yields $x_k = 0$ for any $k \neq j$. This, in turn, implies that $x_j = 1$ because $0 \leq (q + Mx)_i = m_{ij}(x_j - 1)$. By complementarity, we have $0 = (q + Mx)_j = 1$ which is clearly absurd. Consequently, M must be a $\boldsymbol{Z}$-matrix. □

Combining the preceding theorem and Theorem **3.3.7**, we obtain the following characterization of a $\boldsymbol{K}$-matrix.

3.11.8 Corollary. $M \in R^{n\times n}$ is a $\boldsymbol{K}$-matrix if and only if, for all vectors $q \in R^n$, FEA(q, M) has a least element which is the unique solution of the LCP (q, M).

The above corollary can be used to characterize a $\boldsymbol{K}$-matrix in terms of an *antitonicity property* of the unique solution of (q, M) when $M \in \boldsymbol{P}$. This characterization is stated more precisely in the result below.

3.11.9 Proposition. Let $M \in R^{n\times n} \cap \boldsymbol{P}$. Then $M \in \boldsymbol{K}$ if and only if for any two vectors q^1 and q^2 in R^n with $q^1 \geq q^2$, $z(q^1) \leq z(q^2)$ where $z(q^i)$ denotes the unique solution of (q^i, M) for $i = 1, 2$.

Proof. Suppose that $M \in \boldsymbol{K}$ and $q^1 \geq q^2$. Then, it is easy to see that $z(q^2) \in$ FEA(q^1, M). By Corollary **3.11.8**, $z(q^1)$ is the least-element solution of LCP (q^1, M). Hence, $z(q^1) \leq z(q^2)$.

Conversely, suppose the solution function $z(q)$ has the antitonicity property as given. If $m_{ij} > 0$ for some $i \neq j$, define

$$q^1 = e_j - M_{\bullet j}, \quad \text{and} \quad q^2 = -M_{\bullet j}.$$

Clearly, $z(q^2) = e_j$ and $q^1 \geq q^2$. Hence, $z(q^1) \leq e_j$. Now, by the same argument as the proof of Theorem **3.11.7**, we may easily deduce a contradiction. □

Besides the characterization stated in **3.11.8** and **3.11.9**, the class of $\boldsymbol{K}$-matrices admits a wide variety of other useful descriptions. We list several of these in the result below.

3.11.10 Theorem. Let $M \in R^{n\times n}$ be a $\boldsymbol{Z}$-matrix. The following statements are equivalent:

(a) $M \in \boldsymbol{K}$.

(b) All leading principal minors of M are positive.

(c) M^{-1} exists and is nonnegative.

(d) $M \in \boldsymbol{S}$.

(e) $M \in \bar{\boldsymbol{S}}$.

Moreover, any of the above conditions (a) – (e) is further equivalent to each of the conditions in Theorems **3.3.4** and **3.10.7**.

Proof. (a) $\Rightarrow$ (b). This is obvious.

(b) $\Rightarrow$ (c). This is proved by induction on n. Clearly, the implication holds for $n = 1$. Suppose that it holds for all $\boldsymbol{Z}$-matrices of order less than n. Let M be an $n \times n$ $\boldsymbol{Z}$-matrix with the property that all its leading principal minors are positive. Clearly, M must be nonsingular. It remains to show that the inverse of M is nonnegative. With $\alpha = \{1, \ldots, n-1\}$, write M as

$$M = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha n} \\ M_{n\alpha} & m_{nn} \end{bmatrix}.$$

By induction, $M_{\alpha\alpha}^{-1}$ exists and is nonnegative. If, for ease of notation, we let $\rho = m_{nn} - M_{n\alpha}M_{\alpha\alpha}^{-1}M_{\alpha n}$, then it is straightforward to check

$$\rho M^{-1} = \begin{bmatrix} \rho M_{\alpha\alpha}^{-1} + M_{\alpha\alpha}^{-1}M_{\alpha n}M_{n\alpha}M_{\alpha\alpha}^{-1} & -M_{\alpha\alpha}^{-1}M_{\alpha n} \\ -M_{n\alpha}M_{\alpha\alpha}^{-1} & 1 \end{bmatrix}. \tag{3}$$

Now ρ is the Schur complement of $M_{\alpha\alpha}$ in M which, by the Schur determinantal formula (2.3.14), is equal to $\det M / \det M_{\alpha\alpha}$. By assumption, both determinants in this ratio are positive, so $\rho > 0$. As $M \in \boldsymbol{Z}$, both $M_{\alpha n}$ and $M_{n\alpha}$ are nonpositive. Thus, (3) shows M^{-1} to be nonnegative, completing the induction.

(c) $\Rightarrow$ (d). This is obvious because with p taken as any positive vector, the vector $x = M^{-1}p$ must satisfy $x \geq 0$ and $Mx > 0$. Noting (3.1.10), the result follows.

(d) $\Rightarrow$ (e). This is also easy. If x is any vector satisfying $x > 0$ and $Mx > 0$, and if $M_{\alpha\alpha}$ is any principal submatrix of M, then as $M_{\alpha\bar{\alpha}}$ is nonpositive we see that x_α satisfies the required condition for $M_{\alpha\alpha}$ to be an $\boldsymbol{S}$-matrix.

(e) $\Rightarrow$ (a). This can be proved by induction. However, we give a proof that reveals an interesting property of the solution set of an LCP with a $\boldsymbol{Z}$-matrix. Suppose M is a $\boldsymbol{Z}$-matrix which belongs to the class $\bar{\boldsymbol{S}}$. It suffices to show that for every vector q, the LCP (q, M) has a unique solution (see **3.3.7**). Proposition **3.1.5** guarantees that (q, M) is feasible for all q. Since $M \in \boldsymbol{Z}$, Theorem **3.11.6** implies that (q, M) has a least element solution u. We must show u to be the unique solution. Indeed, if z is another solution, then $z \geq u$. Moreover, it is easy to see that the vector $z - u$ is in fact a solution of the LCP (v, M) where $v = q + Mu \geq 0$. By Theorem **3.10.7**, this latter LCP has a unique solution, namely zero. This establishes the uniqueness of u.

The theorem's last assertion is obvious. □

3.11.11 Remark. Theorem **3.11.10** shows that M is in $\boldsymbol{H}$ if and only if the comparison matrix associated with M is in $\boldsymbol{K}$ (see **3.3.11** and **3.3.12**).

As the proof of [(e) ⇒ (a)] in Theorem **3.11.10** shows, the solution set of the LCP (q, M) with a $\boldsymbol{Z}$-matrix M has a rather interesting property. This is formally stated in the next result (see **3.12.29**).

3.11.12 Proposition. Let M be a $\boldsymbol{Z}$-matrix. If (q, M) is a feasible LCP, then

$$\mathrm{SOL}(q, M) = u + \mathrm{SOL}(v, M) \tag{4}$$

where u is the least-element solution of (q, M) and $v = q + Mu$. □

3.11.13 Remark. The representation (4) is noteworthy as $\mathrm{SOL}(v, M)$ is the solution set of an LCP in which the constant column is a *nonnegative* vector.

Theorem **3.11.10** has many implications. For example, the following is an immediate consequence of this result.

3.11.14 Corollary. Let M be a $\boldsymbol{K}$-matrix. If $M_{\alpha\alpha}$ is a principal submatrix of M, then the Schur complement $(M/M_{\alpha\alpha})$ is a $\boldsymbol{K}$-matrix.

Proof. Since $\boldsymbol{K}$ is a complete class, $M_{\alpha\alpha}$ is a $\boldsymbol{K}$-matrix. Thus, the Schur complement $N = (M/M_{\alpha\alpha}) = M_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha}M_{\alpha\alpha}^{-1}M_{\alpha\bar{\alpha}}$ is well-defined. By **3.11.10(c)**, $M_{\alpha\alpha}^{-1}$ is nonnegative. As $M_{\alpha\bar{\alpha}}$ and $M_{\bar{\alpha}\alpha}$ are nonpositive, it follows that $N \in \boldsymbol{Z}$. Finally, if x and y are positive vectors such that $Mx = y$ (such a pair (x, y) must exist by **3.11.10(d)**), then

$$Nx_{\bar{\alpha}} = y_{\bar{\alpha}} - M_{\bar{\alpha}\alpha}M_{\alpha\alpha}^{-1}y_{\alpha} > 0.$$

This completes the proof. □

3.11.15 Remark. If $M \in \boldsymbol{Z}$ and if the principal submatrix $M_{\alpha\alpha}$ is nonsingular with a nonnegative inverse, then the Schur complement $(M/M_{\alpha\alpha})$ remains a $\boldsymbol{Z}$-matrix, although it need not be a $\boldsymbol{K}$-matrix.

We illustrate another application of Theorem **3.11.10** in the context of testing for membership in some matrix classes. Given an arbitrary matrix $M \in R^{n\times n}$, the test of whether or not $M \in \boldsymbol{P}$ can be accomplished by evaluating the principal minors of M. In general, there are 2^n such minors. So this test, albeit finite, is in practice not very effective. When $M \in \boldsymbol{Z}$, this test can be drastically simplified due to condition (b) in Theorem **3.11.10**. The same conclusion can be made with regard to testing for membership in any one of the classes in Theorem **3.10.7**.

Hidden $\boldsymbol{Z}$-matrices and cone orderings

Theorem **3.11.7** shows that the class of $\boldsymbol{Z}$-matrices delimits precisely the least-element approach (least under the usual componentwise ordering in R^n) for the existence of a solution to the LCP. In the sequel, we generalize this approach somewhat by considering partial orderings of R^n induced by simplicial cones.

In general, let $C \subseteq R^n$ be a pointed convex cone. C induces a partial ordering $\preceq_C$ on R^n defined as follows: $x \preceq_C y$ if the vector $y - x$ belongs to C. We define the *least element* u of a set S in R^n, least with respect to this ordering, as an element $u \in S$ such that $u \preceq_C y$ for all vectors $y \in S$. When C is a simplicial cone, say, $C = \text{pos}\, A$ where A is some $n \times n$ nonsingular matrix, then $x \preceq_C y$ if and only if $A^{-1}x \le A^{-1}y$. In other words, the ordering $\preceq_C$ can be thought of as the usual componentwise ordering in a linearly transformed copy of R^n. Consequently, in order to generalize Theorem **3.11.7** to the cone ordering $\preceq_C$, we need to define the notion of a "transformed" $\boldsymbol{Z}$-matrix. This consideration leads to the following definition.

3.11.16 Definition. A matrix $M \in R^{n\times n}$ is called *hidden* $\boldsymbol{Z}$ if there exist $\boldsymbol{Z}$-matrices X and Y and nonnegative vectors r and s such that

(i) $MX = Y$,

(ii) $r^{\mathrm{T}}X + s^{\mathrm{T}}Y > 0$.

A hidden $\boldsymbol{Z}$-matrix which belongs to $\boldsymbol{P}$ is called a *hidden* $\boldsymbol{K}$*-matrix*.

The matrix X in the defining condition of a hidden $\boldsymbol{Z}$-matrix serves to describe the needed transformation of the feasible region of the LCP (q, M)

into a meet semi-sublattice. In order to explain this, we derive a property of a hidden $\boldsymbol{Z}$-matrix.

3.11.17 Theorem. If $M \in R^{n\times n}$ is a hidden $\boldsymbol{Z}$-matrix, and X and Y are any two $\boldsymbol{Z}$-matrices satisfying the conditions in **3.11.16**, then

(i) X is nonsingular, and

(ii) there exists an index set α such that the matrix

$$\begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ Y_{\bar{\alpha}\alpha} & Y_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \tag{5}$$

is in $\boldsymbol{K}$.

Proof. Let $p = X^{\mathrm{T}}r + Y^{\mathrm{T}}s$ where r and s are as given in **3.11.16**. Thus, with $A = (X^{\mathrm{T}}, Y^{\mathrm{T}}) \in R^{n\times 2n}$, the system of linear inequalities

$$Ax = p, \ \ x \geq 0$$

has a solution. It follows from linear programming theory that the system has a basic feasible solution. A basic feasible solution must have no more than n positive variables. However, as $p > 0$ and each column of A has at most one positive element, a basic feasible solution will have exactly n positive variables. Furthermore, as X and Y are $\boldsymbol{Z}$-matrices, it is easy to see that if B is a basis in A corresponding to a basic feasible solution, then B^{T} is of the form (5) for some index set α. Thus, B is a $\boldsymbol{Z}$-matrix and, as B is a feasible basis, **3.11.10** implies that B (and thus B^{T}) is a $\boldsymbol{K}$-matrix. This establishes (ii). To prove (i), suppose that $Xv = 0$. We must have $Yv = 0$. Consequently, if W denotes the matrix in (5), then $Wv = 0$ which implies that $v = 0$. This completes the proof. □

The property (i) in **3.11.17** can be used to explain the word "hidden" in the term "hidden $\boldsymbol{Z}$". Roughly speaking, a hidden $\boldsymbol{Z}$-matrix is a matrix, which although is not a $\boldsymbol{Z}$-matrix itself, can be converted into one by means of a linear transformation which has a certain $\boldsymbol{Z}$-property. This transformation is the key to the proof of the following generalization of Theorem **3.11.7**.

3.11.18 Theorem. The matrix $M \in R^{n\times n}$ is hidden $\boldsymbol{Z}$-matrix if and only if there exists a simplicial cone C in R^n such that for all vectors $q \in \text{pos}\,(I, -M)$, the feasible region of (q, M) contains a least element $\tilde{z}$ with respect to $\preceq_C$ and $\tilde{z}$ satisfies $\tilde{z}^{\mathrm{T}}(q + M\tilde{z}) = 0$.

Proof. Necessity. Suppose M is hidden $\boldsymbol{Z}$; let X and Y be the two $\boldsymbol{Z}$-matrices in **3.11.16**. Let W be the matrix in (5). By **3.11.10**, W^{-1} is nonnegative. Let (q, M) be a feasible LCP. Consider the set

$$S = \{v \in R^n : Xv \geq 0,\ q + Yv \geq 0\}. \tag{6}$$

A vector $z \in \text{FEA}(q, M)$ if and only if the vector $v = X^{-1}z \in S$ (X^{-1} exists by **3.11.17(i)**). Moreover, since X and Y are $\boldsymbol{Z}$-matrices, by the same argument as in **3.11.3**, it is easy to show that S is a meet semi-sublattice. Furthermore, if $v \in S$, then $\tilde{q} + Wv \geq 0$ where

$$\tilde{q} = \begin{bmatrix} 0 \\ q_{\bar{\alpha}} \end{bmatrix}.$$

Thus, as $W^{-1} \geq 0$, the set S is bounded below by $-W^{-1}\tilde{q}$. Therefore, by **3.11.5**, S has a least element $\tilde{v}$ with respect to the componentwise ordering of R^n. It follows that the vector $\tilde{z} = X\tilde{v}$ is the least element of the feasible region of (q, M) with respect to the cone ordering $\preceq_C$ where $C = \text{pos}\,X$. By the same argument as in **3.11.6**, one can show that the vector $\tilde{v}$ satisfies the complementarity condition $(X\tilde{v})^{\mathrm{T}}(q+Y\tilde{v}) = 0$. Consequently, $\tilde{z}$ satisfies the desired property $\tilde{z}^{\mathrm{T}}(q + M\tilde{z}) = 0$.

Sufficiency. Let $C = \text{pos}\,X$ be the simplicial cone, where $X \in R^{n\times n}$ is nonsingular. (X is not necessarily a $\boldsymbol{Z}$-matrix.) For $k \in \{1, \ldots, n\}$, let $q^k = e_k - Me_k$. Clearly, $z = e_k \in \text{FEA}(q^k, M)$, so there exists a least element $\tilde{z}^k$ of the feasible region of (q^k, M) under the $\preceq_C$ ordering. Furthermore, $\tilde{z}^k$ solves the LCP (q^k, M). Notice that $z = e_k$, while feasible, is not a solution to the LCP (q^k, M). Thus, $X^{-1}\tilde{z}^k \leq X^{-1}e_k$ and $\tilde{z}^k \neq e_k$. Letting $v^k = X^{-1}(e_k - \tilde{z}^k)$, we have $0 \neq v^k \geq 0$. Let $Y = MX$. For $i \in \{1, \ldots, n\} \setminus \{k\}$, we see

$$X_{i\bullet}v^k = (e_k - \tilde{z}^k)_i \leq 0\,,$$
$$Y_{i\bullet}v^k = (Me_k - M\tilde{z}^k)_i = -(q^k + M\tilde{z}^k)_i \leq 0\,.$$

Letting $W \in R^{n\times n}$ be defined by having $W_{\bullet k} = v^k$ for all $k \in \{1, \ldots, n\}$, it follows that $\tilde{X} = XW$ and $\tilde{Y} = M\tilde{X} = YW$ are $\boldsymbol{Z}$-matrices. We can show M to be a hidden $\boldsymbol{Z}$-matrix if we can find nonnegative vectors r and s such that $\tilde{X}^{\mathrm{T}}r + \tilde{Y}^{\mathrm{T}}s > 0$. To this end, consider the linear program

$$\begin{array}{ll} \text{minimize} & e^{\mathrm{T}}v \\ \text{subject to} & Xv \geq 0 \\ & Yv \geq 0. \end{array} \tag{7}$$

Notice v is feasible for (7) if and only if Xv is feasible for the LCP $(0, M)$. Since the latter is a feasible LCP, it must have a least element under the $\preceq_C$ ordering and, thus, (7) has an optimal solution. This implies that the dual of the linear program has an optimal (hence feasible) solution. Therefore, there exist nonnegative vectors r and s such that $X^{\mathrm{T}}r + Y^{\mathrm{T}}s = e$. As $W \geq 0$ and no column of W is zero, we have

$$\tilde{X}^{\mathrm{T}}r + \tilde{Y}^{\mathrm{T}}s = W^{\mathrm{T}}(X^{\mathrm{T}}r + Y^{\mathrm{T}}s) = W^{\mathrm{T}}e > 0\,.$$

□

To illustrate Theorem **3.11.18**, the feasible region of an LCP (q, M) with M a hidden $\boldsymbol{Z}$-matrix, along with the least-element solution $\tilde{z}$, is shown in Figure 3.2. The set S (see (6)) for this LCP, along with the least element $\tilde{v}$, is shown in Figure 3.3.

We should point out that in the definition of a hidden $\boldsymbol{Z}$-matrix, the matrix X is required to be $\boldsymbol{Z}$; however, the cone C in **3.11.18** is not required to be generated by a $\boldsymbol{Z}$-matrix. The implication of the theorem is that if the least-element property holds with respect to any simplicial cone, then the same property must hold with respect to one such cone that is generated by a $\boldsymbol{Z}$-matrix.

As in the case of a $\boldsymbol{Z}$-matrix, the least-element solution $\tilde{z}$ in **3.11.18** can be obtained by solving the linear program (2) with any vector p satisfying the condition

$$p^{\mathrm{T}}X > 0$$

where X is the matrix of generators for the cone C. This follows rather easily from the technique by which the least element $\tilde{v}$ of the set S in (6) is obtained and from the relation $\tilde{z} = X\tilde{v}$.

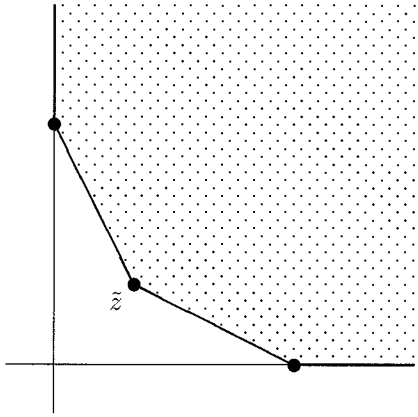

Figure 3.2

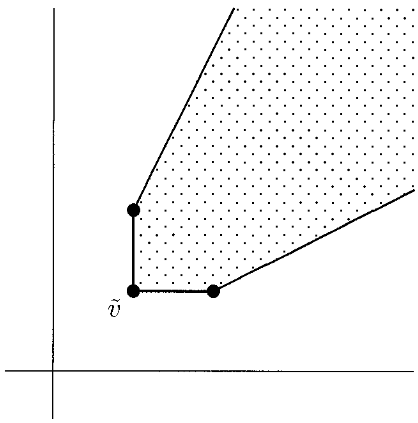

Figure 3.3

Theorem **3.11.10** also admits a generalization to a hidden $\boldsymbol{Z}$-matrix. This is stated below.

3.11.19 Theorem. Let $M \in R^{n\times n}$ be a hidden $\boldsymbol{Z}$-matrix. Let X and Y be any two $\boldsymbol{Z}$-matrices as given in **3.11.16**. The following statements are equivalent:

(a) M is hidden $\boldsymbol{K}$.

(b) $M \in \boldsymbol{S}$.

(c) There exists a vector $v > 0$ such that, for any index set $\alpha \subseteq \{1, \ldots, n\}$, $Wv > 0$ where

$$W = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\bar{\alpha}} \\ Y_{\bar{\alpha}\alpha} & Y_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}. \tag{8}$$

In particular, any such matrix W is in $\boldsymbol{K}$.

(d) M is completely hidden $\boldsymbol{K}$, i.e., for every index set $\alpha \subseteq \{1, \ldots, n\}$, $M_{\alpha\alpha}$ is hidden $\boldsymbol{K}$.

(e) $M \in \bar{\boldsymbol{S}}$.

Moreover, any of the above conditions (a) – (e) is further equivalent to each of the conditions in Theorems **3.3.4** and **3.10.7**.

Proof. (a) $\Rightarrow$ (b). This is obvious because any $\boldsymbol{P}$-matrix must be an $\boldsymbol{S}$-matrix by **3.3.5**.

(b) $\Rightarrow$ (c). Suppose that $M \in \boldsymbol{S}$. Let $x > 0$ be such that $Mx > 0$. Let $v = X^{-1}x$ (X^{-1} exists by **3.11.17(i)**). We have $Xv > 0$ and $Yv > 0$. If we can show $v > 0$, then $Wv > 0$ for any W as given in (8), and (c) would follow from **3.11.10**. To show that v is positive, we consider the particular matrix $W \in \boldsymbol{K}$, as given in **3.11.17(ii)**. We have $Wv > 0$ for this particular W. Furthermore, by **3.11.10**, W^{-1} is nonnegative. Thus, since no row of W^{-1} can equal zero, we have $v > 0$.

(c) $\Rightarrow$ (d). For any $\alpha \subseteq \{1, \ldots, n\}$, let W be as in (8). Clearly, $X_{\alpha\alpha}$ is in $\boldsymbol{K}$. Using **3.11.14** we see that $(X/X_{\alpha\alpha}) \in \boldsymbol{K}$ and $(W/X_{\alpha\alpha}) \in \boldsymbol{K}$. From the relation $MX = Y$, it is easy to deduce

$$M_{\bar{\alpha}\bar{\alpha}}(X/X_{\alpha\alpha}) = (W/X_{\alpha\alpha}). \tag{9}$$

We immediately see that $\det M_{\bar{\alpha}\bar{\alpha}} > 0$. As α was arbitrary, all principal submatrices of M are in $\boldsymbol{P}$. If we can find nonnegative vectors $r_{\bar{\alpha}}$ and $s_{\bar{\alpha}}$

such that $r_{\bar{\alpha}}^{\mathrm{T}}(X/X_{\alpha\alpha}) + s_{\bar{\alpha}}^{\mathrm{T}}(W/X_{\alpha\alpha}) > 0$, then $M_{\bar{\alpha}\bar{\alpha}}$ is hidden $\boldsymbol{K}$ and (d) would follow. To find such vectors, we note that as $(X/X_{\alpha\alpha}) \in \boldsymbol{K}$ then **3.11.10** implies the transpose of $(X/X_{\alpha\alpha})$ is in $\boldsymbol{S}$. Thus, there exists a nonnegative vector $r_{\bar{\alpha}}$ such that $r_{\bar{\alpha}}^{\mathrm{T}}(X/X_{\alpha\alpha}) > 0$. Letting $s_{\bar{\alpha}} = 0$ gives us the desired vectors.

(d) $\Rightarrow$ (e). This is the same as [(a) $\Rightarrow$ (b)].

(e) $\Rightarrow$ (a). We have already shown that [(b) $\Rightarrow$ (c)] and [(c) $\Rightarrow$ (d)]. It follows that [(b) $\Rightarrow$ (d)] which is a stronger statement than [(e) $\Rightarrow$ (a)].

As in **3.11.10**, the theorem's last assertion is obvious. This completes the proof. □

Theorems **3.11.19** and **3.10.7** together imply that the transpose of a hidden $\boldsymbol{K}$-matrix is in $\bar{\boldsymbol{S}}$. However, unlike the case of a $\boldsymbol{K}$-matrix, the transpose of a hidden $\boldsymbol{K}$-matrix is not necessarily hidden $\boldsymbol{K}$ (see the example in **3.11.20**). Interestingly, the transpose of a hidden $\boldsymbol{K}$-matrix plays a very important role in the efficient solution of the LCP by pivoting algorithms (see Section 4.8).

3.11.20 Example. The matrix

$$M = \begin{bmatrix} 2 & -3 & -4 \\ -1 & 2 & 3 \\ 2 & -2 & 5 \end{bmatrix}$$

is hidden $\boldsymbol{K}$, but its transpose is not hidden $\boldsymbol{Z}$. The proof is left as an exercise.

Given an arbitrary square matrix M, it is in general not easy to test whether or not M is hidden $\boldsymbol{Z}$. The difficulty is due to condition (ii) in **3.11.16** which is nonlinear in the unknowns X, Y, r and s. However, the same condition (ii) can be replaced by a linear one if we wish to test for membership in the class of hidden $\boldsymbol{K}$-matrices. Indeed, according to **3.11.10** and **3.11.19**, it is easy to see that a matrix M is hidden $\boldsymbol{K}$ if and only if $M \in \boldsymbol{S}$ and there exists a $\boldsymbol{Z}$-matrix X such that (i) $MX \in \boldsymbol{Z}$ and (ii′) $Xe > 0$. Since the conditions (i) and (ii′) are linear in X, the question of whether a given matrix M is hidden $\boldsymbol{K}$ can be answered by solving two linear programs: one to determine if $M \in \boldsymbol{S}$ and the other to determine if the required X matrix exists.

In concluding this chapter, we remark that the class of $\boldsymbol{H}$-matrices with positive diagonal elements is a subclass of hidden $\boldsymbol{K}$-matrices. (It is a proper subclass as can be seen from the matrix M in Example **3.11.20** which is not in $\boldsymbol{H}$.) The reader is asked to supply the proof for this asserted inclusion in Exercise **3.12.32**.

3.12 Exercises

3.12.1 Show that if $M \in R^{n\times n}$ is positive semi-definite, then M is copositive-plus.

3.12.2 Show that if $M \in R^{n\times n}$ is symmetric and copositive-star, then M is copositive-plus.

3.12.3 Positive semi-definite matrices are both copositive-plus and sufficient. Are there matrices which are both copositive-plus and sufficient, but not positive semi-definite?

3.12.4 Let $M \in R^{n\times n}$ be symmetric and copositive-plus. Let $q \in R^n$.

(a) Show that if there is some vector v such that $q \geq Mv$, then the LCP (q, M) has a solution.

(b) Generalize part (a) to show that if N is a copositive matrix and if the system

$$q + Mx - N^{\mathrm{T}}y \geq 0, \quad y \geq 0$$

is consistent, then $\mathrm{SOL}(q, M + N) \neq \emptyset$.

(c) Let $A, B \in R^{m\times k}$ with $B \geq 0$. Show that if $Q \in R^{m\times m}$ is symmetric positive semi-definite and $\alpha \in (0, 1]$, then the matrix

$$\tilde{M} = \begin{bmatrix} Q & B - A \\ A^{\mathrm{T}} - \alpha B^{\mathrm{T}} & 0 \end{bmatrix}$$

can be written in the form $M + N$ where M is symmetric positive semi-definite and N is copositive. Note, the matrix arising from the optimal invariant capital stock problem (see (1.2.17)) is of this form.

3.12.5 Prove the following two statements about the mixed LCP (1.5.1).

(a) The following conditions are sufficient for this mixed LCP to have a solution for all vectors $a \in R^n$ and $b \in R^m$: (i) A is nonsingular; (ii) the Schur complement $B - DA^{-1}C$ is a $\boldsymbol{Q}$-matrix.

(b) The following conditions are necessary and sufficient for this mixed LCP to have a unique solution for all vectors $a \in R^n$ and $b \in R^m$: (i) A is nonsingular; (ii) the Schur complement $B - DA^{-1}C$ is a $\boldsymbol{P}$-matrix.

3.12.6 Let K be a nonempty closed convex set in R^n. Let $\Pi_K(x)$ denote the projection of the vector x onto the set K under the l_2-norm. Show that a vector z^* solves the problem VI(K, f) if and only if z^* is a fixed point of the mapping $F(z) = \Pi_K(z - f(z))$.

3.12.7 Let $M \in R^{n \times n}$ be a copositive matrix and let $u \geq 0$ be given. Prove there exists a solution to the inequality system

$$Mz \geq -M^{\mathrm{T}}u, \quad z \geq 0.$$

3.12.8 Show that if $M \in R^{n \times n}$ is copositive but not copositive-plus, then $x^{\mathrm{T}}Mx > 0$ for all $x > 0$.

3.12.9 Suppose $M \in R^{n \times n}$ is copositive. Show that if there exists a vector $\bar{x} > 0$ such that $\bar{x}^{\mathrm{T}}M\bar{x} = 0$, then M is positive semi-definite.

3.12.10 Suppose that $M \in R^{n \times n}$ is copositive. Show that the implication

$$[\, v \geq 0,\ Mv \geq 0,\ v^{\mathrm{T}}Mv = 0 \,] \quad \Rightarrow \quad [\, v^{\mathrm{T}}q \geq 0 \,]$$

is valid if and only if the function $f(z) = z^{\mathrm{T}}(q + Mz)$ is bounded below on the set $\{z \in R^n_+ : Mz \geq 0\}$.

3.12.11 Let $M \in R^{n \times n}$ be an $\boldsymbol{E}_0$-matrix. Show that if there exists a vector $z > 0$ such that $Mz = 0$, then there exists a nonzero vector $y \geq 0$ such that $y^{\mathrm{T}}M = 0$. Show if M is a row adequate matrix, then $M \in \boldsymbol{L}$.

3.12.12 Let $M \in R^{n \times n}$ be a column adequate matrix. Show that the following implication holds for each $\alpha \subseteq \{1, \ldots, n\}$

$$[\, M_{\alpha\alpha} y_\alpha = 0 \,] \quad \Rightarrow \quad [\, M_{\cdot\alpha} y_\alpha = 0 \,].$$

3.12.13 Prove that Corollary **3.8.12** is true.

3.12.14 With reference to Definition **3.9.5**, which of the following matrix classes are complete? Positive definite; column adequate; column sufficient; nondegenerate; copositive-star; copositive-plus; $\boldsymbol{H}$; $\boldsymbol{Q_0}$; $\boldsymbol{R_0}$; $\boldsymbol{R}$.

3.12.15 When considered as sets in $R^{n\times n}$, which of the following matrix classes are open? Positive definite; column sufficient; nondegenerate; copositive-star; $\boldsymbol{Q_0}$; $\boldsymbol{Q}$; $\boldsymbol{R_0}$; $\boldsymbol{R}$.

3.12.16 Show that the matrix

$$M = \begin{bmatrix} 0 & -1 & 2 \\ 2 & 0 & -2 \\ -1 & 1 & 0 \end{bmatrix}$$

is sufficient.

3.12.17 Show that if $M \in R^{n\times n}$ is positive, nonnegative, copositive, copositive-plus, strictly copositive, positive semi-definite, row sufficient, or column sufficient, then so, respectively, is

$$\tilde{M} = \begin{bmatrix} M & M \\ M & M \end{bmatrix}.$$

3.12.18 Prove that Theorem **3.9.11** is true.

3.12.19 Prove that Lemma **3.9.12** is true.

3.12.20 Prove that Proposition **3.9.14** is true.

3.12.21 If $M \in R^{n\times n}$ is symmetric and copositive-plus, show that there exist $A, B \in R^{n\times n}$ such that:

(a) $M = A + B$,

(b) A is symmetric and copositive,

(c) B is symmetric and positive semi-definite,

(d) the nullspace of B equals the nullspace of M.

3.12.22 Let $M \in R^{n\times n}$ and $q \in R^n$ be given. Let X be a convex subset of SOL(q, M).

(a) Show that if $z^i \in X$ and $w^i = q + Mz^i$ for $i = 1, 2$, then

$$(z^1)^T w^2 = (z^2)^T w^1 = 0.$$

(b) Suppose that M is symmetric. Let $f(z) = q^T z + \frac{1}{2} z^T M z$. Show that $f(z)$ is a constant for $z \in X$.

(c) Deduce from part (b) that if M is a symmetric matrix, then the quadratic function $f(z)$ attains only a finite number of distinct values on SOL(q, M).

3.12.23 Given $M \in R^{n\times n}$ we define a matrix $H_\alpha \in R^{2n\times 2n}$, for each scalar α, as follows

$$H_\alpha = \alpha \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} + \begin{bmatrix} M^T M & -M^T \\ -M & I \end{bmatrix}.$$

Show that if M is strictly semimonotone, then H_α is strictly copositive for all $\alpha > 0$.

3.12.24 Given a matrix $M \in R^{n\times n}$, consider the $2n \times 2n$ matrix

$$M' = \begin{bmatrix} M & I \\ -I & D \end{bmatrix}.$$

where $D \in R^{n\times n}$ is a diagonal matrix with positive diagonal elements. Show that if $M \in \boldsymbol{E_0}$, then $M' \in \boldsymbol{E_0} \cap \boldsymbol{R_0}$.

3.12.25 Prove that $\boldsymbol{R} \subseteq (\text{int}\, \boldsymbol{Q}) \cap \boldsymbol{R_0}$. Is this inclusion proper?

3.12.26 Let $M \in \boldsymbol{R_0}$. Suppose that for some vector q, the LCP (q, M) has a unique solution that is nondegenerate. Show that there exists a principal pivotal transform of M that belongs to the matrix class $\boldsymbol{R}$. (Hence, M is a $\boldsymbol{Q}$-matrix.)

3.12.27 Let $M \in R^{n\times n}$ be given. Show that $M \in \boldsymbol{P_0}$ $(\boldsymbol{P})$ if and only if for every pair of disjoint index sets α, $\beta \subseteq \{1, \ldots, n\}$, whose union is nonempty, the matrix

$$\begin{bmatrix} M_{\alpha\alpha} & -M_{\alpha\beta} \\ -M_{\beta\alpha} & M_{\beta\beta} \end{bmatrix}$$

is (strictly) semimonotone. What is the analog of this result for a positive semi-definite matrix M?

3.12.28 Let $M \in R^{n\times n}$ be a symmetric and nonsingular matrix. Show that M is positive definite if and only if both M and M^{-1} are strictly copositive.

3.12.29 Give a proof for Proposition **3.11.12**.

3.12.30 Suppose $M \in R^{n\times n}$ is hidden $\boldsymbol{Z}$. Show that if for some index set $\alpha \subseteq \{1, \ldots, n\}$ the matrix $M_{\alpha\alpha}$ is nonsingular, then the principal pivotal transform of $M_{\alpha\alpha}$ in M, i.e.,

$$\begin{bmatrix} M_{\alpha\alpha}^{-1} & -M_{\alpha\alpha}^{-1}M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha}M_{\alpha\alpha}^{-1} & M_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha}M_{\alpha\alpha}^{-1}M_{\alpha\bar{\alpha}} \end{bmatrix},$$

is hidden $\boldsymbol{Z}$.

3.12.31 Show that the matrix M given in Example **3.11.20** is hidden $\boldsymbol{K}$, but that M^{T} is not hidden $\boldsymbol{Z}$.

3.12.32 Let $A \in \boldsymbol{Z} \cap R^{n\times n}$ and $B \in \boldsymbol{K} \cap R^{n\times n}$, with $A \geq B$.

(a) Show that the matrix $M = 2A - B$ is a hidden $\boldsymbol{K}$-matrix.

(b) Let $M \in \boldsymbol{H} \cap R^{n\times n}$ have positive diagonal elements. Show that there exist matrices A and B satisfying the given assumptions such that $M = 2A - B$. Deduce that any $\boldsymbol{H}$-matrix with positive diagonals is a hidden $\boldsymbol{K}$-matrix.

3.12.33 Prove that every copositive $\boldsymbol{Z}$-matrix is positive semi-definite.

3.12.34 Consider the convex quadratic program

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^{\mathrm{T}}Qx + c^{\mathrm{T}}x \\ \text{subject to} & Ax \geq b \end{array}$$

where $Q \in R^{n\times n}$ is symmetric and positive semi-definite. Suppose the problem is feasible. Show that the statements below are equivalent.

(a) The objective function is bounded below on the feasible set.

(b) The implication holds:

$$[Av \geq 0,\ Qv = 0] \quad \Rightarrow \quad c^{\mathrm{T}}v \geq 0.$$

(c) The vector c belongs to $\mathcal{R} + \operatorname{pos} A^{\mathrm{T}}$ where $\mathcal{R}$ is the column space of Q.

3.12.35 Consider the convex quadratic program given in **3.12.34**. Let P denote the feasible region, and θ the objective function. For each scalar $\tau \in R$, define the level set

$$L(\tau) = \{x \in P : \theta(x) \leq \tau\}.$$

Show that if $L(\tau) \neq \emptyset$, then

$$0^+L(\tau) = \{v \in R^n : Av \geq 0,\ Qv = 0,\ c^{\mathrm{T}}v \leq 0\}.$$

Deduce from this representation that the three statements below are equivalent.

(a) The level set $L(\tau)$ is nonempty and bounded for at least one τ.

(b) The level set $L(\tau)$ is bounded for every τ for which $L(\tau)$ is nonempty.

(c) The implication holds:

$$[Av \geq 0,\ Qv = 0,\ v \neq 0] \quad \Rightarrow \quad [c^{\mathrm{T}}v > 0].$$

3.13 Notes and References

3.13.1 Perhaps the earliest existence result for the linear complementarity problem appears in the paper by Samelson, Thrall and Wesler (1958). Motivated by an engineering application, their work was concerned with a geometric problem which can be expressed as a complementarity problem of the form (1.5.2). In the case of the standard LCP, their result yields Theorem **3.3.7**.

3.13.2 A significant portion of the analytic approach to the existence theory presented in this chapter is based upon the association with quadratic programming. One of the important theorems in the early history of quadratic programming is that of Frank and Wolfe (1956). This theorem is the basis for our derivation of all the existence results in the first five sections of this chapter.

3.13.3 The existence results in Section 3.1 are largely due to Dorn (1961) and Cottle (1963, 1964b). For positive semi-definite M, the polyhedral representation of SOL(q, M) stated in Theorem **3.1.7** was pointed out by Adler and Gale (1975). The positive semi-definite LCP is closely related to monotone operator theory. See Minty (1962) and Brézis (1973).

3.13.4 The fundamental matrix classes $\boldsymbol{Q}_0$ and $\boldsymbol{Q}$ were introduced by Parsons (1970) and Murty (1972), respectively. In Parsons' work, however, the class $\boldsymbol{Q}_0$ was denoted $\mathcal{K}$. As noted in Section 3.2, there is no known method for efficiently testing a matrix for membership in one of these classes. Aganagić and Cottle (1978) (see also Cottle (1980a)) describes an inefficient test for membership in $\boldsymbol{Q}$ originally conceived by Gale. This and a finite characterization for $\boldsymbol{Q}_0$-matrices can be found in Murty (1988). Proposition **3.2.1** on the characterization of $M \in \boldsymbol{Q}_0$ in terms of the convexity of $K(M)$ was observed by Eaves (1971a). Watson (1974) described some non-$\boldsymbol{Q}$ matrices through forbidden sign variations of principal minors.

3.13.5 The name "$\boldsymbol{P}$-matrix" and the term "sign reversing" were coined by Gale and Nikaido (1965). The class $\boldsymbol{P}$ had previously been investigated by Fiedler and Pták (1962) who proved Theorem **3.3.4**. Our proof of the uniqueness part of (the Samelson-Thrall-Wesler) Theorem **3.3.7** follows Murty (1972). The same theorem was independently discovered by Ingleton (1966).

3.13.6 Lyapunov (1947) introduced the notion of a (negative) stable matrix (with complex entries) in his study of solution stability of differential equations. His well known characterization of a (complex) stable matrix states that for a complex square matrix A, there exists a (Hermitian) negative definite matrix H such that $AH + HA^*$ is (Hermitian) positive definite if and only if all eigenvalues of A have negative real parts. The last conclusion in our Theorem **3.3.9** is a special case of this famous result. For a contempory review of matrix diagonal stability, see the article by Berman and Hershkowitz (1983).

3.13.7 The notion of an $\boldsymbol{H}$-matrix originates from a paper by Ostrowski (1937/1938) who introduced the class of "comparison matrices" H associated with a given matrix M and whose entries satisfy

$$|h_{ii}| \geq |m_{ii}|, \quad |h_{ij}| \leq -|m_{ij}| \quad (i \neq j).$$

The particular comparison matrix $\bar{M}$ defined in **3.3.12** is just a member of Ostrowski's broader class. Clearly, the $\boldsymbol{H}$-matrices are closely connected with the diagonally dominant matrices which, of course, have a long history.

3.13.8 Example **3.3.10** appears in Dantzig (1967). He points out that R.E. Kalman had identified another such example in 1962.

3.13.9 Parts (a), (b), and (c) of Theorem **3.4.2** appear in a paper by Fiedler and Pták (1966) where the classes $\boldsymbol{P}_0$, $\boldsymbol{P}$, $\boldsymbol{S}_0$ and $\boldsymbol{S}$ are studied as generalizations of positive definite matrices. Part (d) of this theorem is implied by a result in Willson (1971). Moré and Rheinboldt (1973) generalized the matrix classes $\boldsymbol{P}$ and $\boldsymbol{S}$ to nonlinear functions and discussed the connection with other mappings.

3.13.10 Adequate matrices were introduced by Ingleton (1966) in connection with a study of dynamical systems subject to smooth unilateral constraints; the distinction between row and column adequate matrices was made in his later paper, Ingleton (1970), as well as in Eaves (1971a). Ingleton (1970) also characterized the column adequate matrices in terms of w-uniqueness.

3.13.11 For the most part, Section 3.5 is drawn from the paper of Cottle, Pang and Venkatesawaran (1989). The name "sufficient" matrices was brought to mind by Ingleton's "adequate" matrices. Jansen (1983) discusses the structure of the solution set of a general LCP.

3.13.12 The equivalence between statements (a) and (b) in Theorem **3.6.3** was shown by Murty (1972) who first used the term "nondegenerate matrix" in the context of the LCP. Mangasarian (1980) established Theorem **3.6.5**. As a corollary, he derived the implication [(a) $\Rightarrow$ (c)] in **3.6.3**; the reverse implication [(c) $\Rightarrow$ (a)] was observed by Pang (1988).

3.13.13 Except for the generality of an arbitrary positive vector instead of e, the vector of all ones, the formulation of the augmented LCP given in (3.7.2) can be found in the celebrated article of Lemke (1965) who gave a constructive existence proof of Theorem **3.7.3**. (See Section 4.4.) Our proof of this theorem follows Eaves (1971a). In turn, Theorem **3.7.1** was first proved by Hartman and Stampacchia (1966).

3.13.14 Eaves (1971b) proved Theorem **3.7.9** and used it to establish a refinement of the Frank-Wolfe existence theorem for quadratic programming. In the same article, he also obtained a more general version of Proposition **3.7.14** which provided necessary and sufficient conditions for an arbitrary quadratic function to be bounded below on a polyhedron. Pang (1991a), proved Theorem **3.7.17** and used it in the context of interior-point methods. (See Section 5.9.)

3.13.15 It appears that (symmetric) copositive matrices were first studied in Motzkin (1952). There, the emphasis was on the associated quadratic form. The entrance of (asymmetric) copositive matrices into the LCP literature began with Lemke (1965) who made use of copositive-plus matrices, though not by that name. The name "copositive-plus" was introduced in Cottle and Dantzig (1968). The notion of copositive-star matrices is due to Gowda (1989b) who also proved Theorem **3.8.13**.

3.13.16 Starting with Motzkin (1952, 1965), many authors have considered the problem of testing a matrix for copositivity. The criterion given in Theorem **3.8.3** is due to Gaddum (1958). For others see Cottle, Habetler and Lemke (1970b), Pereira (1972), Väliaho (1986), and the references therein. Incidentally, results similar to Propositions **3.9.8** and **3.9.14** were established in the aforementioned work of Pereira.

3.13.17 The class $\boldsymbol{R}_0$ defined in **3.8.7** was first considered by Garcia (1973a, 1973b) under the name $E^*(0)$. Trivially, any nonnegative matrix with positive diagonal entries is an $\boldsymbol{R}_0$-matrix (in fact such a matrix is strictly copositive). The characterization of a nonnegative $\boldsymbol{Q}$-matrix, **3.8.15**, was proved by Murty (1972).

3.13.18 The strictly semimonotone matrices first appeared in Cottle and Dantzig (1968) without a name or a notation. Eaves (1971a) generalized this matrix class to one he denoted $\mathcal{L}_1$ and which we denote $\boldsymbol{E}_0$. The name "semimonotone matrices" was proposed by Karamardian (1972). Theorem **3.9.3** combines results in the cited papers of Eaves and Karamardian.

3.13.19 In essence, Corollary **3.9.13** was noted by Reiman and Williams (1988) whose paper deals with the subject of reflecting Brownian motions. For additional discussion of the latter topic and its connection with the LCP, see Mandelbaum (1989).

3.13.20 The class $\boldsymbol{L}$ defined in **3.9.18** was introduced by Eaves (1971a) who used Lemke's method (see Section 4.4) to prove Corollary **3.9.19**. Our treatment of this result, which relies on Theorem **3.9.16** and Corollary **3.9.17**, follows the approach in the paper by Gowda and Pang (1993) whose main objective is to convey the fundamental importance of Theorem **3.7.3** in the analytic approach for the existence of a solution to the LCP.

3.13.21 Karamardian (1972) defined the concept of "regular mapping" which, when specialized to the affine case, yields that of e-regular matrix. Actually, he used the term "regular" for these matrices. Soon thereafter, Garcia (1973b) developed the more general class of d-regular matrices which he denoted $E^*(d)$.

3.13.22 The characterization of $\boldsymbol{P_0} \cap \boldsymbol{Q}$ given in Theorem **3.9.22** was obtained by Aganagić and Cottle (1979). The effort by Pang (1979b) to extend this result by replacing $\boldsymbol{P_0}$ with $\boldsymbol{E_0}$ was only partially successful. In answer to Pang's closing conjecture in the cited paper, Jeter and Pye (1989) gave an example of a semimonotone $\boldsymbol{Q}$-matrix that is not regular. Gowda (1990b) showed that Pang's conjecture is true in the symmetric case.

3.13.23 Originally defined in the paper of Cottle (1980c), the notion of a completely-$\boldsymbol{Q}$ matrix was motivated by a variable dimension algorithm developed by Van der Heyden (1980), see Section 4.6.

3.13.24 There is a vast literature on the classes $\boldsymbol{Z}$ and $\boldsymbol{K}$, not all of which uses these notations, of course. These symbols appeared in the paper by Fiedler and Pták (1962) which seems to be the earliest systematic study of the subject. Members of $\boldsymbol{K}$ are also known as (nonsingular) $\boldsymbol{M}$-matrices. (The $\boldsymbol{M}$ stands for Minkowski.) The richness of the subject is evident in a remarkable theorem of Fiedler and Pták stating 13 equivalent conditions under which a $\boldsymbol{Z}$-matrix is a $\boldsymbol{K}$-matrix. This feat was surpassed by Berman and Plemmons (1979) who listed 50 such conditions, some of which are contained in Theorem **3.11.10**. Even the Berman-Plemmons collection is not exhaustive; for characterizations in the context of the LCP, see Kaneko (1978d). A result of the latter type is Proposition **3.11.9**. Cottle (1972) and Kaneko (1977a) have established some related characterizations of a $\boldsymbol{K}$-matrix in terms of the isotonicity property of the (unique) solution of the parametric linear complementarity problem; see Proposition **4.8.6** and the note **4.12.24** for more discussion on such characterizations.

3.13.25 The problem of finding the least element of FEA(q, M) was first formulated in Du Val (1940). Studying the more general subject of polyhedral sets having least elements, Cottle and Veinott (1972) obtained results on least-element solutions of linear complementarity problems, including Theorem **3.11.6** and Corollary **3.11.8**. The "if" part of Theorem **3.11.7** was established by Tamir (1974); Bod (1975b) pointed out that the result was anticipated by Wintgen (1964, 1969). In a related paper, Tamir (1976) discussed an application of the least-element property associated with a $\boldsymbol{Z}$-matrix to a class of resource allocation problems.

3.13.26 Minus the name, the class of hidden $\boldsymbol{Z}$-matrices was introduced in a paper by Mangasarian (1976a) which had to do with the idea of solving LCPs as linear programs. In fact, Mangasarian wrote several other papers on this subject. (See Mangasarian (1976b, 1979a, 1979b).) The name "hidden $\boldsymbol{Z}$," which first appeared in Pang (1978), was inspired by a paper of Dantzig and Veinott (1978) that analyzes the class of "hidden totally Leontief" matrices. Dantzig and Veinott attributed Saigal (1971b) for the creation of this nomenclature.

3.13.27 The least-element theory of the LCP with a hidden $\boldsymbol{Z}$-matrix was developed in an effort to explain the phenomenon observed by Mangasarian in the aforementioned papers. This theory was expounded in Cottle and Pang (1978a, 1978b); the former article contains the necessity part of Theorem **3.11.18**. The sufficiency part of this result was proved in Pang (1978). Theorem **3.11.19** was proved in Pang (1979c). The problem of discovering hidden $\boldsymbol{Z}$-matrices was investigated in Pang (1979d).

Chapter 4

PIVOTING METHODS

This is the first of two long chapters on computational methods for solving linear complementarity problems. The algorithms presented in this chapter are finite procedures based on the well known idea of *pivoting* as found in numerical linear algebra and linear programming.

Pivoting can be used to transform the data of a system of linear equations or of a linear program so as to exhibit either a solution or its nonexistence. In the pivoting approach to linear programming, for instance, the goal is to achieve either of two *sign configurations* in the transformed data. The computation terminates when either one is reached, for it is then obvious that an optimal solution is at hand or that no such solution exists.

A similar philosophy can be used in the pivot-theoretic approach to the linear complementarity problem. Let an LCP (q, M) be given. If $q \geq 0$, then $z = 0$ solves the problem. If, however, there exists an index r such that

$$q_r < 0 \quad \text{and} \quad m_{rj} \leq 0 \quad \text{for all } j,$$

then there is no vector $z \geq 0$ such that $q_r + \sum_j m_{rj} z_j \geq 0$. In this case, the LCP (q, M) is infeasible and hence unsolvable. It is rare for the *original* data of a linear complementarity problem to have either of the aforementioned properties. The goal of some pivoting algorithms is to derive an *equivalent* system that *does* have one or the other of these properties. This is particularly so for the principal pivoting algorithms treated in Sections 4.2 and 4.3. Much the same can be said for Lemke's method which is covered in Section 4.4.

The algorithms mentioned in the preceding paragraph are the standard pivoting methods for the LCP. Among their numerous variants are parametric algorithms (Section 4.5) and variable dimension schemes (Section 4.6). The special case of $\boldsymbol{Z}$-matrices (treated in Section 4.7) gives rise to particularly efficient linear complementarity algorithms. These, in turn, are related to a (seemingly) different approach: solution of an LCP as an equivalent linear program and solved by LP methods. In Section 4.8 we will describe a special pivoting method for solving LCP (q, M) where M^{T} is a hidden $\boldsymbol{K}$-matrix. Still other algorithms exist; several of these are discussed in Section 4.12.

It should be noted that, for the LCP, infeasibility is not the precise alternative to the existence of a solution. A feasible linear complementarity problem can fail to possess a solution. In such a case, the global minimum value of the objective function in its quadratic programming formulation (1.4.2) is greater than zero.

When an LCP algorithm necessarily either finds a solution of the LCP (q, M) or indicates that no solution exists, we say the algorithm *processes* the problem. In most cases, the discussion of what problems (q, M) a particular LCP algorithm will process is based on the matrix classes to which the matrices M belong. A notable exception to this is the LCP formulation of the bimatrix game problem. There we single out a specific vector $q = -e$. Other special linear complementarity problems of this sort exist, and will be noted where possible.

In pursuing the pivot-theoretic approach to solving linear complementarity problems, certain key questions arise. As discussed in Section 1.3, the job in solving an LCP (q, M) is to find a complementary submatrix B of $(I, -M)$ such that the vector q belongs to $\operatorname{pos} B$. Each complementary submatrix is associated with an index set α and thereby a principal submatrix of M. This points up the combinatorial question: which α? It may happen that no such α exists. How can this be detected?

Another important question is how to handle degeneracy. Just as in the simplex method for linear programming, it is known that various algorithms for the LCP can cycle on some problems unless special precautions are taken. Techniques for coping with such difficulties will be discussed in Section 4.9.

Two computational considerations are the foci of Section 4.10. The first is the practical matter of updating the information needed to carry out (most of) the algorithms. As in linear programming, this can be done by using basis matrices in factored form instead of pivoting in tableaux. Our discussion of this topic is brief, but suggestions for further reading are given in Section 4.12. The second focus is the theoretical issue of computational complexity of the algorithms covered in this chapter. All of them have the property of *finite termination* when applied to the classes of problems for which they are intended. Generally speaking, in practice, these methods behave rather well on problems of "reasonable" size. Nonetheless, there exist problems that force the algorithms to execute a large number of iterations. We shall encounter some of these LCPs in 4.10.

4.1 Invariance Theorems

In this section, we extend the discussion of pivotal algebra that was begun in Chapter 2. Here we present some important results about the effects that principal pivoting has upon particular matrix classes. These theorems are vital to pivoting methods presented in this chapter.

Certain pivoting algorithms for the linear complementarity problem produce sequences of matrices, each one obtained from its predecessor by a principal pivot and/or a principal rearrangement. In some cases we can show that these operations do not alter some particularly useful or otherwise distinctive property of the matrices. In fact, it is often the case that the preservation of the property under pivotal transformation is instrumental in the justification of the method. Thus, when a property is not disturbed by a particular operation, we say it is *invariant* under that operation. It should, of course, be pointed out that principal pivoting operations can be performed on square matrices without direct reference to algorithms for the LCP. As a very simple example of a property that is invariant under principal pivoting, consider a diagonal matrix, D. Then every principal rearrangement and every (allowable) principal pivotal transform of D is again a diagonal matrix.

The first result along these lines has to do with principal minors of square matrices. It relates the principal minors of a principal pivotal transform to those of the initial matrix and the determinant of the pivot block.

Let A be a square matrix in block partitioned form:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

where the submatrices A_{ii} $(i = 1, 2, 3)$ are square, and the principal submatrix

$$B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is nonsingular. Let C denote the matrix obtained from A by performing a principal pivot on the matrix B. We shall now explore the relationship between the principal minors of C and those of A. For this purpose, suppose C is partitioned in exactly the same way as A. Thus,

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}.$$

Now consider the principal submatrix

$$D = \begin{bmatrix} C_{22} & C_{23} \\ C_{32} & C_{33} \end{bmatrix}.$$

This is the principal submatrix whose determinant we have chosen to analyze. Notice that the rows and columns of A and C associated with the index (subscript) 2 correspond to what we might call the "overlap" between B and D.

4.1.1 Theorem. Under the hypotheses stated above,

$$\det \begin{bmatrix} C_{22} & C_{23} \\ C_{32} & C_{33} \end{bmatrix} = \det \begin{bmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{bmatrix} / \det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{1}$$

Proof. The composition of the linear transformations

$$\begin{bmatrix} y_1 \\ y_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and

$$\begin{bmatrix} y_1 \\ x_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ x_3 \end{bmatrix}$$

is well defined. Moreover, it has the same effect as the linear transformation

$$\begin{bmatrix} y_1 \\ x_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & I & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Consequently, we have the matrix equation

$$\begin{bmatrix} I & 0 & 0 \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & I & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}. \tag{2}$$

Taking determinants on both sides of (2) and dividing by $\det B$, we obtain (1). □

The formula (1) says that the chosen principal minor $\det D$ of the principal pivotal transform C is the quotient of principal minors drawn from A. The denominator is the determinant of the corresponding pivot block and the numerator is the determinant of the principal submatrix of A obtained by selecting the rows and columns (of A) which appear in B but not in D and those which appear in D but not in B.

By a quite similar argument, the exact same formula holds when

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$$

and C is the principal pivotal transform of A with the same partitioning. In this case, there are rows and columns of A and C (namely those corresponding to the index 4) which are not involved in the pivot block and not involved in the principal submatrix whose determinant is in question.

This line of reasoning leads to the *symmetric difference formula.* Recall that if α and β are two subsets of a set Ω, their *symmetric difference* as defined in set theory is

$$\alpha \triangle \beta = (\alpha \cup \beta) \setminus (\alpha \cap \beta). \tag{3}$$

Thus, the symmetric difference of α and β is the set-theoretic difference of their union and their intersection. Let us now regard α and β as subsets of indices (of rows and columns) of a matrix A. The result above can now be expressed in the following rather elegant way:

4.1.2 Theorem. Let C be the matrix obtained from the square matrix A by a principal pivot on the submatrix $A_{\alpha\alpha}$. Then, for any principal submatrix $C_{\beta\beta}$ of C:

$$\det C_{\beta\beta} = \det A_{\gamma\gamma} / \det A_{\alpha\alpha} \tag{4}$$

where $\gamma = \alpha \triangle \beta$. □

We call this the *symmetric difference formula.* It has an immediate consequence of considerable importance.

4.1.3 Theorem. The class $\boldsymbol{P}$ of real square matrices having positive principal minors is invariant under principal pivoting. □

4.1.4 Remark. A somewhat different proof of Theorem **4.1.3** can be given as in the proof of Theorem **4.1.7**.

There is an often-used convention in matrix theory which states that the determinant of the empty matrix [ ] is 1. This practice works out nicely in the symmetric difference formula.

When we speak of matrices with *positive principal minors*, we mean *all principal minors*, not just some of them. (The same sort of usage comes up in phrases like "M has positive eigenvalues" and "M has nonpositive off-diagonal elements".) Thus when M has positive principal minors, it is possible to use any of its principal submatrices as the pivot block in a principal pivotal transformation, and the result will be another matrix of this type because its principal minors are ratios of positive numbers, namely certain principal minors of M.

The symmetric difference formula can be used as above to prove that the class of nondegenerate matrices is also invariant under principal pivoting. Several other classes are also invariant under principal pivoting.

4.1.5 Theorem. The classes of positive definite and positive semi-definite matrices are invariant under principal pivoting.

Proof. Consider the positive semi-definite case first. Let $M_{\alpha\alpha}$ be a nonsingular principal submatrix of the positive semi-definite matrix $M \in R^{n\times n}$. Without loss of generality, we may assume that $M_{\alpha\alpha}$ is a leading principal submatrix. Now for any vector $z \in R^n$, we have

$$w_\alpha = M_{\alpha\alpha}z_\alpha + M_{\alpha\bar{\alpha}}z_{\bar{\alpha}}$$

$$w_{\bar{\alpha}} = M_{\bar{\alpha}\alpha}z_\alpha + M_{\bar{\alpha}\bar{\alpha}}z_{\bar{\alpha}}.$$

Accordingly, and because M is positive semi-definite,

$$z^{\mathrm{T}}Mz = z_\alpha^{\mathrm{T}}w_\alpha + z_{\bar{\alpha}}^{\mathrm{T}}w_{\bar{\alpha}} \geq 0.$$

Let $M' = \wp_\alpha(M)$, the principal transform of M obtained by using $M_{\alpha\alpha}$ as pivot block. Let $z \in R^n$ be arbitrary and define $w \in R^n$ via the equations

$$w_\alpha = M'_{\alpha\alpha}z_\alpha + M'_{\alpha\bar{\alpha}}z_{\bar{\alpha}}$$

$$w_{\bar{\alpha}} = M'_{\bar{\alpha}\alpha}z_\alpha + M'_{\bar{\alpha}\bar{\alpha}}z_{\bar{\alpha}}.$$

It follows that

$$z_\alpha = M_{\alpha\alpha}w_\alpha + M_{\alpha\bar{\alpha}}z_{\bar{\alpha}}$$

$$w_{\bar{\alpha}} = M_{\bar{\alpha}\alpha}w_\alpha + M_{\bar{\alpha}\bar{\alpha}}z_{\bar{\alpha}}.$$

Thus,

$$z^{\mathrm{T}}M'z = z_\alpha^{\mathrm{T}} w_\alpha + z_{\bar\alpha}^{\mathrm{T}} w_{\bar\alpha} = w_\alpha^{\mathrm{T}} z_\alpha + z_{\bar\alpha}^{\mathrm{T}} w_{\bar\alpha} \geq 0.$$

Hence M' is positive semi-definite.

Now suppose that M is positive definite. If $z^{\mathrm{T}}M'z = 0$, then $w_\alpha = 0$ and $z_{\bar\alpha} = 0$. This implies $w_\alpha = M'_{\alpha\alpha} z_\alpha = 0$. As $M'_{\alpha\alpha} = M_{\alpha\alpha}^{-1}$, it follows that $z_\alpha = 0$, so $z = 0$. This implies M' is positive definite. □

According to the last two theorems, the $\boldsymbol{P}$-matrices, the positive definite matrices, and the positive semi-definite matrices are all invariant under principal pivoting. As we know, these classes (and others) are subclasses of another matrix class: the sufficient matrices (see Section 3.5). In the remainder of this section, we shall show that the (row and column) sufficient matrices are also invariant under principal pivoting. Once this is done, we will be in a position to apply this result in an algorithm for the LCP. As it happens, we shall be most interested in the effect of principal pivoting on *row* sufficient matrices, but it is more convenient to treat column sufficiency first.

In order to prove the invariance of the column sufficiency property under principal pivoting, we introduce the notion of the Hadamard product of two vectors of the same size.

4.1.6 Notation. The *Hadamard product* of two vectors $u, v \in R^n$, denoted $u * v$, is the vector given by

$$(u * v)_i = u_i v_i, \quad i = 1, \ldots, n.$$

In terms of this concept, it is easy to see that the column sufficiency of a matrix $M \in R^{n \times n}$ is equivalent to the validity of the implication:

$$x * (Mx) \leq 0 \quad \Rightarrow \quad x * (Mx) = 0$$

for all vectors $x \in R^n$.

4.1.7 Theorem. Let $M_{\alpha\alpha}$ be a nonsingular principal submatrix of the square matrix M. If M is column sufficient and $M' = \wp_\alpha(M)$, then M' is also column sufficient.

Proof. As remarked earlier, it is not restrictive to assume that the pivot block is a leading principal submatrix of M. Let $w = M'z$ and suppose $z * w \leq 0$. We may write

$$\begin{bmatrix} w_\alpha \\ w_{\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} M'_{\alpha\alpha} & M'_{\alpha\bar{\alpha}} \\ M'_{\bar{\alpha}\alpha} & M'_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \begin{bmatrix} z_\alpha \\ z_{\bar{\alpha}} \end{bmatrix}.$$

The condition $z * w \leq 0$ means

$$\begin{bmatrix} z_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} * \begin{bmatrix} w_\alpha \\ w_{\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} z_\alpha * w_\alpha \\ z_{\bar{\alpha}} * w_{\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} w_\alpha * z_\alpha \\ z_{\bar{\alpha}} * w_{\bar{\alpha}} \end{bmatrix} \leq 0.$$

Since $M' = \wp_\alpha(M)$, we have

$$\begin{bmatrix} z_\alpha \\ w_{\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \begin{bmatrix} w_\alpha \\ z_{\bar{\alpha}} \end{bmatrix}.$$

But M is column sufficient, so it follows that

$$\begin{bmatrix} w_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} * \begin{bmatrix} z_\alpha \\ w_{\bar{\alpha}} \end{bmatrix} = 0.$$

Accordingly, $z * w = 0$ which implies that M' is column sufficient. □

We now come to the result for row sufficient matrices.

4.1.8 Theorem. Let $M_{\alpha\alpha}$ be a nonsingular principal submatrix of the square matrix M. If M is row sufficient and $M' = \wp_\alpha(M)$, then M' is also row sufficient.

Proof. It is obvious from first principles that a matrix is row sufficient if and only if its transpose is column sufficient. Thus, it suffices to prove that $(M')^{\mathrm{T}}$ is column sufficient. Our hypothesis implies that M^{T} must be so. Theorem **4.1.7** implies that $\wp_\alpha(M^{\mathrm{T}})$ is column sufficient. By the definition of M' and by equation (2.3.13) we have

$$(M')^{\mathrm{T}} = (\wp_\alpha(M))^{\mathrm{T}} = E_{\bar{\alpha}}(\wp_\alpha(M^{\mathrm{T}}))E_{\bar{\alpha}}.$$

The result now follows from the easily shown fact that if M is row (column) sufficient, then so is EME, for any conformable diagonal matrix E. □

The matrix class $\boldsymbol{P}_1$

We shall now study a special subclass of $\boldsymbol{P}_0$. There are several motivations for considering this matrix class. First, its properties are interesting in their own right. Second, in establishing these properties, we have an opportunity to link up a number of previously developed concepts, one of which is that of invariance under principal pivoting as discussed just above. Third, there are important realizations of this matrix class. See, for instance, Exercise **4.11.4**.

4.1.9 Definition. If $M \in \boldsymbol{P}_0 \cap R^{n\times n}$, then $M \in \boldsymbol{P}_1$ if there exists a unique index set $\alpha \subseteq \{1,\ldots,n\}$ such that $\det M_{\alpha\alpha} = 0$. Members of this class are called $\boldsymbol{P}_1$-matrices.

The definition says that $M \in \boldsymbol{P}_1$ if and only if it has nonnegative principal minors precisely one of which is 0. Matrices of this sort may or may not belong to $\boldsymbol{Q}$. Indeed,

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \in \boldsymbol{P}_1 \cap \boldsymbol{Q},$$

whereas

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \in \boldsymbol{P}_1 \setminus \boldsymbol{Q}.$$

Despite the latter example, it is shown below that $\boldsymbol{P}_1$-matrices must always belong to $\boldsymbol{Q}_0$.

4.1.10 Theorem. For every $\boldsymbol{P}_1$-matrix $M \in R^{n\times n}$, there exists a unique index set β such that $\bar{M} = \wp_\beta(M)$ is a $\boldsymbol{P}_1$-matrix with determinant equal to zero.

Proof. By definition, there exists a unique index set α such that $\det M_{\alpha\alpha} = 0$. Let $\beta = \bar{\alpha}$, the complementary index set. Then $\det M_{\beta\beta} > 0$. It follows from the symmetric difference formula (4) that $\det \bar{M} = 0$. Furthermore, if γ is an index set not equal to $\{1,\ldots,n\}$, then

$$\det \bar{M}_{\gamma\gamma} = \frac{\det M_{\gamma\,\Delta\,\beta,\gamma\,\Delta\,\beta}}{\det M_{\beta\beta}} > 0.$$

The inequality follows (set-theoretically) from the fact that the numerator cannot be the single zero minor of M. □

4.1.11 Corollary. Every $\boldsymbol{P}_1$-matrix is (row and column) sufficient, hence $\boldsymbol{P}_1 \subset \boldsymbol{Q}_0$ (properly).

Proof. By the preceding theorem, every $\boldsymbol{P}_1$-matrix M has a singular principal pivotal transform $\bar{M}$ whose *proper* principal minors are positive. The matrix $\bar{M}$ must be adequate and hence (as noted in Section 3.5) sufficient. The class of sufficient matrices is invariant under principal pivoting (see **4.1.7** and **4.1.8**). Thus, it follows that M is sufficient since it is a principal pivotal transform of the sufficient matrix $\bar{M}$. As we know, all (row) sufficient matrices belong to $\boldsymbol{Q}_0$. The inclusion is obviously proper. □

This corollary implies that when $M \in \boldsymbol{P}_1 \cap R^{n\times n}$, the cone $K(M)$ is convex. The next theorem describes a distinctive feature of such a matrix. In particular, if $K(M) \neq R^n$, then it must be a halfspace. Furthermore, except for vectors q on the boundary of this halfspace, solvable problems have unique solutions. The full statement of the next theorem is facilitated by using the following definition.

4.1.12 Definition. The matrix $M \in R^{n\times n}$ belongs to $\boldsymbol{U}$ if the LCP (q, M) has a unique solution for all $q \in \operatorname{int} K(M)$. Members of this class are called $\boldsymbol{U}$-matrices.

4.1.13 Theorem. If the $n\times n$ matrix $M \in \boldsymbol{P}_1 \backslash \boldsymbol{Q}$, then $M \in \boldsymbol{U}$, and $K(M)$ is a halfspace. If, in addition to the preceding hypothesis, $\det M = 0$, then the normal to the hyperplane $\operatorname{bd} K(M)$ can be chosen as a positive vector.

Proof. The assertion is trivial when $n = 1$, so hereafter we shall assume $n \geq 2$. Under the present conditions, the cone $K(M)$ is a proper convex subset of R^n, hence it must be contained in a halfspace. In particular, suppose $q \in R^n \setminus K(M)$. Then there exists a nonzero vector v such that

$$v \geq 0, \quad v^{\mathrm{T}}M \leq 0, \quad \text{and} \quad v^{\mathrm{T}}q < 0.$$

Note that $\sigma = \operatorname{supp} v$ is nonempty. These inequalities imply that

$$K(M) \subseteq H^+(v) = \{x : v^{\mathrm{T}}x \geq 0\}.$$

Moreover, by the sufficiency of M, it follows that $M_{\sigma\sigma}^{\mathrm{T}} v_\sigma = 0$, hence σ is the index set corresponding to the unique vanishing minor of M. To prove the reverse inclusion, suppose there exists a vector $\tilde{q} \in H^+(v) \setminus K(M)$. Then once again, there exists a nonzero vector u such that

$$u \geq 0, \quad u^{\mathrm{T}} M \leq 0, \quad \text{and} \quad u^{\mathrm{T}} \tilde{q} < 0.$$

The index set $\tau = \operatorname{supp} u$ is likewise nonempty, and it follows that $M_{\tau\tau}^{\mathrm{T}} u_\tau = 0$. But this means that $\tau = \sigma$ and in fact that $v = \lambda u$ for some $\lambda > 0$. Now we have

$$0 \leq v^{\mathrm{T}} \tilde{q} = \lambda u^{\mathrm{T}} \tilde{q} < 0$$

which is impossible. Hence $K(M) = H^+(v)$. Note that in the case where $\det M = 0$, the support of v must be $\{1, \ldots, n\}$ (i.e., $v > 0$).

To show that $M \in \boldsymbol{U}$, note that $\boldsymbol{U}$ is invariant under principal pivoting. This fact and Theorem **4.1.10** allow us to assume that $\det M = 0$, hence M is adequate. Choose an arbitrary $q \in \operatorname{int} K(M)$. From what has been proved above, it must be the case that $v^{\mathrm{T}} q > 0$. Suppose there exist distinct vectors z^1 and z^2 in SOL(q, M). Let $w^i = q + Mz^i$ for $i = 1, 2$. The adequacy of M implies $w^1 = w^2$. This, in turn, means that $M(z^1 - z^2) = 0$, hence the vector $z^1 - z^2$ is a nonzero multiple of $\bar{z}$, the generator of the 1-dimensional nullspace of M. As shown above, $v > 0$ and $M^T v = 0$. Using Exercise **3.12.11** and the adequacy of M, we conclude that $\bar{z} \geq 0$. As $M \in \boldsymbol{P}_1$, we must have $\bar{z} > 0$. It is not restrictive to write

$$z^1 = z^2 + \theta \bar{z}, \quad \theta > 0,$$

which implies that $w^1 = 0$. From this we have the contradiction

$$0 = v^{\mathrm{T}} w^1 = v^{\mathrm{T}} q + v^{\mathrm{T}} M z^1 = v^{\mathrm{T}} q > 0.$$

This contradiction proves that $M \in \boldsymbol{U}$. □

4.1.14 Remark. Membership in $\boldsymbol{U}$ is not guaranteed for matrices in the class $\boldsymbol{P}_1 \cap \boldsymbol{Q}$. This can be seen from the case where

$$q = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in \boldsymbol{P}_1 \cap \boldsymbol{Q}.$$

In this case,

$$\mathrm{SOL}(q, M) = \{(z_1, z_2) : z_1 + z_2 = 1, \ z_1, z_2 \geq 0\}.$$

4.2 Simple Principal Pivoting Methods

In this section and the next we study algorithms based on the idea of principal pivoting. These methods work with principal pivotal transforms of the system

$$w = q + Mz. \tag{1}$$

To distinguish successive pivotal transforms of (1), we shall use the superscript ν as an iteration counter. The initial value of ν will be 0, and the system shown in (1) will be written as

$$w^0 = q^0 + M^0 z^0. \tag{2}$$

In general, after ν principal pivots, the system will be

$$w^\nu = q^\nu + M^\nu z^\nu. \tag{3}$$

Each system (3) can also be represented in the tableau form

$$\begin{array}{c|c|ccc|} & 1 & z_1^\nu & \cdots & z_n^\nu \\ \hline w_1^\nu & q_1^\nu & m_{11}^\nu & \cdots & m_{1n}^\nu \\ \vdots & \vdots & \vdots & & \vdots \\ w_n^\nu & q_n^\nu & m_{n1}^\nu & \cdots & m_{nn}^\nu \\ \hline \end{array} \tag{4}$$

The vectors w^ν and z^ν, which represent the system's basic and nonbasic variables, respectively, may each be composed of the original w- and z-variables. Principal rearrangements can be used to make $\{w_i^\nu, z_i^\nu\} = \{w_i, z_i\}$, $i = 1, \ldots, n$.

Under certain circumstances, one can process linear complementarity problems by methods that use only principal pivots of order 1, that is, pivot elements situated along the main diagonal of the current principal transform of the matrix M. We call these *simple principal pivoting methods*; in the literature they also go by the name *Bard-type methods.*

Perhaps the earliest example of a simple principal pivoting method was proposed by Zoutendijk. The LCP considered by Zoutendijk—and somewhat later by Bard—is of a very special form, namely, $(q, M) = (Pb, PP^{\mathrm{T}})$

for some given vector $b \in R^m$ and some matrix $P \in R^{n \times m}$. Thus, q is in the range of the matrix P. This LCP is equivalent to the problem of finding a point in the cone $K = \{y \in R^m : Py \geq 0\}$ that is closest to the vector $b \in R^m$ under the l_2-norm (cf. Exercise **1.6.1**).

In the sequel, we consider a slight generalization of this class of LCPs by assuming that $M = PAP^{\mathrm{T}}$ where the matrix $A \in R^{m \times m}$ is positive definite. It is not hard to show that a matrix M of this form must be adequate and positive semi-definite (but not necessarily symmetric); moreover, an LCP of this type must have a solution. The following theorem provides the key to the solvability of the LCP (Pb, PAP^{T}) by a simple principal pivoting method, namely **4.2.2**.

4.2.1 Theorem. Let (q, M) be a linear complementarity problem with $M = PAP^{\mathrm{T}}$ and $q = Pb$ where A is positive definite. Let $(q', M') = \wp_\alpha(q, M)$ be obtained from (q, M) by a (possibly vacuous) principal (block) pivot on the principal submatrix $M_{\alpha\alpha}$ of M. Then, $q'_s \neq 0$ only if $m'_{ss} > 0$.

Proof. A precondition for the principal pivotal transform $\wp_\alpha(q, M)$ to be well defined is that the principal submatrix $M_{\alpha\alpha}$ is nonsingular. Since

$$M_{\alpha\alpha} = P_{\alpha\bullet}A(P_{\alpha\bullet})^{\mathrm{T}}$$

and A is positive definite, it follows that $M_{\alpha\alpha}$ must be positive definite and the rows of $P_{\alpha\bullet}$ are linearly independent. Hence $m'_{ss} > 0$ for all indices $s \in \alpha$. Consider an index $s \notin \alpha$. Since principal pivoting preserves positive semi-definiteness, we have $m'_{ss} \geq 0$. Suppose this diagonal entry is equal to zero. Then, the principal submatrix $M_{\alpha'\alpha'}$ where $\alpha' = \alpha \cup \{s\}$ must be singular. In turn, this implies that the rows of $P_{\alpha'\bullet}$ must be linearly dependent. Hence, $P_{s\bullet}$ is linearly dependent on $P_{\alpha\bullet}$. Write

$$P_{s\bullet} = v_\alpha^{\mathrm{T}} P_{\alpha\bullet}$$

for some vector v_α. This yields $P_{s\bullet}A(P_{\alpha\bullet})^{\mathrm{T}} = v_\alpha^{\mathrm{T}} M_{\alpha\alpha}$. By the expression (2.3.10) and a simple manipulation, we obtain

$$q'_s = P_{s\bullet}b - P_{s\bullet}A(P_{\alpha\bullet})^{\mathrm{T}}(M_{\alpha\alpha})^{-1}P_{\alpha\bullet}b = 0.$$

Consequently, $q'_s \neq 0$ only if $m'_{ss} > 0$. □

It is clear from this theorem that in any principal pivotal transform of (Pb, PAP^{T}), it is possible to pivot on the diagonal entry corresponding to any negative entry in the "constant column." This is the essence of Zoutendijk's procedure which can be stated as follows.

4.2.2 Algorithm. (Zoutendijk/Bard)

Step 0. *Initialization.* Input $(q^0, M^0) = (Pb, PAP^{\mathrm{T}})$. Set $\nu = 0$.

Step 1. *Test for termination.* If $q^\nu \geq 0$, then stop: $z^\nu = 0$ solves (q^ν, M^ν). Recover a solution of (Pb, PAP^{T}).

Step 2. *Choose pivot row.* Choose an index r such that $q_r^\nu < 0$.

Step 3. *Pivoting.* Pivot on m_{rr}^ν. Define

$$\begin{aligned} w_r^{\nu+1} &= z_r^\nu \quad & z_r^{\nu+1} &= w_r^\nu \\ w_i^{\nu+1} &= w_i^\nu \quad & z_i^{\nu+1} &= z_i^\nu \quad i \neq r. \end{aligned}$$

Return to Step 1 with $\nu \leftarrow \nu + 1$.

While the steps of this algorithm are executable, there is no guarantee that the method, as stated, is finite. In this case, giving a finiteness proof is tantamount to showing that the algorithm actually solves the problem: the algorithm has only one form of termination, namely with a solution. It is known that cycling (and hence nontermination) can occur when the pivot row (in Step 2) is chosen according to the rule

$$r \in \arg\min_i q_i^\nu. \tag{5}$$

Nevertheless, the use of Algorithm **4.2.2** with r chosen as in (5) has been advocated on the grounds of simplicity and speed. According to Bard (see **4.12.3**), it "has not failed in hundreds of applications." Even so, the algorithm cannot be justified without a suitable modification of Step 2.

How can this be done? Recall that at each iteration of the simplex method for linear programming one has a *feasible basis* of which there are at most a finite number. The finiteness of the simplex method rests on the fact that when suitable precautions are taken, no feasible basis can be used more than once. This property can be assured by introducing a function whose value is uniquely determined by a (feasible) basis and which is strictly monotone in the iteration count.

As applied to a linear complementarity problem (Pb, PAP^{T}) of order n, Algorithm **4.2.2** uses *complementary bases* of which there are at most 2^n. Any technique that prevents a complementary basis from being repeated will render the algorithm finite. To do this one must modify Step 2 of the algorithm where the choice of pivot row is made. Zoutendijk asserted that the finiteness could be established by adapting the linear programming techniques of perturbation and lexicographic ordering. Remarking that these "proposals ... do not seem very efficient," Zoutendijk suggested two other schemes and proved that they lead to finite algorithms. We shall develop the first of these.

Let $B \in R^{n \times n}$ be an invertible matrix with lexicographically positive rows. The columns of this matrix will be transformed by the simple principal pivoting in exactly the same manner as q is transformed. At any iteration ν there will be a triple (q^ν, M^ν, B^ν) which, if desired, can be written in tabular form

	1	z	x
w	q	M	B

(6)

We attach no special meaning to the column headings $x_1, \ldots, x_n$, but we do think of these variables as being equal to zero so that the extra columns do not affect the values of the basic variables. After ν pivots, the triple $(q, M, B) = (q^0, M^0, B^0)$ will become (q^ν, M^ν, B^ν).

The pivot selection rule we have in mind involves a vector-valued function that strictly decreases in the lexicographic sense. This function is given by

$$b^\nu = \frac{B^\nu_{r\bullet}}{q^\nu_r} = \text{lexico max} \left\{ \frac{B^\nu_{i\bullet}}{q^\nu_i} : q^\nu_i < 0 \right\}. \qquad (7)$$

4.2.3 Lemma. Algorithm **4.2.2** with the lexicographic pivot selection rule (7) preserves the property

$$B^\nu_{i\bullet} \succ q^\nu_i \left(\frac{B^\nu_{r\bullet}}{q^\nu_r} \right) \qquad (i \neq r). \qquad (8)$$

Proof. We first check that (8) is satisfied when $\nu = 0$. In the case where $q_i^\nu < 0$, this follows from the definition of r and the invertibility of B. When $q_i^\nu \geq 0$, it follows from the initially given lexicographic positivity of the rows of B.

The first iteration calls for a pivot on m_{rr}^0. The effects this has on q^0 and B^0 are as follows:

$$q_r^1 = \frac{-q_r^0}{m_{rr}^0}, \qquad B_{r\bullet}^1 = \frac{-B_{r\bullet}^0}{m_{rr}^0} \prec 0,$$

while for all $i \neq r$,

$$q_i^1 = q_i^0 - \left(\frac{m_{ir}^0}{m_{rr}^0}\right) q_r^0, \qquad B_{i\bullet}^1 = B_{i\bullet}^0 - \left(\frac{m_{ir}^0}{m_{rr}^0}\right) B_{r\bullet}^0 .$$

The transformed data q^1 and B^1 also satisfy (8) for the index r corresponding to the first iteration. Indeed,

$$\begin{aligned}
q_i^1 \left(\frac{B_{r\bullet}^1}{q_r^1}\right) &= \left[q_i^0 - \left(\frac{m_{ir}^0}{m_{rr}^0}\right) q_r^0\right] \left(\frac{-B_{r\bullet}^0}{m_{rr}^0}\right) \left(\frac{m_{rr}^0}{-q_r^0}\right) \\
&= \left[q_i^0 - \left(\frac{m_{ir}^0}{m_{rr}^0}\right) q_r^0\right] \left(\frac{B_{r\bullet}^0}{q_r^0}\right) \\
&= q_i^0 \left(\frac{B_{r\bullet}^0}{q_r^0}\right) - \left(\frac{m_{ir}^0}{m_{rr}^0}\right) B_{r\bullet}^0 \\
&\prec B_{i\bullet}^0 - \left(\frac{m_{ir}^0}{m_{rr}^0}\right) B_{r\bullet}^0 \\
&= B_{i\bullet}^1 .
\end{aligned}$$

That is,

$$B_{i\bullet}^1 \succ q_i^1 \left(\frac{B_{r\bullet}^1}{q_r^1}\right) . \tag{9}$$

Suppose that $q_i^1 < 0$ for some $i \neq r$. (If no such i exists, the problem is solved.) Then from (9) it follows that

$$\frac{B_{i\bullet}^1}{q_i^1} \prec \frac{B_{r\bullet}^1}{q_r^1} = \frac{B_{r\bullet}^0}{q_r^0}.$$

Consequently,

$$\text{lexico} \max_{\{i:q_i^1<0\}} \frac{B_{i\bullet}^1}{q_i^1} \prec \frac{B_{r\bullet}^1}{q_r^1} = \frac{B_{r\bullet}^0}{q_r^0}. \tag{10}$$

Now for the data of the new tableau, let s denote the index determined by the lexicographic pivot selection rule (7) of Algorithm **4.2.2**. We need to prove

$$B_{i\bullet}^1 \succ q_i^1 \left(\frac{B_{s\bullet}^1}{q_s^1} \right) \qquad (i \neq s). \tag{11}$$

There are three cases, depending on the sign of q_i^1.

Case 1. If $q_i^1 < 0$, then (11) holds by definition of s.

Case 2. If $q_i^1 > 0$, the inequality (9) yields

$$\frac{B_{i\bullet}^1}{q_i^1} \succeq \frac{B_{r\bullet}^1}{q_r^1}.$$

Inequality (10) gives

$$\frac{B_{r\bullet}^1}{q_r^1} \succ \frac{B_{s\bullet}^1}{q_s^1}.$$

Hence

$$\frac{B_{i\bullet}^1}{q_i^1} \succ \frac{B_{s\bullet}^1}{q_s^1}.$$

Multiplication through this inequality by $q_i^1 > 0$ gives (11).

Case 3. If $q_i^1 = 0$, (9) reduces to $B_{i\bullet}^1 \succ 0$ which is the same as (11) in this case.

This shows that the property is inherited by the tableau after the first pivot. The same argument can be repeated to show that each successive tableau inherits the property. □

We can now prove the finiteness of the algorithm.

4.2.4 Theorem. Algorithm **4.2.2** with the lexicographic pivot selection rule (7) is finite. Furthermore, no complementary basis will ever be used more than once.

Proof. As shown in the preceding lemma, the property (8) is preserved from one iteration to the next. In this lexicographic inequality, the subscript $r = r(\nu)$ is defined by (7). By the principal pivoting formulas, it follows that

$$\frac{B^{\nu+1}_{r(\nu)\bullet}}{q^{\nu+1}_{r(\nu)}} = \frac{B^{\nu}_{r(\nu)\bullet}}{q^{\nu}_{r(\nu)}}. \tag{12}$$

Since $q^{\nu+1}_{r(\nu)} > 0$, it follows that $r(\nu+1) \neq r(\nu)$. Applying (8) for iteration $\nu+1$ and (12), we obtain

$$b^{\nu} = \frac{B^{\nu}_{r(\nu)\bullet}}{q^{\nu}_{r(\nu)}} = \frac{B^{\nu+1}_{r(\nu)\bullet}}{q^{\nu+1}_{r(\nu)}} \succ \frac{B^{\nu+1}_{r(\nu+1)\bullet}}{q^{\nu+1}_{r(\nu+1)}} = b^{\nu+1}. \tag{13}$$

The inequality (13) shows that the special vectors b^{ν} decrease in a strict lexicographic sense from one iteration to the next. The data from which these vectors are computed are uniquely determined by the finite collection of complementary bases. It follows that the sequence must terminate after finitely many iterations. □

4.2.5 Remark. As noted above, for problems of the sort under consideration in this theorem, termination occurs only when a solution has been found. In the general positive semi-definite case where the positivity of diagonal entries in M^{ν} corresponding to negative entries in q^{ν} may not hold, it is possible to handle the problem using 2×2 pivots as well as simple principal pivots.

Murty's least-index method

The fact that Bard's version of Algorithm **4.2.2** applied to an LCP (q, M) with $M \in \boldsymbol{P}$ can cycle has been noted by several authors. The following data yield such a problem:

$$q = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \qquad M = \begin{bmatrix} .1 & 0 & .2 \\ .2 & .1 & 0 \\ 0 & .2 & .1 \end{bmatrix}. \tag{14}$$

After a pivot in row 1 (where the pivot row choice is made by arbitrarily breaking the tie), the application of Bard's pivot rule leads to a sequence of pivots in rows 3, 1, 2, 3, 1, and 2 which then produces the complementary basis obtained after the initial pivot. For further discussion of this example, see Exercise **4.11.8**.

When $M \in \boldsymbol{P}$, the LCP (q, M) can be solved by another simple principal pivoting method due to Murty; this algorithm involves choosing the pivot row in Step 2 of Algorithm **4.2.2** according to the *least* (i.e., smallest) of the candidate indices. This variant is stated as follows:

4.2.6 Algorithm. (Murty)

Step 0. *Initialization.* Input $(q^0, M^0) = (q, M)$ with $M \in \boldsymbol{P}$. Set $\nu = 0$.

Step 1. *Test for termination.* If $q^\nu \geq 0$, then stop: $z^\nu = 0$ solves (q^ν, M^ν). Recover a solution of (q, M).

Step 2. *Choose pivot row.* Choose the index r so that

$$r = \min\{i : q_i^\nu < 0\}.$$

Step 3. *Pivoting.* Pivot on m_{rr}^ν. Define

$$\begin{aligned} w_r^{\nu+1} &= z_r^\nu \quad & z_r^{\nu+1} &= w_r^\nu \\ w_i^{\nu+1} &= w_i^\nu \quad & z_i^{\nu+1} &= z_i^\nu \quad i \neq r. \end{aligned}$$

Return to Step 1 with $\nu \leftarrow \nu + 1$.

It should be noted that this algorithm operates on linear complementarity problems with $\boldsymbol{P}$-matrices. This and the special rule for selecting the pivot row are the only ways in which it differs from **4.2.2**. The method is obviously easy to implement; moreover, it is finite, even in the presence of degeneracy.

The finiteness of Murty's least-index method is justified by an inductive argument that uses a string of elementary observations. Their statements will be facilitated by the following definition.

4.2.7 Definition. Let (q, M) be a linear complementarity problem of order n, and let α denote the index set $\{1, \ldots, s\}$ where $s \leq n$. Relative to (q, M), the LCP $(q_\alpha, M_{\alpha\alpha})$ is called the *leading principal subproblem* of order s.

4.2.8 Proposition. Let $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)$ be a solution of the LCP (q, M) of order n. If $\bar{z}_n = 0$, then $\bar{z}_\alpha = (\bar{z}_1, \ldots, \bar{z}_{n-1})$ solves the leading principal subproblem of order $n - 1$. Conversely, if $\bar{z}_\alpha = (\bar{z}_1, \ldots, \bar{z}_{n-1})$ solves the leading principal subproblem of order $n-1$ and $\bar{w}_n = q_n + \sum_{i=1}^{n-1} m_{ni}\bar{z}_i \geq 0$, then $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_{n-1}, 0)$ solves (q, M).

Proof. This is Exercise **4.11.9**. □

Notice that Proposition **4.2.8** is valid for any real $n \times n$ matrix M. When this matrix belongs to $\boldsymbol{P}$, however, the LCP (q, M) always has a unique solution and so does each of its corresponding principal subproblems, in particular the leading principal subproblem of order $n - 1$. Thus, in the converse part of the proposition, $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_{n-1}, 0)$ is the only solution of (q, M).

4.2.9 Theorem. Let (q, M) be an LCP of order n with $M \in \boldsymbol{P}$. Then for any $q \in R^n$, Algorithm **4.2.6** will solve (q, M) in a finite number of steps. Furthermore, no complementary basis will ever be used more than once.

Proof. The proof is by induction on n. For $n = 1$, the theorem is clear. In this case at most one pivot step is required. Inductively, assume that $n > 1$ and that the theorem holds for all linear complementarity problems (of the $\boldsymbol{P}$-matrix type) of order $n - 1$ or less.

The LCP (q, M) under consideration has a unique solution, $\bar{z}$. There are two main cases, according to whether $\bar{z}_n = 0$ or not.

Case 1. $\bar{z}_n = 0$. Suppose Algorithm **4.2.6** is applied to the leading principal subproblem of order $n - 1$. By the inductive hypothesis, the algorithm obtains the unique solution $\hat{z} \in R^{n-1}$ of this problem in a finite number of steps, during the course of which no complementary basis is repeated.

Now when **4.2.6** is applied to the full problem (q, M), it will not call for a pivot in row n unless the leading principal subproblem has already been solved in the process, for n is the *largest* index in the full problem. The assumption $\bar{z}_n = 0$ implies that $(\bar{z}_1, \ldots, \bar{z}_{n-1})$ is *the* solution of the leading principal subproblem, so $\hat{z} = (\bar{z}_1, \ldots, \bar{z}_{n-1})$. Hence $q_n + \sum_{i=1}^{n-1} m_{ni}\hat{z}_i \geq 0$. Therefore in this case, the theorem holds.

Case 2. $\bar{z}_n > 0$. Applying the algorithm to (q, M), we arrive after a finite number of steps at a complementary basis C such that $\left(C^{-1}q\right)_i \geq 0$

for $i = 1, \ldots, n-1$ and $(C^{-1}q)_n < 0$. In other words, the leading principal subproblem of order $n-1$ has been solved—without repetition of a complementary basis. The algorithm then calls for a pivot on the last diagonal element of the current principal pivotal transform of M. The new principal pivotal transform of (q, M) is a linear complementarity problem of the $\boldsymbol{P}$-matrix type having the property covered in Case 1. Accordingly, when the algorithm continues, it solves this new LCP in a finite number of steps without repeating a complementary basis. Hence it solves the original problem in the manner asserted by the theorem. □

4.2.10 Remark. The computational efficiency of **4.2.6** can very well depend on the *order* in which the constraints are written down. Examples have been published showing that for certain (specially constructed) problems of order n, Algorithm **4.2.6** requires $2^n - 1$ pivots, whereas for a suitable principal rearrangement of the problem, the same algorithm finds the solution after executing only one simple principal pivot. (See Section 4.10 for further discussion along these lines.) Thus the ordering of the constraints can have a profound impact on the performance of the method. Disconcerting as this may be, it should be remembered that such strikingly nasty problems are not typical. At least one computational study has found Murty's least-index method to be computationally superior to others that are applicable to the same class of problems. See **4.12.7** for references on the computational behavior of Murty's method.

It should also be noted that Algorithm **4.2.6** can be implemented without actually doing the principal pivoting. The algorithm merely generates a sequence of complementary bases C^ν each of which (except the first) differs from its predecessor in just one column. The passage from C^ν to $C^{\nu+1}$ is determined by the solution of the equation

$$C^\nu x = q.$$

If $x \geq 0$, the algorithm stops. Otherwise, the negative component of x having the smallest index is used to determine the new complementary basis.

Algorithms **4.2.2** and **4.2.6** make no effort to preserve the nonnegativity (if any) of the basic variables. This being so, these algorithms do not require the minimum ratio tests by which such nonnegativity would be maintained, a feature that sets them apart from most of the other algorithms presented in this chapter.

For a better understanding of the material to follow, the reader is encouraged to review the discussion of minimum ratio tests given near the end of Section 2.3. A word of caution is in order, however. Appropriate adjustments will need to be made for the way we express the basic variables in terms of the nonbasic variables. (Cf. (1) and (2.3.20).) Moreover, in the present context, it is not necessary that all basic variables have nonnegative values. We shall heavily use the terms *driving variable* and *blocking variable* which were introduced in Section 2.3, but the latter will take on a slightly more general meaning. The principal pivoting methods of this section and the next generate sequences of *major cycles*; the purpose of each major cycle is to make a particular negative basic variable increase to zero. The latter is called the *distinguished variable* of that major cycle. This distinguished variable can also block the driving variable by reaching zero before any other specified basic variables do. By "specified" we mean that—in some instances—only a subset of the basic variables are taken into consideration when the minimum ratio test is performed. These are called the *eligible blocking variables*. Basic variables that are not in the current set of eligible blocking variables are not "eligible" to block the driving variable, even if they become negative as the driving variable increases.

The symmetric positive semi-definite case

As noted in Section 1.2, a linear complementarity problem (q, M) in which M is symmetric and positive semi-definite is completely equivalent to the quadratic program

$$\begin{array}{ll} \text{minimize} & f(z) = q^{\mathrm{T}}z + \tfrac{1}{2}z^{\mathrm{T}}Mz \\ \text{subject to} & z \geq 0. \end{array} \tag{15}$$

In fact, the KKT conditions of (15) are just the LCP (q, M). The two problems have exactly the same solutions.

Instances of such problems were also mentioned in Section 1.2. To these can be added the "dualized" form of a strictly convex quadratic program

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x + \tfrac{1}{2}x^{\mathrm{T}}Qx \\ \text{subject to} & Ax \geq b \end{array} \tag{16}$$

in which $Q \in R^{m \times m}$ is symmetric and positive definite and $A \in R^{n \times m}$. By eliminating the vector x from the KKT conditions of (16), one obtains an LCP (q, M) with data

$$q = -b - AQ^{-1}c \quad \text{and} \quad M = AQ^{-1}A^{\mathrm{T}}.$$

The variables in this LCP are the Lagrange multipliers (dual variables) associated with the quadratic program (16). Once the LCP is solved, the original x-variables of (16) can be reconstructed from the equation through which they were eliminated, namely

$$x = Q^{-1}(A^{\mathrm{T}}z - c).$$

For this reason, algorithms for solving these special LCPs have been called "dual methods" for solving (16).

Our present aim is to exhibit a principal pivoting method for LCPs of the symmetric positive semi-definite type. The one below is actually a specialization of a quadratic programming algorithm presented in the language of linear complementarity. Specifically, it is a simple principal pivoting method somewhat like Algorithm **4.2.2**, yet different from it in three important respects. First, the vector q is more general. Second the pivot locations are determined through a minimum ratio test that preserves the nonnegativity of basic z-variables. (Algorithm **4.2.2** uses no minimum ratio test.) Third (and quite interestingly), the algorithm *cannot cycle*, even when the given problem is degenerate! The procedure makes strong use of two important results: the invariance of positive semi-definiteness (see **4.1.5**) and of bisymmetry (see Exercise **4.11.2**) under principal pivoting.

4.2.11 Algorithm. (Dantzig; van de Panne and Whinston)

Step 0. *Initialization.* Input (q, M) with M being symmetric and positive semi-definite. Set $(q^0, M^0) = (q, M)$, $\nu = 0$, $\alpha = \emptyset$, and $\beta = \{1, \ldots, n\}$.

Step 1. *Test for termination.* Breaking ties arbitrarily, choose

$$r \in \arg\min \{q_i^\nu : i \in \beta\}.$$

Step 1A. If $q_r^\nu \geq 0$, stop. A solution of (q, M) is given by $\bar{z}$ where $\bar{z}_\alpha = q_\alpha^\nu$, $\bar{z}_\beta = 0$.

Step 1B. If $q_r^\nu < 0$, choose w_r^ν as the distinguished variable and z_r^ν as the driving variable.

Step 1C. If $m_{rr}^\nu = 0$ and $m_{ir}^\nu \geq 0$ for all $i \in \alpha$ stop. There is no solution.

Step 2. *Determination of the blocking variable.* The eligible blocking variables are the distinguished variable and the basic z-variables. Use the minimum ratio test to define the index s of a blocking variable. (Break ties arbitrarily, but if w_r^ν is involved in a tie, choose it as the blocking variable.)

Step 3. *Pivoting.* The driving variable z_r^ν is blocked by w_s^ν. Pivot $\langle w_s^\nu, z_s^\nu \rangle$, getting $(q^{\nu+1}, M^{\nu+1})$. Let

$$\begin{array}{lll} w_s^{\nu+1} = z_s^\nu & z_s^{\nu+1} = w_s^\nu & \\ w_i^{\nu+1} = w_i^\nu & z_i^{\nu+1} = z_i^\nu & i \neq s. \end{array}$$

If $s = r$, transfer r from β to α. Go to Step 1 with $\nu \leftarrow \nu + 1$.

If $s \neq r$, transfer s from α to β. Go to Step 2 with $\nu \leftarrow \nu + 1$.

In justifying this algorithm, we find it convenient to use the schema

$$\begin{array}{c|c|c|} & 1 & z \\ \hline \theta & 0 & q^{\mathrm{T}} \\ \hline w & q & M \\ \hline \end{array} \tag{17}$$

The first row of the schema gives a way of evaluating the objective function. Indeed, it is a simple matter to demonstrate that for *any* vector z satisfying

$$z^{\mathrm{T}}(q + Mz) = 0, \tag{18}$$

the objective function of (15) satisfies

$$f(z) = \tfrac{1}{2} q^{\mathrm{T}} z. \tag{19}$$

Thus, in the the first row of schema (17), we have $\theta = q^{\mathrm{T}}z = 2f(z)$, provided z satisfies (18).

Up to a principal rearrangement, each schema produced by the algorithm has the bisymmetric form

$$
\begin{array}{c|c|c|c|}
 & 1 & w_\alpha & z_\beta \\
\hline
\theta & -q_\alpha^{\mathrm{T}} M_{\alpha\alpha}^{-1} q_\alpha & q_\alpha^{\mathrm{T}} M_{\alpha\alpha}^{-1} & q_\beta^{\mathrm{T}} - q_\alpha^{\mathrm{T}} M_{\alpha\alpha}^{-1} M_{\alpha\beta} \\
\hline
z_\alpha & -M_{\alpha\alpha}^{-1} q_\alpha & M_{\alpha\alpha}^{-1} & -M_{\alpha\alpha}^{-1} M_{\alpha\beta} \\
\hline
w_\beta & q_\beta - M_{\beta\alpha} M_{\alpha\alpha}^{-1} q_\alpha & M_{\beta\alpha} M_{\alpha\alpha}^{-1} & M_{\beta\beta} - M_{\beta\alpha} M_{\alpha\alpha}^{-1} M_{\alpha\beta} \\
\hline
\end{array}
\qquad (20)
$$

The number in the upper left-hand corner of the schema (20) has a special significance. At the basic solution corresponding to this schema, it equals twice the value of the function $f(z)$ as defined in (15). The argument for the finite termination of this algorithm is based (in part) on how this number changes in the course of the procedure; note, however, that the algorithm itself makes no direct use of the information provided by the first row in any of these schemas.

The condition described in Step 1A is clearly the desired outcome; if it holds, the vector $\bar{z}$ so defined would be a solution of the problem. If that condition does not hold, then the distinguished basic variable w_r^ν and the (complementary) driving variable z_r^ν are specified. The reason for the kind of termination indicated in Step 1C is related to the bisymmetry property of the schema. The condition $m_{rr}^\nu = 0$ implies $m_{ri}^\nu = 0$ for all $i \in \beta$, and the condition $m_{ir}^\nu \geq 0$ for all $i \in \alpha$ implies $m_{ri}^\nu \leq 0$ for all $i \in \alpha$. Thus, the rth equation of the transformed system has no nonnegative solution. If the preceding steps do not force termination, the driving variable must be blocked by some eligible basic variable: either the distinguished variable w_r^ν increasing to zero, or a basic z-variable, decreasing to zero (or, in the degenerate case, already at the value zero, but a decreasing function of z_r^ν). The principal pivot $\langle w_s^\nu, z_s^\nu \rangle$ called for in Step 3 is possible. If $s = r$, the diagonal entry m_{rr}^ν is positive. If $s \neq r$, then $s \in \alpha$, and since $M_{\alpha\alpha}^\nu$ is positive definite, its diagonal entries are positive. In either case, the principal pivot can be executed, thereby leading to $m_{rr}^{\nu+1} > 0$. The transfer of the blocking variable's index means that α will be the set of indices of the basic z-variables at iteration $\nu + 1$.

Step 1 marks the beginning of a new major cycle. There can be only finitely many minor cycles (returns to Step 2) within a major cycle because each of them reduces the cardinality of α. Within a minor cycle, the value of θ is given by

$$\theta = -q_\alpha^{\mathrm{T}} M_{\alpha\alpha}^{-1} q_\alpha + q_r^\nu z_r^\nu \tag{21}$$

and is easily seen to be nonincreasing. Indeed, since $q_r^\nu < 0$, it strictly decreases whenever z_r^ν strictly increases. Only basic z-variables whose value is zero can prevent z_r^ν from increasing strictly. When the major cycle ends, and the algorithm returns to Step 1, there is a strict decrease in the value of θ. To complete the argument for finite termination, we observe that at every iteration, there is a complementary basis, and there are only finitely many complementary bases. Each major iteration is finite in length, and no complementary basis can be repeated because of the (eventual) strict decrease property. Hence Algorithm **4.2.11** processes (q, M) after finitely many steps.

4.3 General Principal Pivoting Methods

A given linear complementarity problem and its principal pivotal transforms are all *equivalent* linear complementarity problems in the sense that from a solution of one such transformed problem, a solution of the original can be constructed. A linear complementarity problem (q, M) in which $q \geq 0$ is certainly easy to solve. One immediately obtains a solution by setting the nonbasic variables z equal to zero. Pivoting methods for the LCP are motivated by the idea that although a given LCP (q, M) might not have $q \geq 0$, the problem might still possess a principal pivotal transform that *does* satisfy this condition.

It is tempting to believe that if the LCP (q, M) is solvable, then it *must* possess a principal pivotal transform (q', M') in which $q' \geq 0$. Unfortunately, this is not true as shown by any problem of the form (q, M) where

$$q \simeq \begin{bmatrix} - \\ 0 \end{bmatrix} \quad \text{and} \quad M \simeq \begin{bmatrix} 0 & + \\ 0 & 0 \end{bmatrix}. \tag{1}$$

Such linear complementarity problems have *no* nontrivial principal pivotal transforms at all. (Geometrically speaking, there exist problems (q, M) such that q belongs to $K(M)$ but not to any full complementary cone relative to M.) Nevertheless, many linear complementarity problems have trivially solvable principal pivotal transforms. In fact, it could be said that most LCPs encountered in practice—and certainly the ones treated in this section—*do* have this property.

The algorithms presented in this section involve sequences of principal pivoting operations. The algorithms are "general" in the sense that they do not exclusively involve "simple" principal pivots. They are definitely *not* general with respect to the matrix class to which M must belong. In the so-called "symmetric version" of the principal pivoting method (PPM), all the pivot blocks are either 1×1 or 2×2 principal submatrices of the matrix used in defining the problem. The "asymmetric" variant of the algorithm can (but need not) execute block principal pivots of larger order. The sense in which a particular version of the PPM is symmetric or asymmetric has to do with the index sets of the basic and nonbasic variables rather than with a symmetry property of the matrix M; in fact, these methods require no such symmetry assumption on M.

The principal pivoting method—symmetric version

The symmetric version of the PPM uses principal pivotal transformations (of order 1 or 2) in order to achieve one of two possible terminal sign configurations in the tableau (4.2.4). The first is a nonnegative "constant column", that is, $q_i^\nu \geq 0$ for all $i = 1, \ldots, n$. The other is a row of the form

$$q_r^\nu < 0 \quad \text{and} \quad m_{rj}^\nu \leq 0 \quad j = 1, \ldots, n. \tag{2}$$

The first sign configuration signals the discovery of a solution to (q, M). The second sign configuration reveals that the problem has no feasible solution. The PPM (as originally conceived) does not actually check for this condition. Indeed, it cannot occur when M is a $\boldsymbol{P}$-matrix. When M is positive semi-definite, it can be inferred from the condition

$$q_r^\nu < 0, \quad m_{rr}^\nu = 0 \quad \text{and} \quad m_{ir}^\nu \geq 0 \quad \forall i \neq r, \tag{3}$$

which *is* checked in the "minimum ratio test." A key observation is that the same inference can be made when M is (row) sufficient. (See **3.5.3** and **4.1.8**.)

The PPM consists of a sequence of *major cycles*, each of which begins with the selection of a *distinguished variable* whose value is currently negative. That variable remains the one and only distinguished variable throughout the major cycle. The object during the major cycle is to make the value of the distinguished variable increase to zero, if possible. Each iteration involves the increase of a nonbasic variable in an effort to drive the distinguished variable up to zero. This increasing nonbasic variable is called the *driving variable*. According to the rules of the method, all variables whose values are currently nonnegative must remain so. The initial trial solution is $(w^0, z^0) = (q^0, 0)$, hence at least n of the variables must be nonnegative. For those variables w_i^0 whose initial value is $q_i^0 < 0$, we impose a *negative lower bound* λ where

$$\lambda < \min_{1 \leq i \leq n} \{q_i^0\}.$$

This artifice is used in all cases except that of $M \in \boldsymbol{P}$ where it is not needed; although the artifice is not needed in the $\boldsymbol{P}$-matrix case, it is not *mathematically* incorrect to include it in the statement of the algorithm. Then, in addition to requiring all variables with currently nonnegative values to remain so, the PPM also demands that the variables currently having a negative value remain at least as large as λ. To accommodate this feature, we broaden the notion of *basic solution* by allowing the nonbasic variables to have the value 0 or λ. We also say that a solution of the system (4.2.3) is *nondegenerate* if at most n of its $2n$ variables have the value 0 or λ. Otherwise, the solution is called *degenerate*. In this section we assume the nondegeneracy of all the basic solutions arising in the execution of the methods under discussion. Procedures for handling degenerate problems will be taken up in Section 4.9.

For greater clarity, we introduce the following notations. To distinguish between the names of variables and their values, we use bars over the generic variable names w_i^ν and z_i^ν to indicate definite values of these variables. At the beginning of a major cycle in which negative lower bounds λ are in use, we will have $\bar{z}_i^\nu = 0$ or $\bar{z}_i^\nu = \lambda$ $i = 1, \ldots, n$. Next, we use the notation

$$W^\nu(z^\nu) = q^\nu + M^\nu z^\nu.$$

The definition of the mapping W^ν is identical to that of w^ν; it merely emphasizes the argument z^ν.

4.3.1 Example. A simple example will help to motivate the preceding ideas, especially the need for the negative lower bounds, λ. Consider the LCP of order 2 in which

$$q = \begin{bmatrix} -3 \\ -2 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}.$$

The matrix M is sufficient, i.e., row *and* column sufficient. At the outset we have $(\bar{w}^0, \bar{z}^0) = (-3, -2, 0, 0)$. Suppose we choose w_1^0 as the initial distinguished variable. The PPM then calls for z_1^0 to be used as the initial driving variable. If only nonnegative variables are required to remain nonnegative, there is no limit to the allowable increase of the driving variable. Under ordinary circumstances, such an outcome would indicate that the problem is unsolvable (at least by this method). But notice that this LCP has the solution $(\bar{w}, \bar{z}) = (1, 0, 0, 2)$. Hence some sort of modification is needed. Imposing a lower bound on the negative basic variables leads to blocking unless (3) holds.

If, at the outset of a major cycle, the selected distinguished variable is basic, the first driving variable is the *complement* of the distinguished variable. Thus, if w_r^ν is the distinguished variable for the current major cycle, then z_r^ν is the first driving variable. The distinguished variable need not be a basic variable, however. With the broader definition of basic solution (given above), the current solution $(\bar{w}^\nu, \bar{z}^\nu)$ may have $\bar{z}_r^\nu = \lambda < 0$ at the beginning of a major cycle. In such circumstances, z_r^ν can be the distinguished variable as well as the driving variable. In this event, the increase of the driving variable will always be blocked, either when a basic variable reaches its (current) lower bound (0 or λ) or when z_r^ν reaches zero (in which case the major cycle ends).

The following is a formal statement of this algorithm.

4.3.2 Algorithm. (Symmetric PPM, Nondegenerate Case)

Step 0. *Initialization.* Input (q, M) with M a row sufficient matrix. Define $(q^0, M^0) = (q, M)$ and $(\bar{w}^0, \bar{z}^0) = (q^0, 0)$. Let λ be any number less than $\min_i\{q_i^0\}$. Set $\nu = 0$.

Step 1. *Determine the distinguished variable.* If $q^\nu \geq 0$, stop; a solution can be recovered from the vector $(\bar{w}^{\nu+1}, \bar{z}^{\nu+1}) := (q^\nu, 0)$. Otherwise, determine an index r such that either $\bar{z}_r^\nu = \lambda$ (in which case z_r^ν is both the distinguished variable and the driving variable) or (if $\bar{z}^\nu = 0$) an index r such that $\bar{w}_r^\nu < 0$ (in which case w_r^ν is the distinguished variable, and its complement z_r^ν is the driving variable).

Step 2. *Determine the blocking variable (if any).* With z_r^ν as the driving variable and all other nonbasic variables fixed at their current values, let ζ_r^ν be the largest value of $z_r^\nu \geq \bar{z}_r^\nu$ satisfying the following conditions:

(i) the distinguished variable remains nonpositive;

(ii) all nondistinguished basic variables remain greater than or equal to their current lower bounds.

If $\zeta_r^\nu = +\infty$, stop. No feasible solution exists.
If $\zeta_r^\nu = 0$, no pivoting is necessary. Let $\bar{z}_r^{\nu+1} = 0$, $\bar{z}_i^{\nu+1} = \bar{z}_i^\nu$ for all $i \neq r$, and let

$$\bar{w}^{\nu+1} = W^{\nu+1}(\bar{z}^{\nu+1}) = W^\nu(\bar{z}^{\nu+1}).$$

Return to Step 1 with $\nu \leftarrow \nu + 1$.
If $\zeta_r^\nu < +\infty$, let t be the index of the blocking basic variable.

Step 3. *Pivoting.* If $m_{tt}^\nu > 0$, perform the principal pivot $\langle w_t^\nu, z_t^\nu \rangle$, making w_t^ν nonbasic at its lower bound value.
If $t = r$, return to Step 1 with $\nu \leftarrow \nu + 1$.
If $t \neq r$, return to Step 2 with $\nu \leftarrow \nu + 1$.
If $m_{tt}^\nu = 0$, perform the principal pivot $\{\langle w_t^\nu, z_r^\nu \rangle, \langle w_r^\nu, z_t^\nu \rangle\}$.
Return to Step 2 with $\nu \leftarrow \nu + 1$ and $r \leftarrow t$.

Discussion

Here we discuss what the algorithm does and why it actually processes any (nondegenerate) LCP (q, M) with a row sufficient matrix M.

All major cycles of the PPM begin at Step 1 where the algorithm checks whether it is possible to terminate with a solution. This will be the case if $(\bar{w}^\nu, \bar{z}^\nu) \geq (0,0)$ since $(\bar{w}^\nu, \bar{z}^\nu)$ must then be a nonnegative solution of (4.2.3) with $\bar{z}^\nu = 0$. As illustrated in the example below, it can happen that the constant column q^ν becomes nonnegative before z^ν does. In such a case, resetting z^ν to zero yields a solution. If neither of these forms of termination occurs, there is an index r such that $\bar{z}^\nu_r < 0$ or $\bar{w}^\nu_r < 0$ and it becomes the distinguished variable for the current major cycle.

For a linear complementarity problem (q, M) of order n, there are $2n$ variables in equation (4.2.2). The number of negative components in a solution of (4.2.2) is called its *infeasibility count.* The conditions imposed in Step 2 of the symmetric PPM prevent this number from increasing. Furthermore, with each return to Step 1, the algorithm produces a basic solution having a smaller infeasibility count than its predecessor, hence there can be at most finitely many returns to Step 1. The proof of finiteness therefore boils down to showing that each major cycle consists of at most a finite number of steps.

Termination can also occur in Step 2. In this event, $\zeta^\nu_r = +\infty$. For this to happen, the distinguished variable must be w^ν_r; it must also be true that

$$m^\nu_{rr} = 0 \quad \text{and} \quad m^\nu_{ir} \geq 0 \quad \forall i \neq r.$$

From **3.5.4(c)**, it follows that $m^\nu_{rj} \leq 0 \ j = 1, \ldots, n$. Now, since $\bar{z}^\nu_j \leq 0$ for all j and

$$\bar{w}^\nu_r = q^\nu_r + \sum_{j=1}^{n} m^\nu_{rj} \bar{z}^\nu_j < 0,$$

it follows that $q^\nu_r < 0$, so that the r-th equation

$$w^\nu_r = q^\nu_r + \sum_{j=1}^{n} m^\nu_{rj} z^\nu_j$$

has no nonnegative solution. Another outcome in Step 2 is that $\zeta^\nu_r = 0$ in which case (by nondegeneracy) the distinguished variable and the driving variable must have been z_r^ν which increased to zero. This brings the major cycle to a close without necessitating a pivot. The remaining possibility $0 \neq \zeta_r^\nu < +\infty$ means that *some* basic variable w_t^ν blocked the increase of z_r^ν.

The various alternatives that arise in the latter situation are addressed in Step 3. If $m_{tt}^\nu > 0$, the indicated principal pivot is executable. If $t = r$, the distinguished variable must have increased to 0. This brings about a return to Step 1 and a reduction of the infeasibility count by at least one. If $t \neq r$, the principal pivot is made and the increase of the driving variable continues in accordance with the rules of Step 2. If $m_{tt}^\nu = 0$, then $t \neq r$. The fact that w_t^ν blocked z_r^ν means $m_{tr}^\nu < 0$. The principal pivot of order 2 is executable because the row sufficiency of

$$\begin{bmatrix} m_{rr}^\nu & m_{rt}^\nu \\ m_{tr}^\nu & m_{tt}^\nu \end{bmatrix}$$

and the negativity of m_{tr}^ν implies that $m_{rt}^\nu > 0$. The values of the variables immediately after the pivot are those they had when blocking occurred. At the return to Step 2, the variable w_r^ν becomes $z_t^{\nu+1}$; a principal rearrangement to restore the natural order of subscripts would be possible.

As noted above, the argument that the algorithm will process any nondegenerate LCP with a row sufficient matrix comes down to showing that there can be at most finitely many returns to Step 2. But this is clear from the fact that there are only finitely many principal transformations of the system and finitely many ways to evaluate the nonbasic variables z_i^ν $(i \neq r)$. As for z_r^ν, its value and that of its complement w_r^ν increase monotonically and their sum increases strictly throughout the major cycle. Hence the definition of ζ_r^ν and $\zeta_r^{\nu+\kappa}$ $(\kappa > 0)$ make it impossible to have $z_i^\nu = z_i^{\nu+\kappa}$ $(i = 1, \ldots, n)$ and $\bar{z}_i^\nu = \bar{z}_i^{\nu+\kappa}$ $(i \neq r)$ as would have to be the case with infinitely many steps within a major cycle.

It should also be noted that within a major cycle (whose purpose is to make the distinguished variable w_r^ν or z_r^ν nonnegative) a basic variable w_i^ν with $\bar{w}_i^\nu < 0$ can serendipitously become nonnegative. When this occurs, its lower bound (for all future iterations) becomes zero. That is, once a variable becomes nonnegative, it stays nonnegative.

4.3.3 Example. Consider the LCP (q, M) where

$$q = \begin{bmatrix} -3 \\ 6 \\ -1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & -1 & 2 \\ 2 & 0 & -2 \\ -1 & 1 & 0 \end{bmatrix}.$$

The PPM applies to this problem because the matrix M is sufficient (as shown in **3.12.16**). It is easy to verify that (q, M) has the solution $(\bar{w}; \bar{z}) = (2, 0, 0; 0, 1, 3)$. The discussion below illustrates how this solution can be obtained by the symmetric version of the PPM. For simplicity, the superscripts (iteration counters) and bars (denoting fixed values of variables) have been omitted.

For this choice of data, the problem (q, M) has the tabular form

	1	z_1	z_2	z_3	
w_1	-3	0	-1	2	-3
w_2	6	2	0	-2	6
w_3	-1	-1	1	0	-1
	1	0	0	0	

The number $\lambda = -4$ will serve as the negative lower bound for the initial negative basic variables w_1 and w_3. Choose w_1 as the distinguished variable and its complement z_1 as the driving variable. The blocking variable is w_3 which decreases and reaches its lower bound -4 when z_1 increases to 3. Since the corresponding diagonal entry m_{33} equals 0, it is necessary to perform a principal pivot of order 2: $\langle w_3, z_1 \rangle$ and $\langle w_1, z_3 \rangle$. The new tableau is

	1	w_3	z_2	w_1	
z_3	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0
w_2	1	-2	1	-1	12
z_1	-1	-1	1	0	3
	1	-4	0	-3	

The numbers below the tableau are the current values of the nonbasic variables whereas the numbers to the right of the tableau are the corresponding values of the basic variables. At this stage the distinguished variable w_1 is nonbasic and can be increased directly as the driving variable. In this case, the driving variable blocks itself. Thus, the first major cycle ends with the tableau

	1	w_3	z_2	w_1	
z_3	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
w_2	1	-2	1	-1	9
z_1	-1	-1	1	0	3
	1	-4	0	0	

For the next major cycle, the only possible distinguished variable is w_3 which is nonbasic at value -4. This becomes the driving variable and is blocked when it reaches -1 and z_1 decreases to 0. Once again a principal pivot of order 2 is needed. This time it is $\langle z_1, w_3\rangle$ and $\langle z_3, w_1\rangle$ which leads to

	1	z_1	z_2	z_3	
w_1	-3	0	-1	2	0
w_2	6	2	0	-2	3
w_3	-1	-1	1	0	-1
	1	0	0	$\frac{3}{2}$	

Here the driving variable is z_3 which starts from the value $\frac{3}{2}$; it is blocked when it reaches 3 and w_2 decreases to 0. This time the algorithm performs a different principal pivot of order 2: $\langle w_2, z_3\rangle$ and $\langle w_3, z_2\rangle$. This yields

	1	z_1	w_3	w_2	
w_1	2	1	-1	-1	3
z_3	3	1	0	$-\frac{1}{2}$	3
z_2	1	1	1	0	0
	1	0	-1	0	

The distinguished variable is still w_3 whose current value is -1. If used as the driving variable, it will block itself and a solution will be obtained.

Another option is to observe that the "constant column" is positive. In such a case the negative basic variable(s) can be set equal to zero. Either way, the solution found is $(\bar{w}; \bar{z}) = (2, 0, 0; 0, 1, 3)$.

4.3.4 Remark. The device of imposing artificial negative lower bounds λ on negative variables in the PPM is not needed when the matrix $M \in \boldsymbol{P}$. In such instances, the algorithm executes only simple principal pivots since m^{ν}_{tt} is always positive. In the initial major cycle, the distinguished variable is basic and the driving variable cannot be unblocked because the distinguished variable is bounded above by zero and (in the nondegenerate case) will always increase strictly as the driving variable increases. Subsequent major cycles enjoy the same property.

The asymmetric version of the PPM

The so-called *asymmetric version* of the PPM also consists of a sequence of major cycles, each of which aims to make a distinguished variable become zero. But, instead of executing only principal pivots of order 1 or 2 as in the symmetric version, each major cycle involves a sequence of "simple pivots" whose effect may be a principal pivot of larger order. The rules governing blocking are the same as those in the symmetric PPM. (Nonnegative variables are bounded below by 0, and negative variables are bounded below by λ. A negative driving variable is bounded above by 0.) The main difference between the two versions of the algorithm is that the asymmetric one entails pivotal exchanges between the driving variable and the blocking variable; then it takes the new driving variable to be the complement of the blocking variable. We are assuming the problem (q, M) is nondegenerate and the matrix M is row sufficient. Under these conditions, the distinguished variable and the driving variable increase monotonically and their sum increases strictly.

4.3.5 Algorithm. (Asymmetric PPM, Nondegenerate Case)

Step 0. *Initialization.* Input (q, M) with M a row sufficient matrix. Define $(q^0, M^0) = (q, M)$ and $(\bar{w}^0, \bar{z}^0) = (q^0, 0)$. Let λ be any number less than $\min_i\{q_i^0\}$. Set $\nu = 0$.

Step 1. *Determine the distinguished variable.* If $q^\nu \geq 0$, stop; a solution can be recovered from the vector $(\bar{w}^{\nu+1}, \bar{z}^{\nu+1}) := (q^\nu, 0)$. Otherwise, determine either a nonbasic variable whose current value is λ (in which case this variable is both the distinguished variable and the driving variable) or (if $\bar{z}^\nu = 0$) a basic variable whose current value is negative (in which case it is the distinguished variable, and its complement is the driving variable).

Step 2. *Determine the blocking variable (if any).* Let ζ^ν denote the largest value of the driving variable such that when all other nonbasic variables are fixed at their current values,

(i) the distinguished variable remains nonpositive;

(ii) all basic variables remain greater than or equal to their current lower bounds.

If $\zeta^\nu = +\infty$, stop. No feasible solution exists.
If $\zeta^\nu = 0$, no pivoting is necessary. Set the value of the driving variable to 0 and return to Step 1 with $\nu \leftarrow \nu + 1$.
If $0 \neq \zeta^\nu < +\infty$, some variable blocks the driving variable.

Step 3. *Pivoting.* Pivot

$$\langle \textit{ blocking variable, driving variable } \rangle,$$

making the blocking variable nonbasic at the value it attains when the driving variable equals ζ^ν.
If the blocking variable is the distinguished variable, return to Step 1 with $\nu \leftarrow \nu + 1$.
Otherwise, return to Step 2 with the complement of the blocking variable as the new driving variable and $\nu \leftarrow \nu + 1$.

Discussion

The reader may have noticed that the primary difference between the symmetric and asymmetric versions of the principal pivoting method occurs in Step 3. A precise statement of the asymmetric PPM requires rather elaborate notational apparatus to handle the identification of the distinguished, driving and blocking variables. We have deliberately chosen to use words to express how the algorithm operates and to avoid the symbolic complications that would otherwise be employed in stating it.

The conditions imposed on the driving variable prevent the infeasibility count from increasing. Returns to Step 1 are accompanied by a reduction (by at least one) in this measure. The issue, as before, is the finiteness of the algorithm's major cycles.

When Algorithm **4.3.5** returns to Step 2, the schema is almost complementary. It has a basic pair (the distinguished variable and its complement) and a nonbasic pair (the last blocking variable and its complement which is the new driving variable). Because we are dealing with row sufficient matrices—the class of which is invariant under principal pivoting—the *pair matrix* has the characteristics described in the following proposition.

4.3.6 Proposition. Let A be a 2×2 matrix with the following properties:

(i) $a_{11} \leq 0$;

(ii) $a_{21} \leq 0$;

(iii) $a_{11} + a_{21} < 0$;

(iv) if $a_{11} < 0$, then

$$A_1 := \frac{1}{a_{11}} \begin{bmatrix} -a_{12} & 1 \\ a_{11}a_{22} - a_{12}a_{21} & a_{21} \end{bmatrix}$$

is row sufficient;

(v) if $a_{21} < 0$, then

$$A_2 := \frac{1}{a_{21}} \begin{bmatrix} a_{11} & a_{12}a_{21} - a_{11}a_{22} \\ 1 & -a_{22} \end{bmatrix}$$

is row sufficient.

Then A must have the following properties:

(vi) $a_{12} \geq 0$;

(vii) $a_{22} \geq 0$;

(viii) $a_{12} + a_{22} > 0$.

Proof. By (i)–(iii), at least one of a_{11}, a_{21} must be negative. Suppose $a_{11} < 0$. Then because A_1 is row sufficient, we must have $(-1/a_{11})a_{12} \geq 0$ and $\det A_1 \geq 0$. Hence $a_{12} \geq 0$ and

$$\left(\frac{1}{a_{11}}\right)^2 (-a_{12}a_{21} + a_{12}a_{21} - a_{11}a_{22}) = \left(\frac{-1}{a_{11}}\right) a_{22} \geq 0$$

so that $a_{22} \geq 0$. If $a_{12} = a_{22} = 0$, then

$$A_1 = \frac{1}{a_{11}} \begin{bmatrix} 0 & 1 \\ 0 & a_{21} \end{bmatrix}.$$

It is easily shown that a 2×2 matrix with one zero column and one nonzero off-diagonal element cannot be row sufficient. This contradiction proves $a_{12}+a_{22} > 0$. The case where $a_{21} < 0$ is analogous to that of $a_{11} < 0$, hence we omit the argument and declare that, in either case, $a_{12} + a_{22} > 0$. □

It follows from this proposition that the members of the basic pair are nondecreasing functions of the driving variable. Moreover, in the present nondegenerate case, the driving variable can be increased, and at least one of the variables in the basic pair strictly increases as this happens. The argument for finite termination of the major cycle now follows in the usual way: there are only finitely many almost complementary basic solutions and none of them can be repeated because of the aforementioned strict increase property.

4.3.7 Example. Let us consider what happens when we use Algorithm **4.3.5** to solve the LCP given in Example **4.3.3**. Again we take $\lambda = -4$ as the negative lower bound on w_1 and w_3. Essentially the same preliminary remarks about M and the superscripts apply here too. Recall that the schema is

	1	z_1	z_2	z_3	
w_1	−3	0	−1	2	−3
w_2	6	2	0	−2	6
w_3	−1	−1	1	0	−1
	1	0	0	0	

We may select either w_1 or w_3 as the first distinguished variable. Let us arbitrarily take w_1 for this purpose. Its complement, z_1, will be the first driving variable. It turns out that the first blocking variable is w_3. After the pivot $\langle w_3, z_1 \rangle$, we obtain the schema

	1	w_3	z_2	z_3	
w_1	-3	0	-1	2	-3
w_2	4	-2	2	-2	12
z_1	-1	-1	1	0	3
	1	-4	0	0	

Notice that the values of the basic and nonbasic variables are recorded below and to the right of the schema. The variable w_1 is still negative and distinguished. The next driving variable is z_3, the complement of the previous blocking variable. It can be increased up to $\frac{3}{2}$ at which point w_1 rises to 0. The pivot $\langle w_1, z_3 \rangle$ yields the complementary schema

	1	w_3	z_2	w_1	
z_3	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
w_2	1	-2	1	-1	9
z_1	-1	-1	1	0	3
	1	-4	0	0	

Hereafter, the lower bound on w_1 will be 0, rather than λ.

The first major cycle is complete, but since the termination criteria are not satisfied, it is time to choose a new distinguished variable. In accordance with the statement of the algorithm, we select the nonbasic variable w_3; it is, in fact, the *only* choice for this schema. Here we have the situation where w_3 is the driving variable as well as the distinguished variable. Its increase is blocked by z_1 which decreases when w_3 increases (from -4) to -1. The corresponding pivot $\langle z_1, w_3 \rangle$ leads to the schema

	1	z_1	z_2	w_1	
z_3	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
w_2	3	2	-1	-1	3
w_3	-1	-1	1	0	-1
	1	0	0	0	

The new driving variable is w_1, the complement of the blocking variable. In this instance, the new blocking variable is w_2 which decreases to 0 when w_1 reaches 3. This time the pivot is $\langle w_2, w_1 \rangle$. The resulting schema is

	1	z_1	z_2	w_2	
z_3	3	1	0	$-\frac{1}{2}$	3
w_1	3	2	−1	−1	3
w_3	−1	−1	1	0	−1
	1	0	0	0	

The distinguished variable is still w_3, and the driving variable is now z_2. The minimum ratio test shows that w_3 rises to 0 before the positive variable w_1 decreases to 0. The next pivot is $\langle w_3, z_2 \rangle$, and the major cycle ends with

	1	z_1	w_3	w_2	
z_3	3	1	0	$-\frac{1}{2}$	3
w_1	2	1	−1	−1	2
z_2	1	1	1	0	1
	1	0	0	0	

This complementary schema reveals that we have found a solution of the given LCP.

4.4 Lemke's Method

In this section we turn to the complementary pivoting schemes due to C.E. Lemke. In some respects these algorithms resemble the principal pivoting method inasmuch as they use pivotal exchanges and a choice of driving variable like that of the asymmetric PPM. One advantage of these complementary pivoting schemes is that they are very easy to state. They are also more versatile than the PPM as they do not rely on the invariance of matrix classes under principal pivoting. Over the years, these algorithms have stimulated a considerable amount of research into the classes of matrices M for which they can process the LCP (q, M).

Lemke's complementary pivoting schemes were preceded by the closely related Lemke-Howson algorithm for the bimatrix game problem as formulated in Section 1.2. We shall treat this special class of linear complementarity problems later in this section.

Scheme I

The stage for the algorithm we are about to discuss has already been set in Section 3.7 where an augmented LCP based upon a given LCP is formulated. In fact, this algorithm is mentioned there as one way of proving the existence of a solution to the augmented problem (3.7.2). For the sake of notational uniformity, however, we take the liberty of principally rearranging the system and renaming three of its ingredients. Thus, for a given LCP (q, M) we now write the augmented LCP as $(\tilde{q}, \tilde{M})$ where

$$\tilde{q} = \begin{bmatrix} q_0 \\ q \end{bmatrix} \quad \text{and} \quad \tilde{M} = \begin{bmatrix} 0 & -d^{\mathrm{T}} \\ d & M \end{bmatrix}. \tag{1}$$

In this representation, $q_0 \geq 0$ is a sufficiently large constant.[1] The augmented LCP then takes the form

$$\begin{aligned} w_0 &= q_0 + 0 \cdot z_0 - d^{\mathrm{T}} z \geq 0, \quad z_0 \geq 0, \quad z_0 w_0 = 0 \\ w &= q + d z_0 + M z \geq 0, \qquad z \geq 0, \quad z^{\mathrm{T}} w = 0. \end{aligned} \tag{2}$$

The (user-supplied) vector d is often called the *covering vector*. Recall that $d > 0$. Accordingly, there exists a smallest scalar $\bar{z}_0$ such that

$$w = q + d z_0 \geq 0 \text{ for all } z_0 \geq \bar{z}_0. \tag{3}$$

In fact,

$$\bar{z}_0 = \max_i \{-q_i / d_i\}. \tag{4}$$

Under the reasonable[2] assumption that $q_i < 0$ for some i, it follows that $\bar{z}_0$ will be positive. A solution of $(\tilde{q}, \tilde{M})$ in which $z_0 = 0$ yields a solution of the original problem (q, M). Lemke's complementary pivoting algorithm known as "Scheme I" attempts to find such a solution.

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[1] In (3.7.2), the symbols used for q_0, w_0 and z_0, respectively, are λ, σ, and θ.

[2] In the standard case where only one solution of the LCP is to be found, an LCP (q, M) with $q \geq 0$ is trivial in the sense that $z = 0$ is an obvious solution and nothing more need be done.

Throughout this section, it will be assumed in our discussion of Lemke's method that all basic solutions of the system of equations

$$\begin{bmatrix} w_0 \\ w \end{bmatrix} = \begin{bmatrix} q_0 \\ q \end{bmatrix} + \begin{bmatrix} 0 & -d^{\mathrm{T}} \\ d & M \end{bmatrix} \begin{bmatrix} z_0 \\ z \end{bmatrix} \tag{5}$$

are *nondegenerate.* With suitable precautions (discussed in Section 4.9), this assumption can be relaxed.

4.4.1 Algorithm. (Lemke, Scheme I – Augmented Problem)

Step 0. *Initialization.* Input $(\tilde{q}, \tilde{M})$. If $q \geq 0$, then stop: $z = 0$ solves (q, M). Otherwise, let $\bar{z}_0$ be the smallest value of the (artificial) variable z_0 for which $w = q + dz_0 \geq 0$. Let w_r denote the (unique, by the nondegeneracy assumption) component of w that equals zero when $z_0 = \bar{z}_0$. Pivot $\langle w_r, z_0 \rangle$. (The complements w_0 and z_0 are now basic whereas w_r and z_r are nonbasic.) Choose the driving variable to be the complement of w_r, namely z_r.

Step 1. *Determination of the blocking variable.* Use the minimum ratio test to determine the basic variable that blocks the increase of the driving variable. If w_0 is the blocking variable, then stop. (Interpret this outcome, if possible.)

Step 2. *Pivoting.* The driving variable is blocked.

- If z_0 is the blocking variable, then pivot

  $$\langle\ z_0,\ \textit{driving variable}\ \rangle$$

  and stop. A solution to (q, M) is at hand.

- If some other variable blocks the driving variable, then pivot

  $$\langle\ \textit{blocking variable, driving variable}\ \rangle.$$

  Return to Step 1 using the complement of the most recent blocking variable as the new driving variable.

In Algorithm **4.4.1**, the phrasing of Steps 1 and 2 begs an important question: the *existence* of the blocking variable. The ability to execute the algorithm depends critically on the fact that after the first pivot step $\langle w_r, z_0 \rangle$, the column of the driving variable *always* contains at least one negative entry. This assertion rests on the following lemma.

4.4.2 Lemma. Let $\hat{M}$ denote a square matrix of the form

$$\hat{M} = \begin{bmatrix} a^{\mathrm{T}} & \delta \\ B & c \end{bmatrix}$$

in which $a \leq 0$, $\delta < 0$, and B is nonsingular. Then the pivotal transform of $\hat{M}$ obtained by using B as the pivot block has at least one negative entry in its last column.

Proof. The pivotal transform is the matrix

$$\begin{bmatrix} a^{\mathrm{T}}B^{-1} & \delta - a^{\mathrm{T}}B^{-1}c \\ B^{-1} & -B^{-1}c \end{bmatrix}.$$

Its last column consists of the scalar $\delta - a^{\mathrm{T}}B^{-1}c$ and the vector $-B^{-1}c$. If $-B^{-1}c \geq 0$, then $\delta - a^{\mathrm{T}}B^{-1}c < 0$ by virtue of the sign assumptions on a and δ. □

To apply this lemma to the issue under discussion, we think of $\hat{M}$ as a submatrix of $\tilde{M}$. The submatrix B stands for the *cumulative* pivot block at any stage of the process. The positioning of B as described in the lemma is only for ease of discussion; moreover, it can be moved there by suitable rearrangement of the rows and columns of $\tilde{M}$. The first column of $\hat{M}$ is to be regarded as a subvector of the first column of $\tilde{M}$. This is appropriate since z_0 and its column are involved in the first pivot step. The first row of $\hat{M}$ is likewise a subvector of the first row of $\tilde{M}$. Notice that a and δ derive their sign properties from the assumption $d > 0$ and the definition of $\tilde{M}$. With this interpretation, we immediately obtain the following result.

4.4.3 Theorem. In Algorithm **4.4.1**, when w_0 and z_0 are basic, the column of the driving variable contains at least one negative entry. □

In **3.7.4**, it was noted that it is not necessary to use the vector d twice in defining the augmented LCP. Instead, we could replace the constraint involving w_0 by one of the form $w_0 = q_0 - \bar{d}^{\mathrm{T}} z \geq 0$ where $\bar{d} > 0$. The argument for this was made on the basis of **3.7.3**, but we can see it again as a consequence of **4.4.2**.

Another interesting point can be made here. Recall that we are assuming $\min_i q_i < 0$. In the definition of $\tilde{M}$ we used $d > 0$ (in the first column). Actually, this vector plays the role of making $q + dz_0 \geq 0$ for suitably large z_0. To achieve this we need only $d_i > 0$ when $q_i < 0$. It is not necessary for every component of d to be positive unless $q < 0$. To put this another way, it is enough to choose the column vector d so as to make the $n \times 2$ matrix $[d, q]$ have *lexicographically nonnegative* rows.

Almost complementary paths

Lemke's method (Scheme I) is initialized in a particular way. Making $z_0 \geq \bar{z}_0$ and $z = 0$ yields a set of "feasible solutions" to the augmented problem $(\tilde{q}, \tilde{M})$. These points constitute what is called the *primary ray*.[3] They satisfy the conditions

$$z_0 w_0 > 0 \quad \text{and} \quad z_i w_i = 0 \text{ for all } i \neq 0.$$

Such vectors (z_0, z) are said to be *almost complementary* with respect to (5).[4] Thus, the primary ray consists of almost complementary feasible points for $(\tilde{q}, \tilde{M})$.

The basic solutions generated by Scheme I correspond to almost complementary extreme points of FEA$(\tilde{q}, \tilde{M})$. At each such basic solution generated by **4.4.1**, there is a nonbasic pair w_r, z_r. One of these variables just became nonbasic, and the other is the next driving variable. Notice that even when this driving variable is positive, the corresponding feasible vector will be almost complementary because for all $i \neq 0$, at least one of the two complementary variables z_i, w_i will be nonbasic at value 0. (The same would be true if the other member of the nonbasic pair were made positive.) The point sets generated by the algorithm between extreme points are almost complementary *edges* of FEA$(\tilde{q}, \tilde{M})$.

Altogether, Algorithm **4.4.1** produces an *almost complementary path* of feasible solutions to $(\tilde{q}, \tilde{M})$. In the favorable outcome where the algo-

[3] Strictly speaking, these points form a *halfline* rather than a ray.

[4] In general, if $f : R^N \longrightarrow R^N$, the vector $x \in R^N$ is *almost complementary* with respect to the equation $y = f(x)$ if there exists an index k such that $x_i y_i = 0$ for all $i \neq k$.

rithm terminates with z_0 as the blocking variable and hence at value 0, the solution at hand is actually *complementary*, and the final pivot step exhibits the values of the basic variables. In particular, the z_i that are nonbasic in the final tableau have value 0; the other z_i can be read off from the current update of the system (2).

Finiteness of the algorithm

It remains to be shown that Algorithm **4.4.1** must terminate after finitely many steps.

4.4.4 Theorem. When applied to a nondegenerate instance of the augmented problem $(\tilde{q}, \tilde{M})$, Algorithm **4.4.1** terminates in finitely many steps.

Proof. By **4.4.3**, a suitable pivot entry is available at every iteration. The nondegeneracy assumption implies that the algorithm generates a unique almost complementary path. The almost complementary extreme points of FEA$(\tilde{q}, \tilde{M})$ that occur along this path correspond to almost complementary nonnegative basic solutions of (2). There are at most two almost complementary edges of the path incident to an almost complementary extreme point of FEA$(\tilde{q}, \tilde{M})$. These edges can be swept out by making one of the members of the nonbasic pair increase from the value 0. The nondegeneracy assumption guarantees that *all* nonbasic variables can be made positive before a basic variable decreases to zero. The almost complementary path cannot return to a previously encountered almost complementary extreme point, for otherwise, there would have to be at least three almost complementary edges incident to it. For a given problem, there are only finitely many bases of any kind and *a fortiori* only finitely many almost complementary bases. Hence the algorithm must terminate in a finite number of steps. □

A streamlined version of Scheme I

Choosing a suitable value of q_0 may seem a bit mysterious. When it actually comes down to computing, it is disturbing to be instructed to make q_0 sufficiently large. Fortunately, it is possible to dispense with the constraint $w_0 = q_0 - d^{\mathrm{T}}z \geq 0$ and hence with the question of selecting this constant. The more practical version of the augmented problem is defined as

$$w = q + dz_0 + Mz \geq 0, \quad z_0 \geq 0, \quad z \geq 0, \quad z^T w = 0. \tag{6}$$

We denote this system by (q, d, M). A solution of (q, d, M) with $z_0 = 0$ furnishes a solution to the original LCP (q, M).

The version of Lemke's Scheme I for (q, d, M) is a simple variant of the one for $(\tilde{q}, \tilde{M})$. But, without the constraint $w_0 \geq 0$ that is used in $(\tilde{q}, \tilde{M})$, we have to allow for the possibility that the column of a driving variable may be nonnegative and hence that the variable may be *unblocked*. Such a situation corresponds to the case where q_0 is large and w_0 is the blocking variable in Algorithm **4.4.1** for the full augmented problem. In the event that no blocking variable is found when a driving variable is increased, a *secondary ray* is generated. A secondary ray is also an almost complementary edge of FEA(q, d, M), but it is necessarily unbounded.

The streamlined version of Lemke's Scheme I runs as follows.

4.4.5 Algorithm. (Lemke, Scheme I)

Step 0. *Initialization.* Input (q, d, M). If $q \geq 0$, then stop: $z = 0$ solves (q, M). Otherwise, let $\bar{z}_0$ be the smallest value of the (artificial) variable z_0 for which $w = q + dz_0 \geq 0$. Let w_r denote the (unique, by the nondegeneracy assumption) component of w that equals zero when $z_0 = \bar{z}_0$. Pivot $\langle w_r, z_0 \rangle$. (After this pivot, the complementary variables w_r and z_r are both nonbasic.) Choose the driving variable to be the complement of w_r, namely z_r.

Step 1. *Determination of the blocking variable (if any).* If the column of the driving variable has at least one negative entry, use the minimum ratio test to determine the basic variable that blocks the increase of the driving variable. If the driving variable is unblocked, then stop. (Interpret this outcome, if possible.)

Step 2. *Pivoting.* The driving variable is blocked.

- If z_0 blocks the driving variable, then pivot

$$\langle\ z_0,\ \textit{driving variable}\ \rangle$$

  and stop. A solution to (q, M) is at hand.

- If some other variable blocks the driving variable, then pivot

  $$\langle \textit{ blocking variable, driving variable } \rangle.$$

  Return to Step 1 using the complement of the most recent blocking variable as the new driving variable.

4.4.6 Remarks. (a) The n-vector e whose components are all 1 is often used as the default covering vector. Other choices are possible, however. The choice of the covering vector can drastically affect the outcome of the procedure. This point is illustrated in Exercise **4.11.13** where we consider a $\boldsymbol{Q}$-matrix and match it with different vectors q and d to form systems (q, d, M) for solution by **4.4.5**. The results of the computation differ considerably. A related discussion on the choice of the covering vector is given in Section 4.8. See also **4.12.14**.

(b) After the initial pivot which makes the artificial variable z_0 basic, there is one (and only one) complementary pair of nonbasic variables, initially, w_r and z_r are such a pair. In general, the nonbasic complementary variables make up the *nonbasic pair*.

(c) After the initial increase of z_0 to the value $\bar{z}_0$, all the basic variables are nonnegative and are required to stay so. All of them are *eligible* to block the driving variable. In general, after a pivot occurs, the new driving variable is the complement of the variable that just became nonbasic.

(d) The above discussion is couched in the language of pivoting and tableaux (schemas) — even though none of the latter have actually been written down. This conception of the algorithm is not essential. One can implement the algorithm in a "revised simplex method fashion." Notice that only two columns are needed to execute the minimum ratio test in Step 1. These are the updated constant column and the updated column of the current driving variable. Thus, if the current basis is known, these updated columns can be determined by solving two systems of linear equations. For more on this see Section 4.10.

(e) Interpreting the nonexistence of a blocking variable (as mentioned in Step 1) is a major subject in itself. The essence of the idea is that there are certain classes of matrices for which the nonexistence of a blocking variable implies the infeasibility of the given LCP (q, M).

4.4.7 Example. Consider the LCP that was solved in Example **4.3.3** using the symmetric PPM. In the solution given below, we use the covering vector $e = (1, 1, 1)$. The practical version of Lemke's Scheme I then has the initial tableau

	1	z_0	z_1	z_2	z_3
w_1	-3	1	0	-1	2
w_2	6	1	2	0	-2
w_3	-1	1	-1	1	0

In this case, $\bar{z}_0 = 3$. At this value, the blocking variable is w_1, and the pivot to be performed is $\langle w_1, z_0 \rangle$. The resulting tableau is

	1	w_1	z_1	z_2	z_3
z_0	3	1	0	1	-2
w_2	9	1	2	1	-4
w_3	2	1	-1	2	-2

The driving variable is now z_1, the complement of the preceding blocking variable. There is only one candidate for blocking variable, namely w_3. The pivot operation, then, is $\langle w_3, z_1 \rangle$.

	1	w_1	w_3	z_2	z_3
z_0	3	1	0	1	-2
w_2	13	3	-2	5	-8
z_1	2	1	-1	2	-2

The driving variable is now z_3. This time, the minimum ratio test is needed to determine the blocking variable. We find that

$$\min\{3/2,\ 13/8,\ 2/2\} = 2/2,$$

hence the blocking variable is z_1. Performing the pivot operation $\langle z_1, z_3 \rangle$, we obtain the tableau

	1	w_1	w_3	z_2	z_1
z_0	1	0	1	-1	1
w_2	5	-1	2	-3	4
z_3	1	$\frac{1}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$

The new driving variable w_1 is blocked by w_2, so the pivot operation is $\langle w_2, w_1 \rangle$.

	1	w_2	w_3	z_2	z_1
z_0	1	0	1	-1	1
w_1	5	-1	2	-3	4
z_3	$\frac{7}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$

The column of the driving variable z_2 contains three negative entries. Performing the corresponding minimum ratio test, we find that the blocking variable is z_0. This indicates that after perfoming the pivot step $\langle z_0, z_2 \rangle$, a solution will be recovered from the resulting tableau which is

	1	w_2	w_3	z_0	z_1
z_2	1	0	1	-1	1
w_1	2	-1	-1	3	1
z_3	3	$-\frac{1}{2}$	0	$\frac{1}{2}$	1

Thus, we have found the solution $(\bar{z}_1, \bar{z}_2, \bar{z}_3) = (0, 1, 3)$.

4.4.8 Theorem. When applied to a nondegenerate instance of (q, d, M), Algorithm **4.4.5** will terminate in finitely many steps with either a secondary ray or else a complementary feasible solution of (q, d, M) and hence with a solution of (q, M).

Proof. This follows from **4.4.4**. Termination on a secondary ray in solving (q, d, M) with **4.4.5** is analogous to having w_0 as the final blocking variable in solving the corresponding LCP $(\tilde{q}, \tilde{M})$ with **4.4.1**. In particular, if **4.4.5** does not terminate with a complementary feasible solution of (q, d, M), then it must terminate with a secondary ray. □

It is interesting and fruitful to explore the consequences of termination with a secondary ray. Before doing so, however, a word of caution is in order. We have previously suggested that lexicographic nonnegativity of $[d, q]$ is enough to initiate Algorithm **4.4.5**. Despite this, the meaningful interpretation of termination with a secondary ray usually requires the strict positivity of the covering vector d. Accordingly, this assumption will be in force for the remainder of the discussion of Scheme I.

4.4.9 Theorem. If Algorithm **4.4.5** applied to (q, d, M) terminates with a secondary ray, then M reverses the sign of some nonzero nonnegative vector $\tilde{z}$, that is

$$\tilde{z}_i(M\tilde{z})_i \leq 0 \quad i = 1, \ldots, n. \tag{7}$$

Proof. In general, let $w = q + dz_0 + Mz$ and let (w^*, z_0^*, z^*) be the nonnegative basic solution of this equation corresponding to the almost complementary extreme point incident to the terminal secondary ray. The points of this ray correspond to vectors of the form

$$(w^* + \lambda\tilde{w}, z_0^* + \lambda\tilde{z}_0, z^* + \lambda\tilde{z})$$

where $\lambda \geq 0$ and $(\tilde{w}, \tilde{z}_0, \tilde{z})$ is a nonzero nonnegative solution of the homogeneous system

$$w = dz_0 + Mz. \tag{8}$$

In particular, for all $\lambda \geq 0$

$$w^* + \lambda\tilde{w} = q + d(z_0^* + \lambda\tilde{z}_0) + M(z^* + \lambda\tilde{z}) \tag{9}$$

and

$$(w_i^* + \lambda\tilde{w}_i)(z_i^* + \lambda\tilde{z}_i) = 0 \quad i = 1, \ldots, n. \tag{10}$$

We claim that $\tilde{z} \neq 0$. Indeed, if $\tilde{z} = 0$, then $\tilde{z}_0 > 0$ since $0 \neq (\tilde{w}, \tilde{z}_0, \tilde{z}) \geq 0$. This implies $\tilde{w} = d\tilde{z}_0 > 0$, and hence by (10), $z^* + \lambda\tilde{z} = z^* = 0$. But then the secondary ray is the primary ray, which is a contradiction.

From (10), it follows that

$$z_i^* w_i^* = z_i^* \tilde{w}_i = \tilde{z}_i w_i^* = \tilde{z}_i \tilde{w}_i = 0 \quad i = 1, \ldots, n. \tag{11}$$

Taking the i-th equation of (8) and multiplying it by $\tilde{z}_i$ we obtain

$$\tilde{z}_i(d\tilde{z}_0 + M\tilde{z})_i = 0 \quad i = 1, \ldots, n$$

which implies

$$\tilde{z}_i(M\tilde{z})_i \leq 0 \quad i = 1, \ldots, n.$$

Accordingly, M reverses the sign of the nonzero nonnegative vector $\tilde{z}$. □

More existence results

Theorem **4.4.9** implies that Algorithm **4.4.5** cannot terminate with a secondary ray when $M \in \boldsymbol{P}$, for a $\boldsymbol{P}$-matrix never reverses the sign of a nonzero vector. Hence for any nondegenerate linear complementarity problem of the $\boldsymbol{P}$-matrix type, Lemke's Scheme I will obtain its (necessarily unique) solution. Apart from the nondegeneracy issue, this proves once again that all $\boldsymbol{P}$-matrices belong to the class $\boldsymbol{Q}$.

In fact, we can easily obtain a more general result by recalling the class $\boldsymbol{E}$ of strictly semimonotone matrices introduced in Section 3.9. We noted there that $\boldsymbol{E}$ contains all $\boldsymbol{P}$-matrices and all strictly copositive matrices.

4.4.10 Theorem. Algorithm **4.4.5** will solve any nondegenerate LCP (q, M) such that $M \in \boldsymbol{E}$.

Proof. If **4.4.5** does not solve the LCP (q, M), then there must be a nonzero nonnegative vector $\tilde{z}$ whose sign is reversed by M. But $M \in \boldsymbol{E}$ and (by **3.9.11**) an $\boldsymbol{E}$-matrix cannot do such a thing. □

Theorem **4.4.9** has provided a necessary condition for termination on a secondary ray to occur in Algorithm **4.4.5** applied to (q, d, M). This result relies on no assumption of the matrix M, and we have seen two consequences of the result when the matrix M possesses some special property. In what follows, we show that many of the existence results for the LCP (q, M) that we derived in Chapter 3 using an analytic approach can actually be reproved constructively by means of Algorithm **4.4.5**. The significance of this constructive proof is that not only do solutions exist under the assumptions of these earlier results, they can be computed by an effective method.

Rather than repeating each individual existence conclusion proved in Chapter 3, we single out two fairly general results, **3.8.6** and **3.9.17**, and show why Algorithm **4.4.5** cannot terminate with a secondary ray under the usual nondegeneracy assumption of the LCP. For this purpose, we first derive a sharper ray-termination consequence for the class of semimonotone LCPs.

4.4.11 Theorem. Let $M \in R^{n \times n} \cap \boldsymbol{E}_0$. If Algorithm **4.4.5** applied to (q, d, M) terminates with a secondary ray, then $\mathrm{SOL}(0, M) \neq \{0\}$.

Proof. We continue to use the notation in the proof of **4.4.9**. As shown in that proof, $0 \neq \tilde{z} \in \text{SOL}(d\tilde{z}_0, M)$. As $d > 0$ and M is semimonotone, Theorem **3.9.3** implies that $\tilde{z}_0 = 0$. Consequently, $\tilde{z}$ is a nonzero solution of the homogeneous LCP $(0, M)$. □

If the matrix M is copositive, even more can be said about the vector $\tilde{z}$ obtained in **4.4.11**.

4.4.12 Corollary. Let $M \in R^{n \times n}$ be copositive. If Algorithm **4.4.5** applied to (q, d, M) terminates with a secondary ray, then the vector $y = \tilde{z}/\|\tilde{z}\|$ satisfies: (i) $0 \neq y \in \text{SOL}(0, M)$, and (ii) $q^{\text{T}}y < 0$.

Proof. It suffices to prove that y satisfies the second property (ii). Since $\tilde{z}_0 = 0$ (as proved in **4.4.9**), it follows from equations (9) and (10) that for all $\lambda \geq 0$,

$$0 = (z^* + \lambda\tilde{z})^{\text{T}}(q + dz_0^* + M(z^* + \lambda\tilde{z})) \geq (z^* + \lambda\tilde{z})^{\text{T}}(q + dz_0^*)$$

where the last inequality follows because of the copositivity of M. Normalizing by $\|z^* + \lambda\tilde{z}\|$ and passing to the limit $\lambda \to \infty$, we deduce

$$0 \geq y^{\text{T}}(q + dz_0^*)$$

which implies (ii) because $y^{\text{T}}dz_0^* > 0$. □

An immediate consequence of **4.4.12** is that if M is copositive and q is in $(\text{SOL}(q, M))^*$, then Algorithm **4.4.5** cannot terminate with a secondary ray; therefore, the algorithm must terminate with a solution of the LCP (q, M). Thus, we have proved

4.4.13 Theorem. If M is copositive and $q \in (\text{SOL}(0, M))^*$, Algorithm **4.4.5** will compute a solution of the LCP (q, M) if the problem is nondegenerate.

4.4.14 Remark. Using an analytic argument, we have proved in Theorem **3.8.6** that the LCP (q, M) must have a solution under the assumption of **4.4.13**. The derivation herein therefore provides a *constructive* proof to the same result (except for the nondegeneracy assumption). We also recall from Corollary **3.8.10** that if M is copositive-plus, then $q \in (\text{SOL}(0, M))^*$

if and only if (q, M) is feasible. Hence for a feasible, nondegenerate LCP (q, M) with M being copositive-plus, Algorithm **4.4.5** will always compute a solution of this problem.

The next result shows that the conclusion of Theorem **4.4.13** remains valid for the class of semimonotone LCP (q, M) if q and M satisfy the assumptions of Corollary **3.9.17**.

4.4.15 Theorem. Under the assumptions of Corollary **3.9.17**, Algorithm **4.4.5** will compute a solution of the LCP (q, M) if the problem is nondegenerate.

Proof. By the proof of **4.4.11**, the vector $\tilde{z}$ is a nonzero solution of $(0, M)$. Let $\alpha = \operatorname{supp} \tilde{z}$, $\gamma = \operatorname{supp} \tilde{w}$ and $\beta = \{i : \tilde{z}_i = \tilde{w}_i = 0\}$. Then by the complementarity relations stated in (11), it follows that $w^*_\alpha = 0$ and $z^*_\gamma = 0$. By the assumption of **3.9.17**, there exists a nonzero vector y_α satisfying

$$y_\alpha^T M_{\alpha\alpha} \geq 0, \quad y_\alpha^T M_{\alpha\beta} \geq 0, \quad y_\alpha^T q_\alpha \geq 0.$$

Since $z^* \geq 0$ and $z^*_\gamma = 0$, it follows that

$$y_\alpha^T (q + Mz^*)_\alpha \geq 0.$$

On the other hand, we have

$$0 = y_\alpha^T w^*_\alpha = y_\alpha^T (q + dz_0^* + Mz^*)_\alpha \geq y_\alpha^T d_\alpha z_0^*$$

which is impossible. Consequently, termination on a secondary ray is ruled out and the algorithm must compute a solution of (q, M). □

Further generalization of the above results is possible, see Exercise **4.11.21**.

Caveats

So far, our discussion of Lemke's Scheme I has been restricted to the nondegenerate case. It is well known that this algorithm can *cycle* on certain degenerate linear complementarity problems, and in Section 4.9, we shall treat this topic in detail. At this point, though, we take up a different complication that can occur when degeneracy is present.

Recall that Step 1 of Algorithm **4.4.5** is concerned with the determination of the blocking variable. In a degenerate problem, it can happen that two or more basic variables reach zero at the same time and become choices for the blocking variable. Consider a case where z_0 and at least one other basic variable are involved in a tie for blocking. In a strict implementation of Step 2 of Algorithm **4.4.5**, the artificial variable z_0 will be chosen as the blocking variable, and a solution of the LCP will be at hand. If, in this situation, some other blocking variable is chosen (instead of z_0), the algorithm will choose the complement of the blocking variable as the next driving variable. It might happen that, after the pivot, the current column of this driving variable is nonnegative, in which case the algorithm would terminate with a ray. Such a thing can happen even when Algorithm **4.4.5** is capable of processing the problem. In such circumstances, ray termination would appear to indicate that the problem is infeasible. But as noted above, choosing z_0 as the blocking variable would have led to a solution. This phenomenon is illustrated in the following example.

4.4.16 Example. Consider the LCP (q, M) in which

$$q = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The matrix M is clearly positive semi-definite, so Algorithm **4.4.5** should be able to process the problem. Choosing the covering vector $e = (1, 1)$, we set up the tableau

	1	z_0	z_1	z_2
w_1	1	1	1	−1
w_2	−1	1	−1	1

The critical value of the artificial variable is $\bar{z}_0 = 1$. The first pivot, $\langle w_2, z_0 \rangle$, leads to the tableau

	1	w_2	z_1	z_2
w_1	2	1	2	−2
z_0	1	1	1	−1

Since w_2 just became nonbasic, the driving variable is its complement, z_2. We now find that w_1 and z_0 simultaneously reach zero when $z_2 = 1$. Choosing w_1 as the blocking variable and performing the pivot $\langle w_1, z_2 \rangle$, we obtain the tableau

	1	w_2	z_1	w_1
z_2	1	$\frac{1}{2}$	1	$-\frac{1}{2}$
z_0	0	$\frac{1}{2}$	0	$\frac{1}{2}$

The next driving variable is z_1. Since its column is nonnegative, it is unblocked, and hence the computation terminates with a ray. This illustrates the importance of choosing z_0 as the blocking variable when it is involved in a tie.

The ability of Lemke's Scheme I to process many different classes of linear complementarity problems gives rise to the idea of using it for solving *nonconvex* quadratic programming problems by way of their KKT conditions. By "solving" we mean actually finding *global minima* for quadratic programs with nonconvex minimands. While this approach can sometimes be used to advantage, the following example shows that it does not always work.

4.4.17 Example. Consider the quadratic program

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \text{subject to} & 2x_1 + x_2 \leq 6 \\ & -x_1 + 4x_2 \leq 6 \\ & x_1 \geq 0, \; x_2 \geq 0 \,. \end{array} \tag{12}$$

A global minimum for this nonconvex quadratic program must exist as its feasible region is nonempty and compact. The KKT conditions of this program yield an LCP (q, M) where

$$q = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 6 \\ 6 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} -1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 4 \\ -2 & -1 & 0 & 0 \\ 1 & -4 & 0 & 0 \end{bmatrix}.$$

Let $d = (d_1, d_2, d_3, d_4)$ be the covering vector introduced in an application of Algorithm **4.4.5**. The components of d must be nonnegative and d_2 must be positive. Since any positive multiple of d will produce the same almost complementary path, we may assume that $d_2 = 1$. In tableau form we have

	1	z_0	x_1	x_2	y_3	y_4
y_1	$\frac{1}{2}$	d_1	-1	0	2	-1
y_2	$-\frac{1}{2}$	1	0	1	1	4
x_3	6	d_3	-2	-1	0	0
x_4	6	d_4	1	-4	0	0

Regardless of the (nonnegative) values of d_1, d_3, and d_4, the initial pivot must be $\langle y_2, z_0 \rangle$. The next tableau is

	1	y_2	x_1	x_2	y_3	y_4
y_1	$\frac{1}{2}(1 + d_1)$	d_1	-1	$-d_1$	$2 - d_1$	$-1 - 4d_1$
z_0	$\frac{1}{2}$	1	0	-1	-1	-4
x_3	$\frac{1}{2}(12 + d_3)$	d_3	-2	$-1 - d_3$	$-d_3$	$-4d_3$
x_4	$\frac{1}{2}(12 + d_4)$	d_4	1	$-4 - d_4$	$-d_4$	$-4d_4$

The driving variable is now x_2. It is not difficult to show that no matter what nonnegative value d_1, d_3, and d_4 have, the outcome of the minimum ratio test will be the same: z_0 is the blocking variable. The pivot $\langle z_0, x_2 \rangle$ produces

	1	y_2	x_1	z_0	y_3	y_4
y_1	$\frac{1}{2}$	0	-1	d_1	2	-1
x_2	$\frac{1}{2}$	1	0	-1	-1	-4
x_3	$\frac{11}{2}$	-1	-2	$1 + d_3$	1	4
x_4	4	-4	1	$4 + d_4$	4	16

Thus, we see that no matter how the covering vector for this linear complementarity problem is chosen, the algorithm will find the same solution:

$$(x_1, x_2, y_3, y_4) = (0, \tfrac{1}{2}, 0, 0).$$

The point $(x_1, x_2) = (0, \frac{1}{2})$ happens to be a local minimum for the quadratic program (12) from which this LCP was derived. The corresponding objective function value is $-\frac{1}{8}$. It can be shown that the global minimum of the QP occurs at $(x_1, x_2) = (3, 0)$ where the objective function value is -3. This global minimum is simply *inaccessible* by means of this algorithm.

Scheme II

In some cases, it is possible to process a linear complementarity problem (q, M) without introducing an artificial covering vector d. A prime example of such a problem is one in which the matrix M has a positive column. When this condition obtains, it is a simple matter to construct an almost complementary feasible solution of the underlying system

$$w = q + Mz \geq 0, \quad \text{and} \quad z \geq 0.$$

Assume that $\min_i q_i < 0$. If $M_{\bullet k} > 0$, then putting

$$z_k \geq \bar{z}_k = \max_{1 \leq i \leq n} \frac{-q_i}{m_{ik}} \quad \text{and} \quad z_i = 0 \quad \text{for all } i \neq k$$

yields an almost complementary feasible solution. The set of points (w, z) such that $z_k \geq \bar{z}_k$, $z_i = 0$ for all $i \neq k$ constitutes an almost complementary ray ending at an almost complementary extreme point of FEA(q, M).[5] At this endpoint, there is at least one index r such that $w_r = q_r + m_{rk}\bar{z}_k = 0$. In the nondegenerate case—which we assume for this discussion—there is only one such index.

In formulating Lemke's Scheme II, we postulate that an almost complementary extreme point at the end of an almost complementary ray is available. This assumption is essential since there is such a thing as an almost complementary extreme point that is *not* at the endpoint of an almost complementary ray. In fact, an LCP for which FEA(q, M) is bounded will

[5] In effect, this plays the role of the primary ray in Scheme I.

have no rays of any kind; such a set can have an almost complementary extreme point.

4.4.18 Algorithm. (Lemke, Scheme II – Unaugmented Problem)

Step 0. *Initialization.* Input (q, M). If $q \geq 0$, then stop: $z = 0$ solves (q, M). Otherwise, assume there exists an index k for which $q + M_{\bullet k} z_k \geq 0$ for all $z_k \geq \bar{z}_k$. Let w_r denote the (unique, by the nondegeneracy assumption) component of w that equals zero when $z_k = \bar{z}_k$. Pivot $\langle w_r, z_k \rangle$. (The complements w_k and z_k are now basic whereas w_r and z_r are nonbasic.) Choose the driving variable to be the complement of w_r, namely z_r.

Step 1. *Determination of the blocking variable (if any).* Use the minimum ratio test to determine whether there is a basic variable that blocks the increase of the driving variable. If not, stop. (Interpret this outcome, if possible.)

Step 2. *Pivoting.* The driving variable is blocked. Pivot

$$\langle \textit{ blocking variable, driving variable } \rangle$$

If z_k or w_k is the blocking variable, a solution to (q, M) is at hand. Otherwise return to Step 1 using the complement of the most recent blocking variable as the new driving variable.

With the nondegeneracy assumption in force, Algorithm **4.4.18** terminates in finitely many steps.

4.4.19 Theorem. Let (q, M) be an instance of the LCP to which **4.4.18** applies. Then on this problem, the algorithm will terminate in finitely many steps, either with a solution or a secondary ray.

Proof. The algorithm is initiated at an almost complementary extreme point that is incident to an unbounded edge of almost complementary points. Along the almost complementary path generated by the algorithm, no extreme point can be repeated, for there are exactly two almost complementary edges incident to each such point; a return to a previously visited extreme point would imply the existence of more than two almost complementary edges incident to it. The feasible region possesses finitely many extreme points in all; so, after a finite number of steps, the algorithm must terminate with a complementary feasible solution or an almost complementary ray. □

The following theorem indicates a class of problems that can be solved by Scheme II.

4.4.20 Theorem. Scheme II will solve any (nondegenerate) LCP (q, M) in which $M > 0$.

Proof. The proof is by contradiction. Let k denote the index of the basic pair that arises in the execution of Algorithm **4.4.18**. Suppose it terminates with a ray emanating from $(\bar{w}, \bar{z})$. Then there must exist a pair of vectors $(\tilde{w}, \tilde{z})$ such that

$$\tilde{w} = M\tilde{z} \geq 0, \qquad 0 \neq \tilde{z} \geq 0. \tag{13}$$

Moreover, points along the almost complementary terminal ray are of the form $(\bar{w} + \lambda\tilde{w}, \bar{z} + \lambda\tilde{z})$ where $\lambda \geq 0$ and

$$(\bar{w} + \lambda\tilde{w})_i(\bar{z} + \lambda\tilde{z})_i = 0 \quad \text{for all } \lambda \geq 0 \text{ and all } i \neq k. \tag{14}$$

From this we deduce that

$$\bar{w}_i\bar{z}_i = \tilde{w}_i\bar{z}_i = \bar{w}_i\tilde{z}_i = \tilde{w}_i\tilde{z}_i = 0 \quad \text{for all } i \neq k. \tag{15}$$

Now since $M > 0$, (13) implies that $\tilde{w} > 0$, and from (15), it follows that z_k is the only component of z that can assume a nonzero value along the almost complementary ray. Hence the terminal ray is the initial ray; this means that the almost complementary path must have returned to a previously visited extreme point which is impossible. □

Solving bimatrix games

Bimatrix games were cited in Section 1.2 as one class of LCP source problems. Our aim here is to develop the Lemke-Howson method for solving the type of linear complementarity problem that is derived from a given bimatrix game.

The problem we want to solve was stated in (1.2.5). Repeated here for the sake of convenience, the problem is

$$\begin{aligned} u &= -e_m + Ay \geq 0, \quad x \geq 0, \quad x^{\mathrm{T}}u = 0, \\ v &= -e_n + B^{\mathrm{T}}x \geq 0, \quad y \geq 0, \quad y^{\mathrm{T}}v = 0. \end{aligned} \tag{16}$$

This leads to the LCP with data

$$q = \begin{bmatrix} -e_m \\ -e_n \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & A \\ B^{\mathrm{T}} & 0 \end{bmatrix}. \tag{17}$$

In this formulation, A and B are positive $m \times n$ matrices. Each component of the constant vector q equals -1.

Since no column of M is positive, it is not possible to attain feasibility by increasing the value of just one nonbasic variable (x_i or y_j) to a suitable level. Nevertheless something close to this will work and yield an extreme point at the end of an almost complementary ray. The technique is spelled out in the following algorithm.

4.4.21 Algorithm. (Lemke-Howson)

Step 0. *Initialization.* Input the LCP (q, M) of order $m + n$ given by (17). Select an index $k \in \{1, \ldots, m\}$. Let $s \in \arg\min_{1 \leq j \leq n} b_{kj}$. Pivot $\langle v_s, x_k \rangle$. [This yields an almost complementary, but infeasible, solution.] Let $r \in \arg\min_{1 \leq i \leq m} a_{is}$. Pivot $\langle u_r, y_s \rangle$. [The solution is now almost complementary and feasible.] The basic pair is (x_k, u_k), and the nonbasic pair is (u_r, x_r). If $r = k$, stop. A solution has been found. Otherwise let x_r be the driving variable.

Step 1. *Determine the blocking variable (if any).* Use the minimum ratio test to determine whether there is a basic variable that blocks the increase of the driving variable. If not, stop.

Step 2. *Pivoting.* The driving variable is blocked. Pivot

$$\langle \textit{ blocking variable, driving variable } \rangle.$$

If the blocking variable belongs to the basic pair, a solution to (q, M) is at hand. Otherwise return to Step 1 using the complement of the most recent blocking variable as the new driving variable.

The choice of the index k in Step 0 can be extended. Any column (not necessarily one of the first m columns) can be selected. The appropriate modifications are obvious.

Step 1 of this algorithm can be stated less conservatively, for it can be shown that in this sort of LCP, the driving variable *must* be blocked.

4.4.22 Theorem. Algorithm **4.4.21** finds a solution of every (nondegenerate) instance of the LCP corresponding to a bimatrix game.

Proof. If a solution is not found in Step 0, then an almost complementary extreme point of the feasible set given by

$$\begin{aligned} u &= -e_m + Ay \geq 0, \quad x \geq 0, \\ v &= -e_n + B^{\mathrm{T}}x \geq 0, \quad y \geq 0 \end{aligned}$$

is at hand. The remainder of the algorithm is the same as Steps 1 and 2 of Algorithm **4.4.18**. Thus, it remains to show that in Step 1, the driving variable is always blocked, i.e., that termination with a ray is impossible.

The proof that termination with a ray is impossible is analogous to the one used in Theorem **4.4.20** where it was shown that Scheme II solves LCPs with $M > 0$. In the present circumstances, the structure of M plays more of a role, however. Specifically, we consider the counterparts of (13), (14), and (15). Thus, if termination with a ray occurs, there must exist an almost complementary extreme point $(\bar{u}, \bar{v}, \bar{x}, \bar{y})$ and a vector $(\tilde{u}, \tilde{v}, \tilde{x}, \tilde{y})$ with

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} 0 & A \\ B^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}, \quad 0 \neq (\tilde{x}, \tilde{y}) \geq 0 \tag{18}$$

such that points along the ray are of the form $(\bar{u}+\lambda\tilde{u}, \bar{v}+\lambda\tilde{v}, \bar{x}+\lambda\tilde{x}, \bar{y}+\lambda\tilde{y})$ where $\lambda \geq 0$, and for all such λ

$$(\bar{u} + \lambda\tilde{u})_i(\bar{x} + \lambda\tilde{x})_i = 0 \quad i \neq k \tag{19}$$

$$(\bar{v} + \lambda\tilde{v})_i(\bar{y} + \lambda\tilde{y})_i = 0 \quad i \neq k. \tag{20}$$

This implies that for all $i \neq k$

$$\bar{u}_i \bar{x}_i = \tilde{u}_i \bar{x}_i = \bar{u}_i \tilde{x}_i = \tilde{u}_i \tilde{x}_i = 0, \tag{21}$$

$$\bar{v}_i \bar{y}_i = \tilde{v}_i \bar{y}_i = \bar{v}_i \tilde{y}_i = \tilde{v}_i \tilde{y}_i = 0. \tag{22}$$

It must be the case that either $\tilde{x} \neq 0$ or $\tilde{y} \neq 0$. If $\tilde{x} \neq 0$, then $\tilde{v} = B^{\mathrm{T}}\tilde{x} > 0$. This implies $\bar{y}_j + \lambda \tilde{y}_j = 0$ for all j and all $\lambda \geq 0$. But then $\bar{u} + \lambda \tilde{u} < 0$ which is a contradiction. If $\tilde{x} = 0$, then (18) implies $\tilde{y} \neq 0$ from which it follows that $\tilde{u} = A\tilde{y} > 0$. This implies that $\bar{x}_i = 0$ for all $i \neq k$; from $\tilde{x} = 0$ it follows that $\tilde{v} = B^{\mathrm{T}}\tilde{x} = 0$. Accordingly, $\bar{v}$ must be the same vector as the one defined in Step 0 where the initial value of x_k was specified, i.e., the smallest positive value of x_k so that $-1 + m_{jk}x_k \geq 0$. By the nondegeneracy assumption, only $\bar{v}_s = 0$. The other components of $\bar{v}$ must be positive. Thus $\bar{y}_j + \lambda \tilde{y}_j = 0$ for all $j \neq s$. We now see that the terminating ray is the original ray. This contradiction completes the proof. □

4.4.23 Example. Consider the bimatrix game $\Gamma(A, B)$ with

$$A = B^{\mathrm{T}} = \begin{bmatrix} 10 & 20 \\ 30 & 15 \end{bmatrix}.$$

The corresponding LCP is given by the tableau

	1	z_1	z_2	z_3	z_4
w_1	-1	0	0	10	20
w_2	-1	0	0	30	15
w_3	-1	10	20	0	0
w_4	-1	30	15	0	0

Here we are using the obvious notational scheme,

$$\begin{aligned} z_i &= x_i \quad i = 1, \ldots, m \\ z_{m+j} &= y_j \quad j = 1, \ldots, n \\ w_i &= u_i \quad i = 1, \ldots, m \\ w_{m+j} &= v_j \quad j = 1, \ldots, n. \end{aligned}$$

There are four ways to choose the column of the first pivot in Step 0. Initializing the algorithm with $k = 1$ or $k = 3$ leads to the solution $\bar{z} = (\frac{1}{10}, 0, \frac{1}{10}, 0)$ that corresponds to pivoting on $M_{\alpha\alpha}$ where $\alpha = \{1, 3\}$. In like manner, initializing the algorithm with $k = 2$ or $k = 4$ leads to the solution $\bar{z} = (0, \frac{1}{15}, 0, \frac{1}{15})$ that corresponds to pivoting on $M_{\alpha\alpha}$ where $\alpha = \{2, 4\}$. Hence, from the four choices of initial pivot column, we obtain two—*and only two*—distinct solutions of this LCP. But this problem has another solution: $\bar{z} = (\frac{1}{90}, \frac{2}{45}, \frac{1}{90}, \frac{2}{45})$ which corresponds to a block pivot on M itself. This example illustrates the notion of an *elusive equilibrium point*, an equilibrium point that cannot be reached by application of the Lemke-Howson algorithm.

4.5 Parametric LCP Algorithms

In this section, we take up a slightly different kind of linear complementarity problem, one in which the vector generically denoted q is not constant, but rather moves along a line (segment) in space. As will be seen, problems of this kind have numerous applications, both practical and theoretical.

Formulation

A *parametric linear complementarity problem* (PLCP) is a family of linear complementarity problems

$$(q + \lambda d, M; \lambda \in \Lambda) \tag{1}$$

where d is a nonzero vector and λ is a parameter running over a closed interval $\Lambda \subseteq R$. Thus, to each $\lambda \in \Lambda$, there corresponds an ordinary LCP $(q + \lambda d, M)$.

Geometrically, the set

$$\{q + \lambda d : \lambda \in \Lambda\}$$

is a (possibly unbounded) segment of the line passing through the point q and having direction d. When $\Lambda = R$, the line segment is an entire line. Other choices of Λ can give rise to halflines and closed, bounded line segments.

For a specific value of λ, there may or may not exist a solution to the problem $(q + \lambda d, M)$. That will depend on M and specifically on $K(M)$, the union of the complementary cones determined by M. Thus, $(q+\lambda d, M)$ has a solution if and only if $q + \lambda d \in K(M)$. It is interesting to interpret the existence of solutions in terms of the intersection of the line segment $\{q + \lambda d : \lambda \in \Lambda\}$ with the closed cone $K(M)$.

A propos existence, it is clear that solutions $z(\lambda)$ of the parametric LCP $(q + \lambda d, M; \lambda \in \Lambda)$ must satisfy the linear feasibility relations

$$q + \lambda d + Mz \geq 0, \quad z \geq 0, \quad \lambda \in \Lambda. \tag{2}$$

Except in the case where $\Lambda = R$, the condition $\lambda \in \Lambda$ leads to at least one inequality, and perhaps two of them. It is possible to seek the largest and smallest values of λ for which (2) has a solution. These values can be computed by solving linear programs. Indeed, these largest and smallest values are

$$\begin{aligned}\lambda_* &= \inf\{\lambda \in \Lambda : \mathrm{FEA}(q + \lambda d, M) \neq \emptyset\},\\ \lambda^* &= \sup\{\lambda \in \Lambda : \mathrm{FEA}(q + \lambda d, M) \neq \emptyset\},\end{aligned}$$

and they can be used to refine the interval Λ. In general, it is possible for either $\lambda_* = -\infty$ or $\lambda^* = +\infty$, or both. For example, if $\Lambda = R$ and the matrix $M \in \boldsymbol{S}$, then $\mathrm{FEA}(q + \lambda d, M) \neq \emptyset$ for all λ; in this case, $\lambda_* = -\infty$ and $\lambda^* = +\infty$. There is no point in seeking solutions for parameter values λ such that $\mathrm{FEA}(q + \lambda d, M) = \emptyset$. Recall that when matrix $M \in \boldsymbol{Q_0}$, the LCP $(q + \lambda d, M)$ will have a solution for each finite number $\lambda \in [\lambda_*, \lambda^*]$.

Note also that when λ^0 is an interior point of Λ, the parametric LCP $(q + \lambda d, M; \lambda \in \Lambda)$ can be broken into a pair of problems

$$(q + \lambda d, M; \lambda \in [\lambda_*, \lambda^0]) \quad \text{and} \quad (q + \lambda d, M; \lambda \in [\lambda^0, \lambda^*]).$$

This is an effective way to handle problems in which $\Lambda = R$.

4.5.1 Example. Consider the PLCP $(q+\lambda d, M; \lambda \in \Lambda)$ with the following data:

$$q = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \Lambda = R.$$

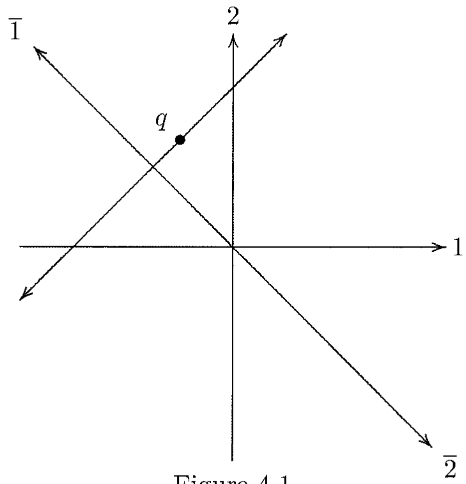

Figure 4.1

Notice that this positive semi-definite matrix M also belongs to the class $\boldsymbol{P_1}$ (see Section 4.1). This implies that $K(M)$ must be a halfspace. In Figure 4.1, we plot $K(M)$, the point q, and the line segment $\{q + \lambda d : \lambda \in \Lambda\}$. In this instance, it is easy to check that

$$\lambda_* = -\tfrac{1}{2} \quad \text{and} \quad \lambda^* = \infty,$$

hence the interval $\Lambda = R$ can be refined to $\Lambda^* = [-\frac{1}{2},\ +\infty)$. Now take $\lambda^0 = 0$, and consider the LCP $(q + \lambda^0 d, M)$. In schematic form we have

	1	λ^0	z_1	z_2	
w_1	-1	1	1	-1	-1
w_2	2	1	-1	1	2
	1	0	0	0	

(3)

The pivot $\langle w_1, z_1 \rangle$ yields the schema

	1	λ^0	w_1	z_2	
z_1	1	-1	1	1	1
w_2	1	2	-1	0	1
	1	0	0	0	

(4)

The corresponding diagram is given in Figure 4.2.

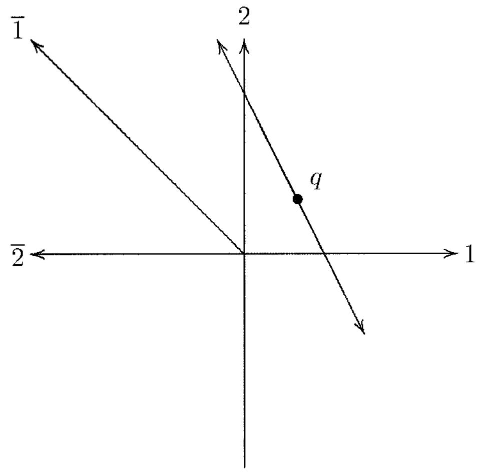

Figure 4.2

For small values of λ, the point $(1,1)+\lambda(-1,2)$ is interior to R^2_+. As λ is changed from $\lambda^0 = 0$, this point approaches the boundary of R^2_+. In fact,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 2 \end{bmatrix} \geq 0 \quad \text{for all } \lambda \in [-\tfrac{1}{2}, 1].$$

For $\lambda \in [-\frac{1}{2}, 1]$, the solution to the PLCP is $(z_1, z_2) = (1-\lambda, 0)$. When $\lambda > \lambda^1 = 1$, the point leaves R^2_+ and enters a neighboring complementary cone. In this particular problem, the line segment remains in that cone for all $\lambda \geq \lambda^1$. The counterpart of this fact is also visible in Figure 4.1.

Sources of parametric LCPs

The PLCP arises naturally from certain kinds of parametric linear programming and quadratic programming problems. Parametrization of the right-hand side or the (linear term of the) objective function (or both) in the usual way[6] leads to parametric analogues of the LCP formulation of these problems as described in Section 1.2 under the heading "Quadratic programming."

[6]See S.I. Gass (1985) and T. Gal (1979).

A well known example of a parametric quadratic programming problem is the *portfolio selection problem* (see **4.12.18**). In its most elementary form, the problem is to determine the percentage of the portfolio to allocate to each of n given securities. Let x_j denote the percentage allocated to the jth security. The expected return is given by $r^{\mathrm{T}}x$ where r_j is the average return on the jth security. One goal is to maximize this return. At the same time, the investor wishes to minimize the risk associated with the portfolio x. The risk is given by the value of a positive semi-definite quadratic form $x^{\mathrm{T}}Dx$. To balance the conflicting goals of maximizing expected return and minimizing risk, one can consider the function

$$f_\lambda(x) = -\lambda r^{\mathrm{T}}x + x^{\mathrm{T}}Dx \tag{5}$$

where λ is a nonnegative parameter. The larger the value of λ, the more importance is attached to the expected return. The constraints of the problem, as stated, must include the conditions

$$e^{\mathrm{T}}x = 1, \quad x \geq 0. \tag{6}$$

For simplicity, let us assume there are no other constraints. Converting the equality constraint $e^{\mathrm{T}}x = 1$ to a pair of inequality constraints, we arrive at the parametric convex quadratic program of minimizing $f_\lambda(x)$ subject to (6). This gives rise to the parametric linear complementarity problem $(q + \lambda d, M; \lambda \in R_+)$ where

$$q = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad d = \begin{bmatrix} -r \\ 0 \\ 0 \end{bmatrix}, \quad M = \begin{bmatrix} 2D & -e & e \\ e^{\mathrm{T}} & 0 & 0 \\ -e^{\mathrm{T}} & 0 & 0 \end{bmatrix}. \tag{7}$$

Notice that, in this case, the constraint set given by (6) is nonempty and compact. Since the objective function defined in (5) is continuous, the minimum exists for every $\lambda \in R$. Accordingly, the PLCP $(q+\lambda d, M; \lambda \in R)$ has a solution for every value of λ and, *a fortiori*, those belonging to R_+.

As another closely related example, consider a quadratic program of the form

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x + \tfrac{1}{2}x^{\mathrm{T}}Qx \\ \text{subject to} & a^{\mathrm{T}}x = b \\ & x \geq 0. \end{array} \tag{8}$$

Let us assume that the objective function is strictly convex (Q is positive definite and—without loss of generality—symmetric). Assume also that the single linear equality constraint is both feasible and nontrivial. If a is not a positive vector, the feasible region can be unbounded. But its assumed nonemptiness and the assumed strict convexity of the objective function ensure the existence of a *unique* optimal solution to this quadratic program. The Karush-Kuhn-Tucker conditions for this problem are

$$\begin{aligned} &u = c + Qx - \lambda a \geq 0, \quad x \geq 0, \quad x^{\mathrm{T}}u = 0, \\ &v = -b + a^{\mathrm{T}}x = 0, \qquad \lambda \text{ free}, \quad (\lambda v = 0). \end{aligned} \tag{9}$$

Notice that for any fixed value of λ, the first line of the above KKT conditions amounts to a linear complementarity problem $(c - \lambda a, Q)$ in which the matrix Q is symmetric and positive definite. The LCP has a unique solution $\bar{x} = \bar{x}(\lambda)$. For this given value of λ, the solution vector $\bar{x}(\lambda)$ may or may not satisfy the equality constraint of (8). If it does, then the original quadratic programming problem is solved. If it does not, the value of λ can be changed. The idea is to solve the PLCP $(c - \lambda a, Q; \lambda \in R)$, at least to the point where a value of λ corresponding to the solution of (8) is discovered.

Later in this section, we shall exhibit a *parametric* form of Lemke's Scheme I for the *ordinary* linear complementarity problem. As may be anticipated, the idea in this approach is to use the covering vector as the direction vector, and the artificial variable z_0 as the parameter, λ.

An interesting application of the parametric linear complementarity problem arises from a structural mechanics problem that seeks conditions under which the solution of (1) is an isotone function of the parameter λ. We will address this question in Section 4.8 in connection with a related issue. Further applications involving parametric linear complementarity problems can be found in actuarial science, spatial price equilibrium theory, and the traffic equilibrium problem. We shall return to the latter problem later. Information on the other applications is provided in the exercises and in the notes and references at the end of the chapter.

The parametric principal pivoting method

The symmetric principal pivoting method (Algorithm **4.3.2**) presented in Section 4.3 processes nondegenerate instances of the row sufficient LCP

(q, M). Our aim now is to discuss a parametric variant of that procedure. In this case, we shall assume that M is (row and column) sufficient.

Let $(q + \lambda d, M; \lambda \in \Lambda^*)$ be a given PLCP in which M is a sufficient matrix and $(q + \lambda d, M)$ is feasible for all $\lambda \in \Lambda^*$. Under these conditions,[7] each of these LCPs is solvable, and when they are nondegenerate,[8] the principal pivoting method can be used to solve them.

The interval Λ^* may have been obtained from another interval Λ by calculating λ_* and λ^* as described above. Whatever the case may be, we assume that this bit of preprocessing has already been done. If $\Lambda^* = R$, we may choose $\lambda^0 = 0$ as above and split the problem into a pair of PLCPs. If Λ^* has a finite least element or greatest element, it is a simple matter to convert the problem at hand to one in which the least element of the parameter set is zero. Accordingly, we assume that $\lambda_* = 0$.

Starting with $\lambda^0 = 0$, we may solve $(q + \lambda^0 d, M) = (q, M)$. By the nondegeneracy assumption, the corresponding principal pivotal transform of q will be positive. For this reason, we start from the assumption that $q > 0$.

4.5.2 Algorithm. (Symmetric PPPM)

Step 0. *Initialization.* Input $(q + \lambda d, M; \lambda \in \Lambda)$ with M being (row and column) sufficient. Preprocess the data (as described above) or assume that $q > 0$, $d \neq 0$, and $\Lambda = \Lambda^* = [0, \lambda^*]$. Define $\nu = 0$ and

$$q^\nu = q, \; d^\nu = d, \; M^\nu = M, \; w^\nu = w, \; z^\nu = z, \text{ and } \lambda^\nu = 0.$$

Step 1. *Determine next critical value of* λ. Define

$$\lambda^{\nu+1} = \min\{\min_i\{\frac{-q_i^\nu}{d_i^\nu} : d_i^\nu < 0\}, \lambda^*\}$$

and set

$$(\bar{w}^\nu(\lambda), \bar{z}^\nu(\lambda)) = (q^\nu + \lambda d^\nu, 0) \text{ for all } \lambda \in [\lambda^\nu, \lambda^{\nu+1}].$$

---

[7] That is, because of the *row* sufficiency.

[8] In this context, it seems that we want to regard a solution of the system of equations with $\lambda \neq 0$ as being nondegenerate if it has at most $n + 1$ of its $2n$ variables equal to zero. Thus, a basic solution would have at most one of its basic variables equal to zero.

If $\lambda^{\nu+1} = \lambda^*$, stop. Otherwise, let

$$r = \arg \min_i \{ \frac{-q_i^\nu}{d_i^\nu} : d_i^\nu < 0 \}.$$

The new critical value of λ is $\lambda^{\nu+1} = -q_r^\nu / d_r^\nu$.

Step 2. *Pivoting.* If $m_{rr}^\nu > 0$, pivot $\langle w_r^\nu, z_r^\nu \rangle$. Put

$$w_r^{\nu+1} = z_r^\nu, \quad z_r^{\nu+1} = w_r^\nu,$$
$$w_i^{\nu+1} = w_i^\nu, \quad z_i^{\nu+1} = z_i^\nu, \quad i \neq r.$$

Return to Step 1 with ν replaced by $\nu + 1$.

If $m_{rr}^\nu = 0$, use z_r^ν as a driving variable and determine the basic blocking variable w_s^ν (in the usual way). Pivot $\langle w_s^\nu, z_r^\nu \rangle$, $\langle w_r^\nu, z_s^\nu \rangle$. Put

$$w_s^{\nu+1} = z_r^\nu, \quad z_s^{\nu+1} = w_r^\nu,$$
$$w_r^{\nu+1} = z_s^\nu, \quad z_r^{\nu+1} = w_s^\nu,$$
$$w_i^{\nu+1} = w_i^\nu, \quad z_i^{\nu+1} = z_i^\nu, \quad i \neq r, \ s.$$

Return to Step 1 with ν replaced by $\nu + 1$.

The justification for Algorithm **4.5.2** rests heavily on the nondegeneracy assumption and on the row and column sufficiency properties of M. By nondegeneracy, only the r-th component of the vector $\bar{w}^\nu(\lambda^{\nu+1})$ is 0; the other components are positive. It follows that in Step 2, if $m_{rr}^\nu > 0$, the simple principal pivot $\langle w_r^\nu, z_r^\nu \rangle$ makes w_r^ν nonbasic (as desired) and also gives $\bar{w}^{\nu+1}(\lambda^{\nu+1}) = \bar{w}^\nu(\lambda^{\nu+1})$. If $m_{rr}^\nu = 0$, it is not possible to make w_r^ν nonbasic with one simple principal pivot; under the present assumption, it can be done with a principal block pivot of order 2. Using z_r^ν as a driving variable, we use the minimum ratio test to determine the index of a blocking variable, say w_s^ν. The uniqueness of this index s is a consequence of the nondegeneracy assumption. Its existence can be argued as follows. We need to show that there is a negative entry in the column of z_r^ν. Now, if the column of z_r^ν were nonnegative, the row sufficiency of M^ν would imply that $M_{r\cdot}^\nu \leq 0$. But in Step 1, we had $\lambda^{\nu+1} < \lambda^*$. This means that $q^\nu + \lambda d^\nu \in K(M^\nu)$ for at least some $\lambda > \lambda^{\nu+1}$. Since $d_r^\nu < 0$, this cannot be true if $M^\nu_{r\bullet} \leq 0$. Because its column contains a negative entry, the driving variable must be blocked. Now the column sufficiency implies that

$$\begin{bmatrix} m^\nu_{rr} & m^\nu_{rs} \\ m^\nu_{sr} & m^\nu_{ss} \end{bmatrix} \simeq \begin{bmatrix} 0 & + \\ - & \oplus \end{bmatrix}$$

and hence must be nonsingular (otherwise it has a zero row which is forbidden in such a column sufficient matrix). The two off-diagonal pivots indicated in Step 2 accomplish the task of making w^ν_r nonbasic, and (by the nondegeneracy assumption) they do not alter the signs of the components of the vector $q(\lambda^{\nu+1})$. That is, $q^{\nu+1} + \lambda^{\nu+1}d^{\nu+1} \simeq q^\nu + \lambda^{\nu+1}d^\nu$. It is now possible to increase λ to its next critical value.

4.5.3 Remark. The upshot of the algorithmic process is a finite sequence of *critical values* $\lambda^0, \lambda^1, \ldots, \lambda^\ell$ and a corresponding sequence of solution vectors $(\bar{w}^\nu(\lambda), \bar{z}^\nu(\lambda))$ where $\lambda \in [\lambda^\nu, \lambda^{\nu+1}]$ for $\nu = 0, 1, \ldots, \ell - 1$. In the case where $M \in \boldsymbol{P}$, the individual components of $\bar{w}$ and $\bar{z}$ reveals that they are continuous *piecewise linear* functions of λ (cf. **1.4.6**). The *breakpoints* at which the slopes may change are among the critical values $\lambda^1, \ldots, \lambda^{\ell-1}$.

Lemke's Scheme I in parametric form

Algorithm **4.4.5**, the streamlined version of Lemke's Scheme I, for the linear complementarity problem (q, M) uses the augmented problem (4.4.6) and, in particular, the underlying system

$$w = q + dz_0 + Mz \tag{10}$$

where $d > 0$ is the covering vector. The goal of the method is to obtain a nonnegative solution of the underlying system (10) in which $z_0 = 0$ and $z^{\mathrm{T}}w = 0$. In the standard version of Algorithm **4.4.5**, the artificial variable z_0 is immediately made basic at a positive level; it remains so until it either decreases to zero and can be made nonbasic again, in which case a solution of (q, M) has been found, or else termination with a secondary ray occurs.

As an alternative to doing this, it is possible to treat z_0 as a *parameter*. In terms of the notation established at the beginning of this section, we can let $\lambda = z_0$ and take Λ to be the interval $[0, \bar{z}_0]$ where, as in (4.4.4),

$$\bar{z}_0 = \max_i\{-q_i/d_i\}.$$

We then try to *reduce* the value of z_0 from $\bar{z}_0$ to 0 while retaining non-negativity and complementarity of the variables z and w. Accordingly, no pivoting is done in the z_0-column of the schema

$$\begin{array}{c|c|c|c|c} & 1 & z_0 & z & \\ \hline w & q & d & M & \bar{w} \\ \hline & 1 & \bar{z}_0 & \bar{z} & \end{array} \tag{11}$$

The symbols below and to the right of the schema represent *values* of the column labels and row labels, respectively. As an extra notation, let

$$\bar{q} = q + \bar{z}_0 d.$$

After ν pivots have occurred, the schema can be denoted

$$\begin{array}{c|c|c|c|c} & 1 & z_0^\nu & z^\nu & \\ \hline w^\nu & q^\nu & d^\nu & M^\nu & \bar{w}^\nu \\ \hline & 1 & \bar{z}_0^\nu & \bar{z}^\nu & \end{array} \tag{12}$$

where $z_0^\nu = z_0$ for all ν and (w^ν, z^ν) is a permutation of (w, z).

The parametric version of Algorithm **4.4.5** involves a sequence of *major cycles*, each of which is associated with a *critical value* of the parameter z_0. To describe the algorithm, we need to specify the steps taken within a major cycle and then show how to pass from one major cycle to the next.

At the beginning of each major cycle there will be a complementary schema (12) and a unique index r such that $\bar{w}_r^\nu = 0$ and $\bar{w}_i^\nu > 0$ for $i \neq r$. It is useful to keep in mind that if the value of z_0 were slightly smaller than $\bar{z}_0^\nu$, the basic variable w_r^ν would be negative while all the other basic variables would (by nondegeneracy) be positive. Under such circumstances, the natural goal would be to make w_r^ν nonbasic at value zero. Ideally, this would be accomplished with a simple principal pivot $\langle w_r^\nu, z_r^\nu \rangle$. This can be done only if $m_{rr}^\nu \neq 0$. If this is the case, the sign of m_{rr}^ν is not important because $\bar{w}_r^\nu = \bar{q}_r^\nu$ is actually zero (rather than negative), and hence the values of the basic variables do not change under such a principal pivot operation. If $m_{rr}^\nu = 0$, however, the simple principal pivot is not possible; the next best thing would be a suitable (block) principal pivot leading to a different complementary schema and another solution of the LCP. Thus, each major cycle of the parametric scheme can be thought of as an ordinary linear complementarity problem in its own right.

It is interesting to interpret the algorithm geometrically. While this will be done in greater detail in Section 6.3, it is appropriate to spend some time with this now. The nature of the LCP at the beginning of a major cycle is that its "q vector" (say, $\bar{q}^\nu = q^\nu + \bar{z}_0^\nu d^\nu$) lies on the *boundary* of the complementary cone spanned by the complementary basis matrix currently being used to represent the system. If this point lies on the boundary of $K(M)$ as well, it will be impossible to decrease the parameter and generate another point $\bar{q} \in K(M)$. This condition would force the termination of the algorithm. If $\bar{q}^\nu$ lies on a proper facet, i.e., one that is common to a pair of nondegenerate (solid) complementary cones that also lie on opposite sides of the affine hull of this common facet, then the line segment (of q vectors) will enter (the interior of) this neighboring complementary cone. Thus, when z_0 is made smaller than $\bar{z}_0^\nu$, the corresponding point still lies in $K(M)$.

The geometric description just given is illustrated in Figure 4.3. There, we see a line segment running from the edge of the first quadrant, R^2_+, to the point q indicated by the heavy dot in the the third quadrant. The line segment traverses two complementary cones before it reaches the complementary cone that contains q. When a point moving along the line segment reaches the boundary of a complementary cone, a new representation of the point is needed. Pivoting provides this new representation.

In a nutshell, then, the idea in solving the LCP of a major cycle is to use a simple principal pivot if possible, and if it is not possible, then generate an almost complementary path as in Lemke's method (or the *asymmetric* version of the principal pivoting method). Once the LCP is solved, the next step is to adjust the parameter value in an appropriate manner.

Generically, r will denote the index of the *distinguished variable*, s will denote the index of the *driving variable*, and t will denote the index of the *blocking variable*. In the first iteration of a major cycle, $s = r$. In other words, the first driving variable of the major cycle is z_r^ν, the complement of the distinguished variable.

The aim of the major cycle is to make w_r^ν nonbasic. When this occurs, the value of z_0 is adjusted and a new major cycle is begun unless termina-

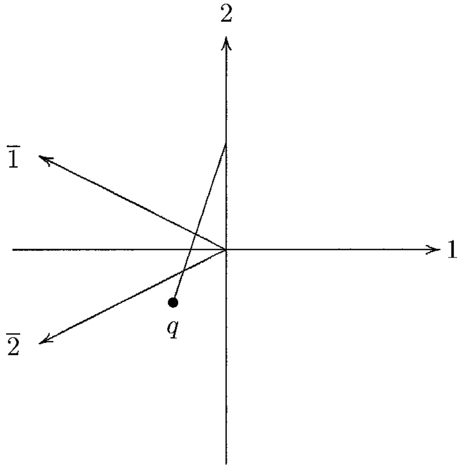

Figure 4.3

tion is indicated. Since $z^\nu = 0$ in (12), we have $\bar{w}^\nu = \bar{q}^\nu = q^\nu + \bar{z}_0^\nu d^\nu \geq 0$ for all ν. Whenever made, the next choice of value for z_0 will preserve this condition. If $t \neq r$, then after the pivot, the schema will be almost complementary, and the next driving variable will be the complement of the blocking variable. Thus, within a major cycle, if the first pivot is not a simple principal pivot, the algorithm will generate an almost complementary path, just as in Algorithm **4.4.5**.

4.5.4 Algorithm. (Parametric version of Lemke's Scheme I)

Step 0. *Initialization.* Input the augmented LCP (q, d, M) with $d > 0$. If $q \geq 0$, stop: $z = 0$ solves (q, M). Otherwise, define $\bar{z}_0 = \max_i\{-q_i/d_i\}$ and $r = \arg\max_i\{-q_i/d_i\}$. Set $\nu = 0$ and define

$$q^\nu = q,\ d^\nu = d,\ M^\nu = M,\ w^\nu = w,\ z^\nu = z,\ \text{and}\ \bar{z}_0^\nu = \bar{z}_0.$$

Step 1. *Finding another complementary cone.* Define w_r^ν as the distinguished variable and its complement z_r^ν as the driving variable.

Step 1A. If $m_{rr}^\nu \neq 0$, pivot $\langle w_r^\nu, z_r^\nu \rangle$ and go to Step 2 with $\nu \leftarrow \nu + 1$. Otherwise, go to Step 1B.

Step 1B. If the entry in the row of the distinguished variable and the column of the driving variable is nonzero, then a new complementary cone has been found. Pivot

$$\langle \textit{ distinguished variable, driving variable } \rangle.$$

and go to Step 2 with $\nu \leftarrow \nu + 1$.

If the entry in the row of the distinguished variable and the column of the driving variable is zero, perform the minimum ratio test using the column of the driving variable. If the driving variable is unblocked, stop. (The algorithm terminates with a ray.) Otherwise, pivot

$$\langle \textit{ blocking variable, driving variable } \rangle.$$

If the blocking variable is the complement of the distinguished variable, a new complementary cone has been found. Go to Step 2 with $\nu \leftarrow \nu + 1$ and the complement of the last blocking variable as the driving variable.

Step 2. *Traversing the new complementary cone.* Find the next critical value of z_0 (if possible). If $q^\nu \geq 0$, then decrease z_0 to 0 and stop; a solution to (q, M) has been found. If $d^\nu \geq 0$, stop; the algorithm terminates with a z_0-ray. Otherwise, d^ν must have both positive and negative components. The current value of z_0 is either $\max\{-q_i^\nu/d_i^\nu : d_i^\nu > 0\}$ or $\min\{-q_i^\nu/d_i^\nu : d_i^\nu < 0\}$.

If $\bar{z}_0^{\nu-1} = \max\{-\dfrac{q_i^\nu}{d_i^\nu} : d_i^\nu > 0\}$, set $\bar{z}_0^\nu = \min\{-\dfrac{q_i^\nu}{d_i^\nu} : d_i^\nu < 0\}$;

if $\bar{z}_0^{\nu-1} = \min\{-\dfrac{q_i^\nu}{d_i^\nu} : d_i^\nu < 0\}$, set $\bar{z}_0^\nu = \max\{-\dfrac{q_i^\nu}{d_i^\nu} : d_i^\nu > 0\}$.

Return to Step 1 with r as the unique index i such that $\bar{w}_i^\nu = 0$.

The parametric version of Lemke's Scheme I can be justified by noting its equivalence with the original version. When a major cycle, say the first one, begins with the parameter z_0 at a positive critical value $\bar{z}_0$, there is (by the nondegeneracy assumption) a unique index r for which the basic variable $w_r^\nu = 0$. Hence d_r^ν, the coefficient of z_0 in the corresponding equation, must be nonzero (otherwise the value of z_0^ν could be changed). Accordingly, it is possible to use the pivot $\langle w_r^\nu, z_0^\nu \rangle$ to make $z_0^\nu = z_0$ *basic* as it would be in **4.4.5**. In other words, for every schema of **4.5.4** there is a schema of **4.4.5** in which the z_0 and the original system variables w_i, z_i have the same values.

4.5.5 Example. Consider the LCPs (q, M) and $(q, \tilde{M})$ where

$$q = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \qquad M = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}, \qquad \tilde{M} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Notice that the q-vector is the same in both problems, and that $\tilde{M}$ is gotten by permuting columns of M. In each case, we shall use $d = e$ as the covering vector. The geometry of these problems is indicated in Figures 4.4(a) and 4.4(b).

For problem (q, M), the first critical value of z_0 is 2. The schema is

	1	z_0	z_1	z_2	
w_1	−2	1	1	−1	0
w_2	4	1	−1	−1	6
	1	2	0	0	

The algorithm calls for the pivot $\langle w_1, z_1 \rangle$, and the resulting schema is

	1	z_0	w_1	z_2	
z_1	2	−1	1	1	0
w_2	2	2	−1	−2	6
	1	2	0	0	

Notice how the line segment of q-vectors runs from $\bar{q}^0$ to q. The line segment does not encounter any other facet of a complementary cone. It is clear from the schema that z_0 can be reduced to 0, giving the solution $(\bar{z}_1, \bar{z}_2) = (2, 0)$.

For problem $(q, \tilde{M})$, the first critical value of z_0 is 2 also. (The vectors q and d are the same for both problems.) This problem's initial schema is

	1	z_0	z_1	z_2	
w_1	-2	1	-1	1	0
w_2	4	1	-1	-1	6
	1	2	0	0	

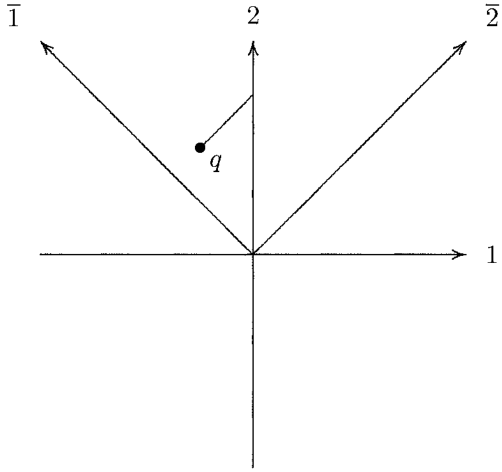

Figure 4.4(a)

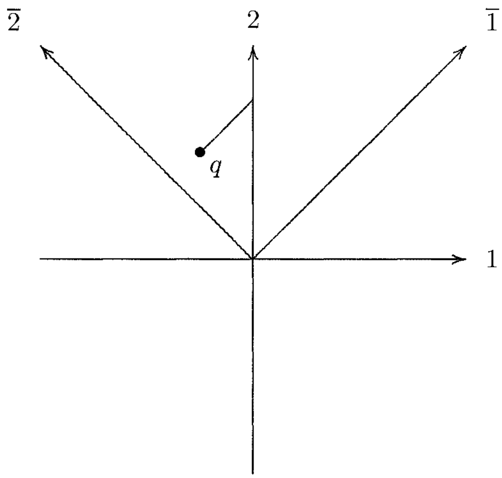

Figure 4.4(b)

The first pivot is $\langle w_1, z_1\rangle$; it results in

	1	z_0	w_1	z_2	
z_1	−2	1	−1	1	0
w_2	6	0	1	−2	6
	1	2	0	0	

In this case, the algorithm finds that the column headed by z_0 is nonnegative, meaning that the parameter can be increased indefinitely. The procedure terminates with a z_0-ray, even though the problem itself is solvable. Indeed, the problem is feasible and $\tilde{M} \in \boldsymbol{Q}_0$.

The following example illustrates a feature of Step 2 in Algorithm **4.5.4**.

4.5.6 Example. Monotonic decrease of the parameter z_0 is not always possible. This fact is illustrated by the augmented LCP (q, d, M) in which

$$q = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \quad d = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad M = \begin{bmatrix} -1 & -2 & -1 \\ -1 & 1 & -2 \\ 4 & -2 & 2 \end{bmatrix}.$$

A schematic representation of this problem is given by

	1	z_0	z_1	z_2	z_3	
w_1	2	1	−1	−2	−1	5
w_2	1	1	−1	1	−2	4
w_3	−3	1	4	−2	2	0
	1	3	0	0	0	

$3 \leq z_0 < \infty.$

It is easy to see that if the nonbasic variables z_1, z_2, and z_3 are all zero, then the basic variables w_1, w_2, and w_3 will be nonnegative if and only if $3 \leq z_0 < \infty$. Accordingly, we set $\bar{z}_0 = 3$ and record this in the last line of the schema. At this value of the parameter, $\bar{w}_3 = 0$. Since the corresponding diagonal entry of the matrix M is nonzero, the principal pivot $\langle w_3, z_3\rangle$ is possible, and we obtain

$$
\begin{array}{c|c|c|ccc|c}
 & 1 & z_0 & z_1 & z_2 & w_3 & \\
\hline
w_1 & \frac{1}{2} & \frac{3}{2} & 1 & -3 & -\frac{1}{2} & 2 \\
w_2 & -2 & 2 & 3 & -1 & -1 & 0 \\
z_3 & \frac{3}{2} & -\frac{1}{2} & -2 & 1 & \frac{1}{2} & 1 \\
\hline
 & 1 & 1 & 0 & 0 & 0 &
\end{array}
\qquad 1 \leq z_0 \leq 3.
$$

This time we execute the principal pivot $\langle w_2, z_2 \rangle$ obtaining

$$
\begin{array}{c|c|c|ccc|c}
 & 1 & z_0 & z_1 & w_2 & w_3 & \\
\hline
w_1 & \frac{13}{2} & -\frac{9}{2} & -8 & 3 & \frac{5}{2} & 0 \\
z_2 & -2 & 2 & 3 & -1 & -1 & \frac{8}{9} \\
z_3 & -\frac{1}{2} & \frac{3}{2} & 1 & -1 & -\frac{1}{2} & \frac{15}{9} \\
\hline
 & 1 & \frac{13}{9} & 0 & 0 & 0 &
\end{array}
\qquad 1 \leq z_0 \leq \tfrac{13}{9}.
$$

Notice that in this schema, the value of the parameter z_0 is larger than it was in the preceding schema; it has been set at the *largest* number for which the basic variables are nonnegative. This illustrates the assertion made about nonmonotone behavior of the parameter. Since $\bar{w}_1 = 0$ and the corresponding diagonal entry is nonzero, we carry out the principal pivot $\langle w_1, z_1 \rangle$. This produces the schema

$$
\begin{array}{c|c|c|ccc|c}
 & 1 & z_0 & w_1 & w_2 & w_3 & \\
\hline
z_1 & \frac{13}{16} & -\frac{9}{16} & -\frac{2}{16} & \frac{6}{16} & \frac{5}{16} & \frac{13}{16} \\
z_2 & \frac{7}{16} & \frac{5}{16} & -\frac{6}{16} & \frac{2}{16} & -\frac{1}{16} & \frac{7}{16} \\
z_3 & \frac{5}{16} & \frac{15}{16} & -\frac{2}{16} & -\frac{10}{16} & -\frac{3}{16} & \frac{5}{16} \\
\hline
 & 1 & 0 & 0 & 0 & 0 &
\end{array}
\qquad 0 \leq z_0 \leq \tfrac{13}{9}.
$$

The LCP (q, M) is now solved since the parameter z_0 has been reduced to zero.

Paradoxes in traffic equilibrium problems

In a previous subsection, we have mentioned several sources of parametric LCPs. In the sequel, we discuss a highly simplified instance of the *traffic equilibrium problem* and sketch how the theory of the PLCP can be used to analyze the prevalence of some traffic paradoxes.

Consider a *congested transportation network* modelled as a digraph with node set $\mathcal{N}$ and arc set $\mathcal{A}$. Assume for simplicity, that a pair of nodes i and $j \in \mathcal{N}$ is singled out as the *origin-destination* (OD) pair. We are interested in moving $D > 0$ units of flow (the traffic demand) from i to j via certain specified paths in the network. Let $p_1, p_2, \ldots, p_n$ be an enumeration of these paths that connect the node i to the node j. Let F_k denote the units of traffic flow along the path p_k ($k = 1, \ldots, n$), and $F = (F_k) \in R^n$ be the vector of path flows. A vector $F \in R^n_+$ of path flows is said to be *feasible* if it satisfies the demand condition:

$$\sum_{k=1}^{n} F_k = D.$$

Associated with each arc $a \in \mathcal{A}$ is a cost that measures the *traffic congestion* on that arc. Typically, this cost is an asymmetric function of the vector of flows on all the arcs in the network. A simplification of the model is obtained by assuming that each such arc cost depends linearly and only on the flow of the arc involved; i.e.,

$$c_a(f) = b_a + d_a f_a \quad \text{for all } a \in \mathcal{A}$$

where $c_a(\cdot)$ and f_a are, respectively, the cost function and flow amount on the arc a, and b_a and d_a are scalars with $d_a > 0$. The relation between the arc flows and the path flows is governed by the following expression:

$$f_a = \sum_{k_a} F_{k_a}$$

where the summation ranges over those paths p_{k_a} that contain the arc a. In the *additive model* of the traffic equilibrium problem, it is further assumed that the cost $C_k(F)$ on each path p_k is equal to the sum of the arc costs which ranges over those arcs contained in the path p_k. Let us introduce the *arc-path incidence matrix* $\Delta = (\delta_{ak})$: for $a \in \mathcal{A}$ and $k = 1, \ldots, n$,

$$\delta_{ak} = \begin{cases} 1 & \text{if arc } a \text{ is contained in path } p_k \\ 0 & \text{otherwise.} \end{cases}$$

Then, by letting $C(F) = (C_k(F))$ be the vector of path costs, we have

$$C(F) = \Delta^{\mathrm{T}} c(f) = \Delta^{\mathrm{T}} b + \Delta^{\mathrm{T}} D \Delta F$$

where $b = (b_a)$ and $D = \mathrm{diag}(d_a)$.

Central to the traffic equilibrium problem is *Wardrop's principle* which defines the *equilibrium flow patterns.* Specifically, a feasible flow vector $F = (F_k) \in R^n$ is said to be *in equilibrium* if the costs on the paths with positive flows are all equal and are less than or equal to the costs on the remaining paths. Mathematically, this states that the vector $F \in R^n$ is an equilibrium flow if there exists a scalar G (the minimum cost between the OD-pair (i, j)) such that the following conditions are satisfied:

$$F \geq 0$$

$$\sum_{k=1}^{n} F_k = D$$

$$C(F) - Ge \geq 0$$

$$F_k(C_k(F) - G) = 0 \quad k = 1, \ldots, n.$$

Clearly, these conditions define a mixed linear complementarity problem with a single equality constraint ($\sum_k F_k = D$) and one free variable G.

There are several well-studied paradoxes occurring in traffic equilibrium theory. The most famous among these is the one due to D. Braess which demonstrates that in a congested transportation network, it is possible for the elimination of a path in the network to decrease the cost of all the paths with positive flows. This is somewhat counter-intuitive because one would expect that by eliminating a path, at least one (and hence every, by the equilibrium principle) used path in the network would suffer an increase in cost.

A network that gives rise to Braess's paradox is depicted in Figure 4.5 and the data of the problem are given in Figure 4.6. The demand D between the OD pair is 6 and there are 3 paths joining the origin node to the destination node. Note that the cost on the arc e_{32} contains a nonnegative parameter λ whose value is permitted to change.

Substituting the data into the mixed LCP formulated above, we obtain for this network,

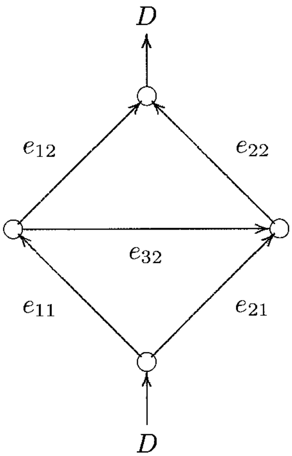

Figure 4.5

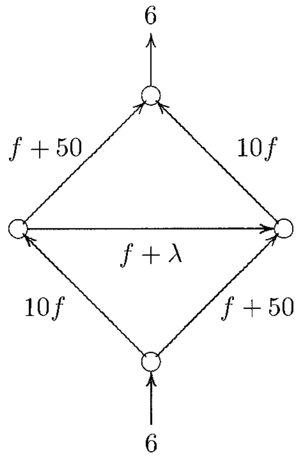

Figure 4.6

$$q = \begin{bmatrix} \Delta^{\mathrm{T}} b \\ -D \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \\ \lambda \\ -6 \end{bmatrix},$$

$$M = \begin{bmatrix} \Delta^{\mathrm{T}} D \Delta & -e \\ e^{\mathrm{T}} & 0 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 10 & -1 \\ 0 & 11 & 10 & -1 \\ 10 & 10 & 21 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

(For this problem, it can be shown that the mixed LCP formulation is equivalent to the standard LCP formulation in which the constraint $F_1 + F_2 + F_3 = 6$ is turned into an inequality, the variable G is restricted to be nonnegative, and a complementarity condition is imposed between G and the resulting demand constraint. The data (q, M) given above refers to the standard LCP formulation.)

Solving the above LCP as a parametric problem, we obtain the (unique) solution as a function in the parameter λ: for $\lambda \in [0, 23]$,

$$F_1 = F_2 = \frac{\lambda + 16}{13}, \ F_3 = \frac{2(23 - \lambda)}{13}, \ G = \frac{(1286 - 9\lambda)}{13};$$

and for $\lambda \geq 23$,

$$F_1 = F_2 = 3, \ F_3 = 0, \ G = 83.$$

Notice that in the latter case, the third path is no longer used; yet the minimum cost on the other two paths is equal to 83 units which is less than that in the previous case when all three paths are used.

From the point of view of a parametric LCP, Braess's paradox and many others are not particularly surprising. The reason for this is that the matrix $\Delta^{\mathrm{T}} D \Delta$ and the vector $\Delta^{\mathrm{T}} b$ in the LCP formulation can be fairly general and do not seem to possess any special properties that would allow one to predict with certainty the change in the equilibrium flows and the minimum path cost.

4.6 Variable Dimension Schemes

The idea behind the methods described in this section is to attempt to solve linear complementarity problems (q, M) by solving smaller *principal subproblems*. When (q, M) is an LCP of order n, the latter are linear complementarity problems associated with the (index) subsets of $\{1, \ldots, n\}$. Thus, if $\alpha \subseteq \{1, \ldots, n\}$, the corresponding principal subproblem of (q, M) is

$$(q, M)_\alpha = (q_\alpha, M_{\alpha\alpha}).$$

Now suppose z_α solves the principal subproblem $(q, M)_\alpha$. If

$$q_{\bar{\alpha}} + M_{\bar{\alpha}\alpha} z_\alpha \geq 0, \tag{1}$$

then by putting $z_{\bar{\alpha}} = 0$, the given solution of principal subproblem $(q, M)_\alpha$ can be completed to a solution z of the original problem (q, M). Naturally, one hopes that this approach reduces the total computational effort. It is not clear that this can always be realized.

Apart from the computational efficiency matter is the question of the *existence* of solutions for the principal subproblems. Fortunately, there is a class of matrices for which the question has an affirmative answer. Recall that, in Section 3.10, we studied completely-$\boldsymbol{Q}$ matrices. Also known as strictly semimonotone matrices, they have the property that every principal submatrix belongs to $\boldsymbol{Q}$. This guarantees the existence of at least one solution to every principal subproblem that can be formed.

In cases where the solutions to principal subproblems are not unique, there arises the question of whether the choice of solution vector affects the outcome of the test (1). This is a matter of w-uniqueness, which we studied in Section 3.4. The key property is column adequacy (see **3.4.6**). Notice that if we assume this property along with strict semimonotonicity, we have matrices belonging to $\boldsymbol{P}_0 \cap \boldsymbol{Q}$. The latter class came up in **3.9.22**.

Van der Heyden's method

The variable dimension scheme of Van der Heyden is designed to work with nondegenerate linear complementarity problems having strictly semimonotone matrices. An essential concept for this method is that of a leading principal subproblem (see **4.2.7**), i.e., a principal subproblem (as defined above) with the additional property that $\alpha = \{1, \ldots, k\}$ for some $k \leq n$. The corresponding principal subproblem (or any principal pivotal transform thereof) is called the k-subproblem.

The class $\boldsymbol{E}$ of strictly semimonotone matrices is unaffected by principal rearrangement of its members. Thus, if (q, M) is an LCP of order n with $M \in \boldsymbol{E}$ and $P \in R^{n \times n}$ is a permutation matrix, then (Pq, PMP^{T}) is an equivalent LCP whose matrix is also strictly semimonotone. This observation enables us to *preprocess* the problem so as to place the nonnegative[9] elements of q (if any) as the leading components of Pq. The motivation for this step is to simplify the statement of the algorithm rather than to reduce the number of its iterations.

Algorithm **4.6.3** below is another principal pivoting method. It consists of a sequence of major cycles, each of which ends with a solution to a k-subproblem and a principal pivotal transform of the original LCP. Solving (q, M) is a matter of solving the n-subproblem. At all times, the algorithm

[9] When nondegeneracy is assumed (as it is here), the vector q will have no zero components.

works with vectors w and z satisfying the fundamental equation

$$w = q + Mz. \tag{2}$$

Initially, z is the zero vector. Throughout the procedure, the components of z are required to remain nonnegative. In general, once a k-subproblem is defined, its basic variables are all eligible to block the current driving variable.

The vectors generated by this algorithm are basic solutions of (2). They are of two types as defined below.

4.6.1 Definition. A solution (w, z) of (2) is of *type I* if:

(a) there exists an index $k \in \{1, \ldots, n\}$ such that $z_k = 0$ and $w_k < 0$;

(b) $z_j = 0$ for all $j > k$;

(c) if $k > 1$ and $\alpha = \{1, \ldots, k-1\}$, then (w_α, z_α) solves $(q, M)_\alpha$.

4.6.2 Definition. A solution (w, z) of (2) is of *type II* if:

(a) there exists an index $k \in \{1, \ldots, n\}$ such that $z_k > 0$ and $w_k < 0$;

(b) $z_j = 0$ for all $j > k$;

(c) there exists an index $\ell < k$ such that $w_\ell = 0$ and $z_\ell = 0$;

(d) if $\alpha = \{1, \ldots, k-1\}$, then (w_α, z_α) solves $(q, M)_\alpha$.

A solution of type I satisfies the complementarity condition $z_i w_i = 0$ for all i, but is *not* a feasible solution of (2). A solution of type II is almost complementary; it satisfies the complementarity condition for every index except k. Note that the index k associated with a solution of type II must be greater than 1, since the definition calls for the existence of an index $\ell < k$.

4.6.3 Algorithm. (Van der Heyden)

Step 0. *Initialization.* Input the nondegenerate LCP (q, M) with M strictly semimonotone. Preprocess (q, M) so that the positive components of q (if any) come first.

Step 1. *Feasibility test; definition of the subproblem.* If $w \geq 0$, stop. [A solution has been found.] Otherwise, determine the index

$$k = \min\{i : w_i < 0\}.$$

[If $k > 1$ and $\alpha := \{1, \ldots, k-1\}$, then (w_α, z_α) solves $(q, M)_\alpha$.] Let the driving variable be z_k and define the eligible blocking variables to be w_k and the other basic variables of the k-subproblem.

Step 2. *Determination of the blocking variable.* Let v_j denote the blocking variable. Pivot

$$\langle \textit{ blocking variable, driving variable } \rangle.$$

Perform one of the following operations:

(a) If $j < k$, increase the complement of v_j; go to Step 2.

(b) If $j = k$ and $v_k = w_k$, the k-subproblem is solved; go to Step 1.

(c) If $j = k$ and $v_k = z_k$, another solution of a *smaller* subproblem has been found; go to Step 3.

Step 3. *Determination of a new driving variable.* Define

$$k = \max\{i : z_i > 0\}.$$

[The variable w_k will be nonbasic at value 0.] Use $-w_k$ as the new driving variable. [That is, *decrease* w_k.] Go to Step 2.

Since the solutions generated by the algorithm are *basic* solutions of (2) and these finite in number, it follows that the algorithm can generate only finitely many distinct basic solutions of type I or type II. The stipulation that the solutions be basic is essential to the truth of this assertion; there is nothing in the preceding two definitions that implies the finiteness of the number of such solutions. See Exercise **4.11.22**.

At any stage of the algorithm, there is a specified k-subproblem (possibly of a principal rearrangement of the original LCP), and when $k > 1$, a solution of the $(k-1)$-subproblem is at hand. It follows from this and the strict semimonotonicity of M that the algorithm cannot generate unbounded edges (equivalently, unblocked driving variables). To see this, let $\alpha = \{1, \ldots, k\}$ and suppose there is an unblocked driving variable encountered in solving $(q, M)_\alpha$. Then there must be a nonzero vector $(\tilde{w}_\alpha, \tilde{z}_\alpha)$ such that

$$\tilde{w}_\alpha = M_{\alpha\alpha}\tilde{z}_\alpha, \qquad \tilde{z}_\alpha \geq 0, \qquad \tilde{w}_k \leq 0.$$

Furthermore, $\tilde{w}_i\tilde{z}_i = 0$ for all $i < k$. Since $\tilde{z}_\alpha \neq 0$ as $(\tilde{w}_\alpha, \tilde{z}_\alpha)$ is nonzero, it follows that $\tilde{z}_\alpha$ violates the definition of strict semimonotonicity relative to $M_{\alpha\alpha}$. This contradiction shows that in Algorithm **4.6.3** driving variables are always blocked.

The nondegeneracy assumption assures that driving variables can always be increased or decreased, and since driving variables are always blocked, the algorithm generates a sequence of basic solutions of type I or type II. There are only finitely many basic solutions of any kind and a fortiori only finitely many of these two types. Thus, to prove that the process is finite, it remains to show that no basic solution is ever visited more than once. The argument for this is analogous to the one given in Theorem **4.4.4**. The details are omitted.

4.6.4 Example. Consider the LCP given by the following tableau.

	1	z_1	z_2	z_3	z_4
w_1	−10	2	3	3	2
w_2	−12	2	2	2	3
w_3	−9	2	3	3	1
w_4	−8	1	1	1	2

Here the 4×4 matrix M, being positive, is strictly copositive and hence strictly semimonotone although not a $\boldsymbol{P}$-matrix. Since all the components of q are negative, there is no need to preprocess the problem. We start with $k = 1$ and solve the leading 1-subproblem by pivoting $\langle w_1, z_1 \rangle$. This results in the schema

	1	w_1	z_2	z_3	z_4
z_1	5	$\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{3}{2}$	-1
w_2	-2	1	-1	-1	1
w_3	1	1	0	0	-1
w_4	-3	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1

At this stage, $k = 2$. The driving variable is z_2 is blocked by z_1, hence the next pivot step is $\langle z_1, z_2 \rangle$. This produces the almost complementary schema

	1	w_1	z_1	z_3	z_4
z_2	$\frac{10}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	-1	$-\frac{2}{3}$
w_2	$-\frac{16}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	0	$\frac{5}{3}$
w_3	1	1	0	0	-1
w_4	$-\frac{14}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{4}{3}$

Using w_1 as the driving variable, we find that w_2 is the blocking variable and the cycle ends with a solution of the 2-problem. The pivot $\langle w_2, w_1 \rangle$ gives the complementary schema

	1	w_2	z_1	z_3	z_4
z_2	6	$\frac{1}{2}$	-1	-1	$-\frac{3}{2}$
w_1	8	$\frac{3}{2}$	-1	0	$-\frac{5}{2}$
w_3	9	$\frac{3}{2}$	-1	0	$-\frac{7}{2}$
w_4	-2	$\frac{1}{2}$	0	0	$\frac{1}{2}$

Here we see that $k = 4$. The driving variable is z_4 and the next blocking variable is w_3. Performing the corresponding pivot, we get

	1	w_2	z_1	z_3	w_3
z_2	$\frac{15}{7}$	$-\frac{1}{7}$	$-\frac{4}{7}$	-1	$\frac{3}{7}$
w_1	$\frac{11}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	0	$\frac{5}{7}$
z_4	$\frac{18}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	0	$-\frac{2}{7}$
w_4	$-\frac{5}{7}$	$\frac{5}{7}$	$-\frac{1}{7}$	0	$-\frac{1}{7}$

The driving variable for this schema is z_3. Its increase is blocked by z_2. The pivot $\langle z_2, z_3 \rangle$ gives the schema

	1	w_2	z_1	z_2	w_3
z_3	$\frac{15}{7}$	$-\frac{1}{7}$	$-\frac{4}{7}$	-1	$\frac{3}{7}$
w_1	$\frac{11}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	0	$\frac{5}{7}$
z_4	$\frac{18}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	0	$-\frac{2}{7}$
w_4	$-\frac{5}{7}$	$\frac{5}{7}$	$-\frac{1}{7}$	0	$-\frac{1}{7}$

The new driving variable is w_2, and it is blocked by w_4. Performing the pivot $\langle w_4, w_2 \rangle$ brings the computation to a close with the feasible complementary schema

	1	w_4	z_1	z_2	w_3
z_3	2	$-\frac{1}{5}$	$-\frac{3}{5}$	-1	$\frac{2}{5}$
w_1	2	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{4}{5}$
z_4	3	$\frac{3}{5}$	$-\frac{1}{5}$	0	$-\frac{1}{5}$
w_2	1	$\frac{7}{5}$	$\frac{1}{5}$	0	$\frac{1}{5}$

This schema reveals that $(z_1, z_2, z_3, z_4) = (0, 0, 2, 3)$ solves the problem.

Lemke's variable dimension scheme

Just as Van der Heyden's method is related to Algorithm **4.3.2** (the symmetric principal pivoting method), Lemke's variable dimension scheme, which we now describe, is based on Algorithm **4.5.4** (the *parametric version* of his Algorithm, Scheme I).

For consistency, we continue to use the language and notation introduced above. Thus, for a given LCP (q, M) of order n and a suitable covering vector $d \in R^n$, we have the system (q, d, M) defined in (4.4.6). Using this, we can define the subproblem

$$(q, d, M)_\alpha = (q_\alpha, d_\alpha, M_{\alpha\alpha}).$$

When α is of the form $\{1, \ldots, k\}$ for $k \leq n$, we call $(q, d, M)_\alpha$ a *leading subproblem* of (q, d, M). More specifically, it is also called the *(leading) k-subproblem* of (q, d, M).

The algorithm discussed below deals with successively larger leading subproblems, each of which is treated by Algorithm **4.5.4**. A key feature is its *transition mechanism* for passing from the k-subproblem to the $(k+1)$-subproblem. This feature is needed because in solving the k-problem, the algorithm ignores the nonnegativity constraints that would normally be imposed on *all* the (basic) variables, not just those of the k-problem. When $k < n$ and it becomes necessary to move on to the $(k+1)$-problem for which feasibility has been lost, the covering vector can be modified in the $(k+1)$-st coordinate. The following observations are pertinent to this process. First, recall that the basic requirement of any covering vector d is that $(d, q) \succ 0$. Thus, any vector $\tilde{d} \geq d$ will also qualify as a covering vector. In particular, if d is a covering vector for (q, M) and $k < n$,

$$\tilde{d} = d + \theta e_{k+1}, \quad \theta > 0,$$

can also be used as a covering vector. Of course, we won't know a priori how large θ will need to be in order to cover negativity that might be present in the $(k+1)$-problem. What's more, (in the tableau implementation of the algorithm) the determination of such a value will be based on the *updated* form of the data. Nevertheless, because of the special structure of the modification to d and the fact that only the leading k-problem is being solved, we have the identity

$$\tilde{d}^\nu = d^\nu + \theta e_{k+1}.$$

As stated earlier, the k-problems are treated with Algorithm **4.5.4**. The initial schema for the algorithm is given by (4.5.11) which at the ν-th iteration is given by (4.5.12). In the following schema we depict the situation at iteration ν and a partitioning of the problem. In the upper left-hand corner we have the leading k-problem. The leading $k \times k$ submatrix of M is denoted $M_{\alpha\alpha}$, and so forth.

$$
\begin{array}{c|c|c|c|c|c}
 & 1 & z_0^\nu & z_\alpha^\nu & z_{\bar\alpha}^\nu & \\
\hline
w_\alpha^\nu & q_\alpha^\nu & d_\alpha^\nu & M_{\alpha\alpha}^\nu & M_{\alpha\bar\alpha}^\nu & \bar w_\alpha^\nu \\
\hline
w_{\bar\alpha}^\nu & q_{\bar\alpha}^\nu & d_{\bar\alpha}^\nu & M_{\bar\alpha\alpha}^\nu & M_{\bar\alpha\bar\alpha}^\nu & \bar w_{\bar\alpha}^\nu \\
\hline
 & 1 & \bar z_0^\nu & \bar z_\alpha^\nu & \bar z_{\bar\alpha}^\nu &
\end{array}
\qquad (3)
$$

Let us assume that Algorithm **4.5.4** has been applied to the leading k-problem and terminates at iteration ν. There are three possibilities which we list below along with the steps to be taken in each. These steps constitute the "transition algorithm."

Case 1. $q_\alpha^\nu > 0$, (3) is a complementary schema, and the variable z_0 has decreased from a positive value $\bar z_0^\nu$ to 0.

- If $q_{k+1}^\nu > 0$, the $(k+1)$-problem also terminates in Case 1.
- Otherwise, by nondegeneracy, $q_{k+1}^\nu < 0$. Set

$$\bar z_0^* = -q_{k+1}^\nu/(d_{k+1}^\nu + \theta).$$

  Regard this as the parameter value for a new major cycle of the $(k+1)$-problem. In this case, $0 < \bar z_0^* < \bar z_0^\nu$.

Case 2. $d_\alpha^\nu \geq 0$, (3) is a complementary schema, the end of a major cycle for the k-problem has been reached, and z_0 has increased from a positive value $\bar z_0^\nu$ to ∞.

- The parameter z_0 can be increased to ∞ in the $(k+1)$-problem which also terminates in Case 2.

Case 3. $(M_{\alpha\alpha}^\nu)_{\bullet s} \geq 0$ for some s, the schema (3) is almost complementary, and z_0 is at a positive value $\bar z_0^\nu$.

- If $m_{k+1,\,s}^\nu \geq 0$, the $(k+1)$-problem terminates in Case 3.
- If $m_{k+1,\,s}^\nu < 0$, the major cycle continues but in the $(k+1)$-problem; the parameter value remains at $\bar z_0^\nu$, and the pivot $\langle w_{k+1}^\nu, z_s^\nu \rangle$ is carried out.

4.7 Methods for $\boldsymbol{Z}$-Matrices

In Section 3.11, we considered the special, but important, class of $\boldsymbol{Z}$-matrices and discussed its intimate connection with the linear complementarity problem. Members of this class and its subclass, $\boldsymbol{K}$, have interesting properties that contribute to the qualitative analysis of LCPs and to their efficient solution. Our purpose here is to develop pivoting algorithms for problems of this class.

It is a consequence of Theorem **3.11.6** that $\boldsymbol{Z} \subset \boldsymbol{Q}_0$. Thus, if $M \in \boldsymbol{Z}$ and the LCP (q, M) is feasible, then (q, M) must have a solution which is a least element of its feasible region. Under these conditions, the problem can have more than one solution, although only one of them can be a least-element solution. This is illustrated by the case where

$$q = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{1}$$

In this particular problem, the feasible region is the half-line

$$\{z \in R^2_+ : z_2 = z_1 - 1\}$$

and each of its members is a solution of (q, M). The least-element solution is $(z_1, z_2) = (1, 0)$.

Preprocessing

In general, solving the linear complementarity problem (q, M) can be thought of as the task of identifying an appropriate index set α (corresponding to the set of positive z-variables) such that

$$\begin{aligned} w_\alpha &= q_\alpha + M_{\alpha\alpha} z_\alpha = 0 \\ w_{\bar\alpha} &= q_{\bar\alpha} + M_{\bar\alpha\alpha} z_\alpha \geq 0. \end{aligned} \tag{2}$$

Once α is known, the rest is a matter of solving the system (2) for $z_\alpha > 0$. The combinatorial challenge of the LCP is to find such an α. When M is a $\boldsymbol{Z}$-matrix, however, the identification of α is greatly simplified by the following observation.

4.7.1 Proposition. If $M \in \boldsymbol{Z}$, the LCP (q, M) is feasible only if $m_{ii} > 0$ when $q_i < 0$, and if z is a feasible solution to this problem, then $z_i > 0$. If z solves (q, M), then $w_i = q_i + \sum_j m_{ij} z_j = 0$, if $q_i < 0$.

Proof. The feasibility part is obvious from the sign patterns of the data in the inequality

$$w_i = q_i + \sum_{j=1}^{n} m_{ij} z_j \geq 0,$$

which for nonnegative values of the variables implies

$$m_{ii} z_i \geq -(q_i + \sum_{j \neq i} m_{ij} z_j) > 0.$$

If z solves (q, M) and $q_i < 0$, then $z_i > 0$ and hence $w_i = 0$. □

The preceding proposition enables one to do some preliminary analysis of a problem in this class. An LCP (q, M) with $M \in \boldsymbol{Z}$ will be called *nontrivial* if q is not nonnegative and the constraints are not *prima facie* infeasible, i.e., there is no subscript i such that $q_i < 0$ and $m_{ii} \leq 0$. If this is found, the problem can be declared infeasible. If it is not found, the problem can still be infeasible, but in a subtler way as for example in the problem (q, M) with

$$q = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}.$$

Second, after ruling out *prima facie* infeasibility, one can concentrate on finding a set of variables that must be positive (and a set of inequalities that must be binding) in any solution to the problem (if one exists at all). This idea is crucial to several algorithms.

As we shall see, the next proposition also has algorithmic importance.

4.7.2 Proposition. Let M be a $\boldsymbol{Z}$-matrix having nonsingular principal submatrix $M_{\alpha\alpha}$ and let $M' = \wp_\alpha(M)$ be the principal transform of M obtained by using $M_{\alpha\alpha}$ as pivot block. If $M_{\alpha\alpha}^{-1} \geq 0$, then

(a) $M'_{\alpha\alpha} \geq 0$,

(b) $M'_{\alpha\bar{\alpha}} \geq 0$,

(c) $M'_{\bar{\alpha}\alpha} \leq 0$,

(d) $M'_{\bar{\alpha}\bar{\alpha}} \in \boldsymbol{Z}$.

Proof. This is a simple consequence of the block pivot formulas (2.3.9) and the sign patterns of the submatrices involved. □

This proposition fits in beautifully with principal pivoting approaches to solving the LCPs of the $\boldsymbol{Z}$-matrix type. If $M_{\alpha\alpha}$ is the (cumulative) pivot block, it follows from (b) in **4.7.2** that each component of the basic subvector z_α is a *nondecreasing* function of each nonbasic variable z_j for $j \in \bar{\alpha}$. Thus, once a variable becomes basic, it will not decrease and become nonbasic when a nonbasic driving variable is increased. Moreover, the columns corresponding to the nonbasic subvector w_α play no role in computing the updates needed for execution of the algorithm. Hence these columns can be ignored. A further benefit of this proposition stems from (d) which states that the Schur complement $(M/M_{\alpha\alpha}) = M'_{\bar{\alpha}\bar{\alpha}} \in \boldsymbol{Z}$. This fact makes it possible to apply the same reasoning to the reduced LCP $(q'_{\bar{\alpha}}, M'_{\bar{\alpha}\bar{\alpha}})$ where $q'_{\bar{\alpha}} = q_{\bar{\alpha}} - M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1} q_\alpha$.

Chandrasekaran's method

The following algorithm can be construed as a special simple principal pivoting method. In light of the remarks made above, it does not attempt to determine a blocking variable. Instead of preventing an increase in the number of negative basic variables (which *can* happen), it assures strict lexicographic increase in the current z-vector.

4.7.3 Algorithm. (Chandrasekaran)

Step 0. *Initialization.* Input (q, M) with $M \in \boldsymbol{Z} \cap R^{n \times n}$. Let $\alpha = \emptyset$ and define $(q^0, M^0) = (q, M)$. Let $\nu = 0$.

Step 1. *Test for termination.* If $q^\nu_{\bar{\alpha}} \geq 0$, stop. A solution is given by the vector z such that $z_\alpha = q^\nu_\alpha$ and $z_{\bar{\alpha}} = 0$.

Step 2. *Choose distinguished variable.* Choose $r \in \bar{\alpha}$ such that $q^\nu_r < 0$. If $m^\nu_{rr} \leq 0$, stop. The problem is infeasible.

Step 3. *Pivoting*. Execute the pivot $\langle w_r^\nu, z_r^\nu \rangle$. Define

$$\begin{aligned} w_r^{\nu+1} &= z_r^\nu \\ z_r^{\nu+1} &= w_r^\nu \\ w_i^{\nu+1} &= w_i^\nu \qquad i \neq r \\ z_i^{\nu+1} &= z_i^\nu \qquad i \neq r \end{aligned}$$

Return to Step 1 with $\alpha \leftarrow \alpha \cup \{r\}$ and $\nu \leftarrow \nu + 1$.

Chandrasekaran's method is an example of a *greedy algorithm* as the following result shows.

4.7.4 Theorem. For any $M \in \boldsymbol{Z} \cap R^{n \times n}$ and $q \in R^n$, Algorithm **4.7.3** will process (q, M) in at most n iterations.

Proof. The argument rests on Propositions **4.7.1** and **4.7.2**. To apply the latter, we need to verify that the nonnegativity hypothesis holds. At each iteration, the algorithm has, in effect, transformed the original system by a block pivot on some principal submatrix $M_{\alpha\alpha}$. It must be shown that $M_{\alpha\alpha}^{-1} \geq 0$.

Consider the first iteration. Either the procedure stops for want of a needed positive diagonal entry in M or else there is a first pivot. At this stage, the index set α is a singleton, and the corresponding principal submatrix is a positive matrix of order 1. At the next iteration, an index $r \in \bar{\alpha}$ is chosen. The corresponding diagonal entry of the Schur complement $(M/M_{\alpha\alpha})$ is a ratio of the form

$$\frac{\det M_{\beta\beta}}{\det M_{\alpha\alpha}}$$

where $\beta = \alpha \cup \{r\}$. If this fraction is nonpositive, the procedure stops. If it is positive, then up to a principal rearrangement

$$M_{\beta\beta}^1 \simeq \begin{bmatrix} + & \oplus \\ \ominus & + \end{bmatrix},$$

and by virtue of having this sign pattern, it follows that the 2×2 matrix $M_{\beta\beta}^{-1} \geq 0$. The next thing that happens is that r is adjoined to α, which means that the next iteration (if any) begins with $M_{\alpha\alpha}^{-1} \geq 0$. More generally, suppose k iterations have been executed so that $|\alpha| = k$. Then, by **3.11.10**, it follows that $M_{\alpha\alpha} \in \boldsymbol{K}$. Suppose the diagonal entry of interest is m_{rr}^k, and let $\beta = \alpha \cup \{r\}$. From the theory of principal pivoting, we have

$$m_{rr}^k = \frac{\det M_{\beta\beta}}{\det M_{\alpha\alpha}}.$$

Since the denominator $\det M_{\alpha\alpha}$ is positive, the sign of m_{rr}^k is determined by that of $\det M_{\beta\beta}$. If it is nonpositive, the problem is infeasible and the algorithm stops. Suppose it is positive. Then, up to a principal rearrangement,

$$M_{\beta\beta}^k \simeq \begin{bmatrix} \oplus & \oplus \\ \ominus & + \end{bmatrix}.$$

Pivoting on the element in the lower right-hand corner yields $M_{\beta\beta}^{-1}$ which is easily seen to be a nonnegative matrix. Thus, at each iteration, the cumulative pivot block has a nonnegative inverse, and the conclusions of **4.7.2** are applicable. Since the steps of the algorithm are executable and the only changes to α arise by adjoining new elements, the algorithm must terminate after at most n iterations. □

Applicability of Lemke's method

In Theorem **3.11.6** the fact that $\boldsymbol{Z} \subset \boldsymbol{Q}_0$ was demonstrated with an analytical technique, whereas Chandrasekaran's method, Algorithm **4.7.3**, and Theorem **4.7.4** amount to a constructive proof of this result. Another such constructive proof can be based on the fact that Lemke's method will process any LCP (q, M) such that $M \in \boldsymbol{Z}$.

It might be imagined that this can be proved by showing that $\boldsymbol{Z}$ is contained in the class $\boldsymbol{L}$ defined in **3.9.18**. (See the end of **4.12.15**.) But this is not possible, however, because a $\boldsymbol{Z}$-matrix is not necessarily semimonotone: $\boldsymbol{Z}$-matrices need not have nonnegative diagonal entries, whereas all semimonotone matrices must.

To simplify the discussion, we assume the covering vector chosen for Lemke's method is e, the vector of ones. Other positive covering vectors can be used provided appropriate notational changes are made. We shall also assume that the problem (q, M) to which the algorithm is applied is nontrivial (as defined above).

Lemke's method deals with a class of basic solutions of the system

$$Iw - Mz - z_0 e = q. \tag{3}$$

Initially, z_0 becomes basic at a positive level. As long as z_0 remains basic, there is a nonbasic pair of variables which for purposes of this discussion will be denoted w_n and z_n. It should be emphasized that the actual index corresponding to the nonbasic pair changes from one iteration to the next. However, it eases this discussion to assume that the variables are continually relabelled so as to reserve the index n for the index of the nonbasic pair.

We also use another generic notation in this discussion. Let α denote the indices of the basic w-variables and β the indices of the basic z-variables. Note that according to the convention regarding the common index of the nonbasic pair, we have

$$\alpha \cup \beta = \{1, \ldots, n-1\}, \quad \alpha \cap \beta = \emptyset.$$

Up to principal rearrangement, we have

$$[I, \ -M, \ -e] = \begin{bmatrix} I_{\alpha\alpha} & 0 & 0 & -M_{\alpha\alpha} & -M_{\alpha\beta} & -M_{\alpha n} & -e_\alpha \\ 0 & I_{\beta\beta} & 0 & -M_{\beta\alpha} & -M_{\beta\beta} & -M_{\beta n} & -e_\beta \\ 0 & 0 & 1 & -M_{n\alpha} & -M_{n\beta} & -M_{nn} & -1 \end{bmatrix} \tag{4}$$

and

$$q = \begin{bmatrix} q_\alpha \\ q_\beta \\ q_n \end{bmatrix}.$$

The corresponding almost-complementary basis is the matrix

$$B = \begin{bmatrix} I_{\alpha\alpha} & -M_{\alpha\beta} & -e_\alpha \\ 0 & -M_{\beta\beta} & -e_\beta \\ 0 & -M_{n\beta} & -1 \end{bmatrix}.$$

Thus, $M_{\alpha\alpha}$ and $M_{\beta\beta}$ are $\boldsymbol{Z}$-matrices, and all the entries of $-M_{\alpha n}$, $-M_{\beta n}$, $-M_{\alpha\beta}$, $-M_{\beta\alpha}$, $-M_{n\alpha}$ and $-M_{n\beta}$ are nonnegative.

Notice that the almost-complementary basic feasible solution to (3) associated with B is

$$\begin{bmatrix} w_\alpha \\ z_\beta \\ z_0 \end{bmatrix} = B^{-1}q.$$

In canonical form, the column of the nonbasic variable z_n is $-B^{-1}M_{\bullet n}$. From this it follows that

$$\partial z_0 / \partial z_n = (B^{-1}M_{\bullet n})_n. \tag{5}$$

As we shall see, the sign of this particular number plays a crucial role in the solvability of the problem.

The following theorem gives another constructive proof—via Lemke's method—of the inclusion $\boldsymbol{Z} \subset \boldsymbol{Q_0}$.

4.7.5 Theorem. Suppose $M \in \boldsymbol{Z} \cap R^{n \times n}$. If (q, M) is a nontrivial LCP, then Lemke's method applied to (q, e, M) will process it.

Proof. We must show that if the algorithm fails to determine a solution, then the problem (q, M) must be infeasible. In keeping with the choice of covering vector and the notational conventions established above, we may assume that

$$q_n = \min_i q_i < 0.$$

Thus, after the initial pivot $\langle w_n, z_0 \rangle$, we obtain an almost-complementary basis B in which $\alpha = \{1, \ldots, n-1\}$ and $\beta = \emptyset$. Because (q, M) is nontrivial, $m_{nn} > 0$; and since $M \in \boldsymbol{Z}$, it follows that (with respect to the basis B) the updated column of z_n is positive. Using z_n as the driving variable—as dictated by the algorithm—makes *all* the basic variables decrease. (Recall that in this discussion, z_0 and all the other variables are on the left-hand side of the equation. See (3).) This implies that z_n must be blocked by some basic variable. The problem is solved when z_0 can be chosen as the blocking variable, so assume that some other basic variable blocks z_n. Performing the appropriate pivot and relabeling the variables, we obtain new index sets α and β. As a result of this operation, we know that $M_{\beta\beta} \in \boldsymbol{P} \cap \boldsymbol{Z}$. (Actually, it is just a single positive number at this stage, but the statement is true nonetheless.) In the relabelled system, z_n is the driving variable.

Relative to the *new* basis B, we are interested in the number indicated in (5). To this end, consider the system of equations

$$\begin{bmatrix} I_{\alpha\alpha} & -M_{\alpha\beta} & -e_\alpha \\ 0 & -M_{\beta\beta} & -e_\beta \\ 0 & -M_{n\beta} & -1 \end{bmatrix} \begin{bmatrix} u_\alpha \\ u_\beta \\ u_n \end{bmatrix} = \begin{bmatrix} -M_{\alpha n} \\ -M_{\beta n} \\ -M_{nn} \end{bmatrix}. \tag{6}$$

More concisely, this is

$$Bu = -M_{\bullet n}.$$

Here B, being a basis, is nonsingular. Suppose (as is initially the case) that

$$M_{\beta\beta} \in \boldsymbol{P} \cap \boldsymbol{Z}. \tag{7}$$

Then, using an elimination procedure, it is easy to see that

$$(1 - M_{n\beta}M_{\beta\beta}^{-1}e_\beta)u_n = M_{nn} - M_{n\beta}M_{\beta\beta}^{-1}M_{\beta n}. \tag{8}$$

In terms of Schur complements, (8) says

$$\left(\begin{bmatrix} M_{\beta\beta} & e_\beta \\ M_{n\beta} & 1 \end{bmatrix} / \, M_{\beta\beta}\right) u_n = \left(\begin{bmatrix} M_{\beta\beta} & M_{\beta n} \\ M_{n\beta} & M_{nn} \end{bmatrix} / \, M_{\beta\beta}\right).$$

Our assumptions imply

$$1 - M_{n\beta}M_{\beta\beta}^{-1}e_\beta > 0,$$

so (8) implies that u_n behaves the same way as $M_{nn} - M_{n\beta}M_{\beta\beta}^{-1}M_{\beta n}$. Since

$$\partial z_0 / \partial z_n = -u_n,$$

the variation of z_0 with respect to an increase of z_n as a driving variable depends on $(M_{\gamma\gamma}/M_{\beta\beta})$ where $\gamma = \beta \cup \{n\}$. Moreover, our assumptions enable us to claim that

$$M_{\gamma\gamma} \in \boldsymbol{P} \cap \boldsymbol{Z} \quad \Leftrightarrow \quad (M_{\gamma\gamma}/M_{\beta\beta}) > 0. \tag{9}$$

When $(M_{\gamma\gamma}/M_{\beta\beta}) > 0$, the driving variable z_n cannot be unblocked. Now suppose $(M_{\gamma\gamma}/M_{\beta\beta}) \leq 0$ while z_0 is still basic at a positive level. We now have

$$v^{\mathrm{T}} := (0, -M_{n\beta}M_{\beta\beta}^{-1}, 1) \geq 0. \tag{10}$$

From the assumption $(M_{\gamma\gamma}/M_{\beta\beta}) \leq 0$, direct calculation shows that

$$v^{\mathrm{T}}M \leq 0. \tag{11}$$

From the equation

$$\begin{bmatrix} I_{\alpha\alpha} & -M_{\alpha\beta} & -e_{\alpha} \\ 0 & -M_{\beta\beta} & -e_{\beta} \\ 0 & -M_{n\beta} & -1 \end{bmatrix} \begin{bmatrix} w_{\alpha} \\ z_{\beta} \\ z_0 \end{bmatrix} = \begin{bmatrix} q_{\alpha} \\ q_{\beta} \\ q_n \end{bmatrix}$$

we obtain

$$-(1 - M_{n\beta}M_{\beta\beta}^{-1}e_{\beta})z_0 = q_n - M_{n\beta}M_{\beta\beta}^{-1}q_{\beta}.$$

Since z_0 is positive and its coefficient is negative, we have

$$v^{\mathrm{T}}q = q_n - M_{n\beta}M_{\beta\beta}^{-1}q_{\beta} < 0. \tag{12}$$

The inequalities (10), (11) and (12), imply that (q, M) is infeasible.

The proof is complete but for one detail. The argument is predicated on the assumption that the driving variable is never a w-variable. To see that this is so, note that it is true at the outset. If $u_n \leq 0$ at some stage, we have a signal that (q, M) is infeasible. If $u_n > 0$, however, then from (6), we have

$$\partial z_i/\partial z_n = -(M_{\beta\beta}^{-1})(M_{\beta n} - e_{\beta}u_n)_i > 0 \quad \text{for all } i \in \beta.$$

Thus, except for z_0, once a z-variable becomes basic, it remains so and can never be a blocking variable—hence its complement can never be a driving variable. □

The reader may have noticed that the customary assumption about nondegeneracy was not made here. This is because no such assumption is required in the present circumstances. In a degenerate problem, it may be necessary to make many w-variables nonbasic before the driving variable can actually be increased by a positive amount. But apart from the extra computational effort this causes, there is no danger of cycling.

A word of caution is in order, though. The implementation of the algorithm must include the provision that z_0 *will* be chosen as the blocking variable whenever it *can* be chosen. Failure to do so can lead to false termination as illustrated in **4.4.16**.

When $M \in \boldsymbol{Z}$, there is an interesting connection between the processing of (q, M) by Lemke's method and the solution of the *linear program*

$$\begin{array}{ll} \text{minimize} & z_0 \\ \text{subject to} & Iw - Mz - ez_0 = q \\ & w \geq 0,\ z \geq 0,\ z_0 \geq 0 \end{array} \tag{13}$$

by the simplex method. Let us see why this is so. We assume that the notations established above are still in force.

4.7.6 Lemma. Let B denote an almost-complementary basis generated by Lemke's method in the solution of (q, M). Then $(B^{-1})_{n\bullet} \leq 0$.

Proof. An easy calculation shows that

$$(B^{-1})_{n\bullet} = (0,\ (1 - M_{n\beta}M_{\beta\beta}^{-1}e_\beta)^{-1}M_{n\beta}M_{\beta\beta}^{-1},\ -(1 - M_{n\beta}M_{\beta\beta}^{-1}e_\beta)^{-1})$$

which is nonpositive because $M_{\beta\beta} \in \boldsymbol{P} \cap \boldsymbol{Z}$. □

4.7.7 Theorem. If $M \in \boldsymbol{Z} \cap R^{n \times n}$ and (q, M) is a nontrivial problem, then the sequences of basic solutions generated by the simplex method[10] and Lemke's method are the same.

Proof. We may assume that immediately after the first pivot, Lemke's method and the simplex method have the same almost complementary basis at hand. In general, suppose the first ν (almost complementary) feasible bases are the same. Let z_n and w_n denote the nonbasic pair (with notations and rearrangements as above). The simplex multipliers for any such matrix are given by the last row of its inverse. This row is nonpositive as shown in the lemma above. It now follows that there is only one negative reduced cost associated with this basis, and it corresponds to the nonbasic variable z_n which must be made basic. The mechanism for removing a basic variable is the same in both algorithms. □

4.8 A Special n-Step Scheme

In the preceding section, we have described a special pivoting method for solving the linear complementarity problem with a $\boldsymbol{Z}$-matrix. The interesting feature of this method is that it terminates in at most n iterations.

[10] We assume that $q_n = \min_i q_i < 0$ and initialize the simplex method with $w_1, \ldots, w_{n-1}, z_0$ as the first set of basic variables.

In turn, this termination property rests on the fact that once a z-variable has become basic, it will stay basic and will never become nonbasic again. The latter property also holds for Lemke's method applied to the LCP of the same type (see the proof of Theorem **4.7.5**). Note that in both instances, the assumption of a nondegenerate LCP is not required for the success of the methods.

In this section, we describe a special pivoting method that exploits the idea outlined above—namely, that once a z-variable becomes basic, it stays basic. The backbone of the method is the parametric LCP $(q+\lambda d, M)$ with a specially chosen parametric vector $d > 0$. The parameter λ is initially set (implicitly) at a sufficiently large positive value so that $z = 0$ is a solution of the LCP $(q + \lambda d, M)$. The goal is to decrease λ until it reaches zero, at which point, a solution of the original LCP (q, M) is obtained. The decrease of λ is accomplished by the parametric principal pivoting method discussed in **4.5.2**. As a matter of fact, the resulting scheme is just a variant of this previous method. Each iteration of the scheme consists of the same minimum ratio test (slightly simplified due to the special property of the vector d) and an update of the necessary data (i.e., the pivot step).

The basic assumptions underlying the special method are (a) M is a nondegenerate matrix, and (b) there exists a positive vector $d > 0$ such that

$$M_{\alpha\alpha}^{-1} d_\alpha > 0 \tag{1}$$

for all $\alpha \subseteq \{1, \ldots, n\}$.

An example of a matrix M satisfying these two conditions is any $\boldsymbol{K}$-matrix. Indeed, if $M \in \boldsymbol{K}$, then an arbitrary positive vector d will satisfy (b) in view of the fact that all principal submatrices of a $\boldsymbol{K}$-matrix are $\boldsymbol{K}$-matrices and such matrices must have a nonnegative inverse. Nevertheless, there are matrices $M \notin \boldsymbol{K}$ that satisfy these two properties. An example is the matrix

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

A vector d of the required sort is $(3, 4)$. Note that $M \notin \boldsymbol{P}$ and $M \notin \boldsymbol{Z}$.

4.8.1 Definition. A positive vector $d \in R^n$ that satisfies (1) for all index sets α is called an *n-step vector* for M.

Using such a vector d, we now describe the special pivoting method for solving the LCP (q, M) with a nondegenerate matrix M.

4.8.2 Algorithm. The n-Step Scheme

Step 0. *Initialization.* Input (q, d, M) where M is a nondegenerate matrix, d is an n-step vector for M, and q is arbitrary. Let $\alpha = \emptyset$ and $\nu = 0$. Define $(q^0, d^0) = (q, d)$.

Step 1. *Determine pivot element.* If $d^\nu_{\bar{\alpha}} \leq 0$, terminate, a solution of (q, M) is given by

$$z_\alpha = -M^{-1}_{\alpha\alpha} q_\alpha, \; z_{\bar{\alpha}} = 0. \tag{2}$$

Otherwise, compute

$$\lambda = \max_i \{ \frac{-q^\nu_i}{d^\nu_i} : i \in \bar{\alpha}, \; d^\nu_i > 0 \}.$$

If $\lambda \leq 0$, terminate; a solution z of (q, M) is given by (2). Otherwise, let

$$r \in \arg\max_i \{ \frac{-q^\nu_i}{d^\nu_i} : i \in \bar{\alpha}, \; d^\nu_i > 0 \}.$$

Step 2. *Pivoting.* Insert the index r into the set α, and replace ν by $\nu + 1$. If (the new) $\alpha = \{1, \ldots, n\}$, terminate with the solution z given in (2). Otherwise, compute (using the updated index sets α and $\bar{\alpha}$)

$$(q^\nu_{\bar{\alpha}}, d^\nu_{\bar{\alpha}}) = (q_{\bar{\alpha}}, d_{\bar{\alpha}}) - M_{\bar{\alpha}\alpha} M^{-1}_{\alpha\alpha} (q_\alpha, d_\alpha). \tag{3}$$

Return to Step 1.

As we have mentioned, the above algorithm is a variant of **4.5.2** that takes into account the assumptions (a) and (b) about the matrix M and vector d. There are several noteworthy points in **4.8.2**. One is the fact that only simple diagonal pivots are performed; these are possible because of the nondegeneracy assumption on the matrix M. Another point (which we have already noted) is that the parameter λ starts at the first critical value and decreases; this is different from **4.5.2** where λ is nondecreasing in value at each iteration, but is similar to the parametric version of Lemke's Scheme I (see Algorithm **4.5.4**). It should be emphasized that the change in value of λ has very little to do with the n-step termination of Algorithm **4.8.2**; as we shall see shortly, the fact that d is an n-step vector is the principal reason for the success of the algorithm. A final point to note is that the ratio test in Step 1 involves only the nonbasic components $i \in \bar{\alpha}$; this is because the special property (b) of the vector d implies that the basic components d_i^ν for $i \in \alpha$ are always negative, (see (4)), hence they are not qualified to be included in the ratio test.

Note that in the pivoting step (Step 2), we update only the vectors q and d and completely ignore the change of the matrix M. Although the displayed formula (3) gives only the updated (nonbasic) components $q_{\bar{\alpha}}^\nu$ and $d_{\bar{\alpha}}^\nu$, their computation implicitly involves the basic components q_α^ν and d_α^ν which are given by

$$(q_\alpha^\nu, d_\alpha^\nu) = -M_{\alpha\alpha}^{-1}(q_\alpha, d_\alpha). \tag{4}$$

Furthermore, since the index set α increases by one element at each iteration, the vector pair (q^ν, d^ν) can be computed by updating the previous pair $(q^{\nu-1}, d^{\nu-1})$.

4.8.3 Theorem. Let $M \in R^{n\times n}$ be a nondegenerate matrix and $q \in R^n$ be arbitrary. Suppose that there exists an n-step vector d for M. Then Algorithm **4.8.2** computes a solution of the LCP (q, M) in at most n iterations.

Proof. It suffices to prove $m_{rr}^\nu > 0$ at each iteration of the algorithm. Once this is established, the argument used previously to justify Algorithm **4.5.2** and the fact that the index set α always increases by one element can then be combined to yield the desired n-step conclusion in the present situation.

We start with the first iteration. Since d is a positive vector and M is nondegenerate, the condition (b) implies that all diagonal entries of M are positive. Hence, $m_{rr}^0 > 0$. Inductively, suppose that the algorithm is at iteration $\nu \geq 1$ with a current index set α. The index $r \in \bar{\alpha}$ has just been determined in Step 1. We have $d_r^\nu > 0$. The quantities d_r^ν and m_{rr}^ν can be expressed as follows:

$$d_r^\nu = d_r - M_{r\alpha}M_{\alpha\alpha}^{-1}d_\alpha, \quad m_{rr}^\nu = m_{rr} - M_{r\alpha}M_{\alpha\alpha}^{-1}M_{\alpha r}.$$

Since M is nondegenerate, m^{ν}_{rr} is nonzero. To show that it is positive, write

$$\begin{bmatrix} M_{\alpha\alpha} & M_{\alpha r} \\ M_{r\alpha} & M_{rr} \end{bmatrix} \begin{bmatrix} \bar{d}^{\nu}_{\alpha} \\ \bar{d}^{\nu}_{r} \end{bmatrix} = \begin{bmatrix} d_{\alpha} \\ d_{r} \end{bmatrix}.$$

By assumption, $\bar{d}^{\nu}_{r} > 0$. By an easy calculation, it follows that

$$\bar{d}^{\nu}_{r} = \frac{d^{\nu}_{r}}{m^{\nu}_{rr}}$$

which implies $m^{\nu}_{rr} > 0$ as desired. □

Theorem **4.8.3** implies that if M is an $n \times n$ nondegenerate matrix for which there exists an n-step vector d, then $M \in \boldsymbol{Q}$. Obviously, every principal submatrix of M must possess the same two properties as M. Consequently, the following result is obvious.

4.8.4 Corollary. If $M \in R^{n\times n}$ satisfies the assumptions of **4.8.3**, then M is a completely-$\boldsymbol{Q}$ matrix, and hence is strictly semimonotone. □

When $M \in \boldsymbol{P}$, the condition (1) for d to be an n-step vector can be slightly weakened without affecting the validity of the conclusion of Theorem **4.8.3**. This weakening amounts to replacing the strict inequality in (1) by

$$M^{-1}_{\alpha\alpha} d_{\alpha} \geq 0. \tag{5}$$

We state the resulting conclusion more precisely in the corollary below whose proof is evident.

4.8.5 Corollary. Let $M \in R^{n\times n} \cap \boldsymbol{P}$ and $q \in R^n$ be arbitrary. Suppose that there exists a vector $d \geq 0$ such that $q + \lambda d \geq 0$ for some $\lambda > 0$ and that (5) holds for all α. Then, using this vector d, Algorithm **4.8.2** computes the unique solution of the LCP (q, M) in no more than n iterations. □

The validity of the above corollary (and the more general **4.8.3**) can be linked to a certain monotonicity property of the solution of the parametric LCP (4.5.1). In order to explain this property, we let $z(\lambda, q, b)$ denote the (unique) solution of the LCP $(q + \lambda b, M)$ for $M \in \boldsymbol{P}$. The significant thing about the notation $z(\lambda, q, b)$ is that $\lambda \in R$ is the parameter of the LCP, $q \in R^n$ is the constant vector, and $b \in R^n$ is the parametric vector.

According to Algorithm **4.5.2** (see also Remark **4.5.3**), there exist a finite number of breakpoints $-\infty < \lambda^1 \leq \lambda^2 \leq \ldots \lambda^k < \infty$ such that within each interval $[\lambda^\nu, \lambda^{\nu+1}]$, there is an index set α (not necessarily unique) in terms of which the solution function $z(\cdot, q, d)$ is given by

$$z_\alpha(\lambda, q, d) = -M_{\alpha\alpha}^{-1}(q_\alpha + \lambda d_\alpha), \quad z_{\bar{\alpha}}(\lambda, q, d) = 0 \tag{6}$$

for all $\lambda \in [\lambda^\nu, \lambda^{\nu+1}]$; a similar expression for $z(\lambda, q, d)$ is also valid within the two unbounded intervals $(-\infty, \lambda^1]$ and $[\lambda^k, \infty)$.

4.8.6 Proposition. Let $M \in R^{n\times n} \cap \boldsymbol{P}$ and $d \in R^n_+$. The following statements are equivalent.

(a) For each index set α, (5) holds.

(b) For each vector $q \in R^n$, the function $z_q(\lambda) = z(\lambda, q, d)$ is *antitone* in $\lambda \in R$, i.e., $\lambda \leq \lambda' \Rightarrow z_q(\lambda) \geq z_q(\lambda')$.

(c) For each vector $b \in R^n$, the function $\tilde{z}_b(\lambda) = z(\lambda, d, b)$ is *isotone* for $\lambda \geq 0$ (i.e., $0 \leq \lambda \leq \lambda' \Rightarrow \tilde{z}_b(\lambda) \leq \tilde{z}_b(\lambda')$) and antitone for $\lambda \leq 0$.

Proof. (a) $\Rightarrow$ (b). This is obvious from the expression for $z(\lambda, q, d)$ (see (6)).

(b) $\Rightarrow$ (a). Suppose that for some index set $\alpha \neq \emptyset$, $M_{\alpha\alpha}^{-1} d_\alpha$ contains a negative component, say $(M_{\alpha\alpha}^{-1} d_\alpha)_i < 0$ for $i \in \alpha$. Pick a vector $q \in R^n$ such that

$$-M_{\alpha\alpha}^{-1} q_\alpha > 0, \quad \text{and} \quad q_{\bar{\alpha}} - M_{\beta\alpha} M_{\alpha\alpha}^{-1} q_\alpha > 0.$$

Then for this vector q, we have $\alpha = \operatorname{supp} z_q(0)$. Moreover, for $\lambda < 0$ sufficiently close to zero, we have $(z_q(\lambda))_i < (z_q(0))_i$, contradicting the antitonicity of the function $z_q(\cdot)$.

(a) $\Rightarrow$ (c). We prove only the isotonicity of $\tilde{z}_b(\lambda)$ for $\lambda \geq 0$. Take two values $\lambda' > \lambda'' \geq 0$. Without loss of generality, we may assume that λ' and λ'' belong to the same interval of linearity of $z_b(\lambda)$, say $[\lambda^\nu, \lambda^{\nu+1}]$, and that the corresponding index set α for this interval is nonempty. We have

$$0 \leq (\tilde{z}_b(\lambda'))_\alpha = -M_{\alpha\alpha}^{-1}(d_\alpha + \lambda' b_\alpha) \leq -\lambda' M_{\alpha\alpha}^{-1} b_\alpha.$$

Since $\lambda' > 0$, it follows that $M_{\alpha\alpha}^{-1} b_\alpha \leq 0$. Hence, the isotonicity of $\tilde{z}_b(\lambda)$ for $\lambda \geq 0$ follows.

(c) $\Rightarrow$ (a). Suppose that for some index set $\alpha \neq \emptyset$, $M_{\alpha\alpha}^{-1}d_\alpha$ contains a negative component, say $(M_{\alpha\alpha}^{-1}d_\alpha)_i < 0$ for $i \in \alpha$. Since $M_{\alpha\alpha}$ is a $\boldsymbol{P}$-matrix, there exists a vector $f_\alpha > 0$ such that $M_{\alpha\alpha}f_\alpha > 0$. Let $\varepsilon > 0$ be such that

$$(M_{\alpha\alpha}^{-1}d_\alpha)_i + \varepsilon f_i < 0.$$

Define

$$b_\alpha = -d_\alpha - \varepsilon M_{\alpha\alpha}f_\alpha,$$

and let $b_{\bar{\alpha}}$ be such that

$$(d_{\bar{\alpha}} - M_{\bar{\alpha}\alpha}M_{\alpha\alpha}^{-1}d_\alpha) + (b_{\bar{\alpha}} - M_{\bar{\alpha}\alpha}M_{\alpha\alpha}^{-1}b_\alpha) > 0.$$

Then for this vector b, we have $\alpha = \operatorname{supp} \tilde{z}_b(1)$. Moreover, it is easy to see that $-(M_{\alpha\alpha}^{-1}b_\alpha)_i < 0$. Consequently, for $\lambda < 1$ but sufficiently close to 1, we have $(z_b(\lambda))_i > (z_b(1))_i$, contradicting (c). □

Hidden $\boldsymbol{Z}$-matrices and n-step vectors

It turns out that if M is a $\boldsymbol{P}$-matrix to start with, then the existence of an n-step vector for M can be characterized in terms of the hidden $\boldsymbol{Z}$-property for the transpose of M. We divide the proof of this result into two parts; the following theorem asserts the first part.

4.8.7 Theorem. Suppose that $M^{\mathrm{T}} \in R^{n\times n}$ is a hidden $\boldsymbol{K}$-matrix. Let X and Y be two $\boldsymbol{Z}$-matrices satisfying

(a) $M^{\mathrm{T}}X = Y$,

(b) $r^{\mathrm{T}}X + s^{\mathrm{T}}Y > 0$ for some $r, s \geq 0$.

If $d \in R^n$ is any vector such that $d^{\mathrm{T}}X > 0$, then d is an n-step vector for M.

Before proving this theorem, we compare its implication with the results derived for an LCP with a hidden $\boldsymbol{Z}$-matrix (see Section 3.11). Under the assumptions of Theorem **4.8.7**, it follows that for any vector $q \in R^n$, the LCP (q, M) can be solved by the n-step pivoting scheme described in **4.8.2** with the parametric vector d being an arbitrary vector satisfying $d^{\mathrm{T}}X > 0$. According to Theorem **3.11.18** and the subsequent discussion, a solution of the LCP (q, M^{T}) can be obtained by solving the linear program

(3.11.2) with the vector $p = d$. This is an extremely peculiar situation in that a very special pivoting method solves an LCP, whereas a certain linear program solves a related LCP that is defined by the transpose of the matrix involved in the former problem; moreover, the objective "cost vector" in the linear program is the same as the parametric vector in the special pivoting scheme. Perhaps some kind of "duality" relationship can be used to explain this mysterious phenomenon. Unfortunately, such an explanation has yet to be given.

Proof of 4.8.7. First of all, we note that Theorem **3.11.19** implies $X \in \boldsymbol{K}$. Hence X^{-1} exists and is nonnegative; thus $d > 0$. Let $\alpha \subseteq \{1, \ldots, n\}$ be arbitrary. We verify the condition:

$$M_{\bar{\alpha}\bar{\alpha}}^{-1} d_{\bar{\alpha}} > 0. \tag{7}$$

According to the expression (3.11.9), we have

$$M_{\bar{\alpha}\bar{\alpha}}^{-\mathrm{T}} = (X/X_{\alpha\alpha})(W/X_{\alpha\alpha})^{-1}$$

where W is the matrix given in (3.11.8). Consequently,

$$M_{\bar{\alpha}\bar{\alpha}}^{-1} = (W/X_{\alpha\alpha})^{-\mathrm{T}}(X/X_{\alpha\alpha})^{\mathrm{T}}.$$

Put $\bar{d} = X^{\mathrm{T}} d$. It is then a simple matter to show

$$(X/X_{\alpha\alpha})^{\mathrm{T}} d_{\bar{\alpha}} = \bar{d}_{\bar{\alpha}} - ((X_{\alpha\alpha})^{-1} X_{\alpha\bar{\alpha}})^{\mathrm{T}} \bar{d}_{\alpha}$$

which is positive because $\bar{d}$ is a positive vector and $X \in \boldsymbol{K}$. Moreover, the Schur complement $(W/X_{\alpha\alpha})$ is a $\boldsymbol{K}$-matrix, hence has a nonnegative inverse; in particular, its inverse cannot have a zero row or column. Consequently, the desired inequality (7) follows. Since α is arbitrary, it follows that d is an n-step vector. □

In order to prove the reverse of Theorem **4.8.7**, we first derive a useful property of an n-step vector.

4.8.8 Lemma. Suppose that d is an n-step vector for the $\boldsymbol{P}$-matrix $M \in R^{n \times n}$. Then, for each index set $\alpha \subset \{1, \ldots, n\}$, and each index $t \notin \alpha$,

$$d_t - M_{t\alpha} M_{\alpha\alpha}^{-1} d_{\alpha} > 0.$$

Proof. Let $\alpha' = \alpha \cup \{t\}$, and write

$$\bar{d}_{\alpha'} = M_{\alpha'\alpha'}^{-1} d_{\alpha'}$$

which by assumption, is a positive vector. It is easy to see that

$$\bar{d}_t = \frac{d_t - M_{t\alpha} M_{\alpha\alpha}^{-1} d_\alpha}{m_{tt} - M_{t\alpha} M_{\alpha\alpha}^{-1} M_{\alpha t}}.$$

Since $M \in \boldsymbol{P}$, the denominator is positive. Hence, it follows that the numerator is positive also. □

Our goal is to show that if M is a $\boldsymbol{P}$-matrix possessing an n-step vector, then M^{T} must be a hidden $\boldsymbol{Z}$-matrix. In order to exhibit the required $\boldsymbol{Z}$-matrices for this purpose, we define the matrix M^i obtained from M^{T} by first negating the i-th column and then negating the i-th row. Note that this leaves the i-th diagonal of M unchanged. In terms of the sign-changing matrix introduced in Section 2.3, we have

$$M^i = E_i M^{\mathrm{T}} E_i \tag{8}$$

where E_i is equal to the diagonal matrix with all diagonal entries equal to 1 except for the i-th diagonal entry which is equal to -1. It is easy to see that $M^i \in \boldsymbol{P}$. Let p^i denote the i-th column of M^i. The LCP (p^i, M^i) has a unique solution which we denote by z^i. We claim that $z^i_i = 0$. Indeed, if z^i_i were positive, then the vector $e_i + z^i$ would be a nonzero solution of the homogeneous LCP $(0, M^i)$, contradicting the $\boldsymbol{P}$-property of M^i. We summarize this discussion in the lemma below.

4.8.9 Lemma. For each $i = 1, \ldots, n$, the LCP (p^i, M^i) has a unique solution z^i with $z^i_i = 0$. □

The following result formally states the reverse of Theorem **4.8.7**.

4.8.10 Theorem. Let $M \in R^{n \times n} \cap \boldsymbol{P}$. If an n-step vector exists for M, then M^{T} is a hidden $\boldsymbol{Z}$-matrix.

Proof. For each $i = 1, \ldots, n$, let z^i be the solution as described in Lemma **4.8.9**, and define $X_{\bullet i} = e_i - z^i$. Then, the matrix $X \in \boldsymbol{Z}$. We show that the matrix $Y := M^{\mathrm{T}}X \in \boldsymbol{Z}$. By an easy manipulation and using the fact that $z_j^j = 0$, it is not difficult to deduce that for $i \neq j$,

$$y_{ij} = -(p^j + M^j z^j)_i \leq 0$$

where the last inequality holds because $z^j \in \mathrm{SOL}(p^j, M^j)$. Hence, the matrix $Y \in \boldsymbol{Z}$. Let $d > 0$ be an n-step vector for M. It remains to be shown that $d^{\mathrm{T}}X > 0$, or equivalently, $d^{\mathrm{T}}X_{\bullet j} > 0$ for all $j = 1, \ldots, n$. Fix an index j. Let $\alpha = \operatorname{supp} z^j$, and let

$$\bar{d}_\alpha = M_{\alpha\alpha}^{-1} d_\alpha.$$

Then, $d_\alpha = M_{\alpha\alpha}\bar{d}_\alpha$. Since $z_j^j = 0$, it follows that $j \in \alpha$; moreover, we have

$$0 = (p^j + M^j z^j)_\alpha = -(M_{j\alpha})^{\mathrm{T}} + (M^{\mathrm{T}})_{\alpha\alpha} z_\alpha^j.$$

Consequently,

$$\begin{aligned} d^{\mathrm{T}}X_{\bullet j} &= d_j - d^{\mathrm{T}}z^j = d_j - d_\alpha^{\mathrm{T}} z_\alpha^j \\ &= d_j - (\bar{d}_\alpha)^{\mathrm{T}}(M_{\alpha\alpha})^{\mathrm{T}} z_\alpha^j \\ &= d_j - M_{j\alpha}\bar{d}_\alpha > 0 \end{aligned}$$

where the last inequality follows from Lemma **4.8.8**. □

Combining Theorems **4.8.7** and **4.8.10**, we obtain the following characterization.

4.8.11 Corollary. Let $M \in R^{n\times n} \cap \boldsymbol{P}$. Then, M^{T} is hidden $\boldsymbol{Z}$ if and only if M has an n-step vector. □

The proofs of the two Theorems **4.8.7** and **4.8.10** reveal several interesting facts. First, if $M \in \boldsymbol{P}$, then there must exist $\boldsymbol{Z}$-matrices X and Y, with X having positive diagonal entries, such that $M^{\mathrm{T}}X = Y$, regardless of whether an n-step vector exists for M. Moreover, a vector $d > 0$ is an n-step vector for M if and only if $d^{\mathrm{T}}X > 0$. This last observation implies that if an n-step vector for M exists, then the matrix $X \in \boldsymbol{K}$ and the set of the n-step vectors is equal to $\operatorname{int}(\operatorname{pos} X^{-\mathrm{T}})$; in particular, the closure of this set of vectors is equal to the simplicial cone $\operatorname{pos} X^{-\mathrm{T}}$.

4.9 Degeneracy Resolution

Throughout most of this chapter on pivoting methods, we have assumed the nondegeneracy of certain linear systems and their basic solutions. Two exceptions occurred in Section 4.2 where we saw a lexicographic scheme—used in conjunction with Algorithm **4.2.2**—and a least-index rule that was formally part of Murty's Algorithm **4.2.6**. In addition to these, we noted that nondegeneracy assumptions are unnecessary in the Dantzig/van de Panne-Whinston Algorithm **4.2.11** for the symmetric positive semi-definite case, in the methods for $\boldsymbol{Z}$-matrices presented in Section 4.7, and in the n-step scheme of Section 4.8.

In Section 4.2 we gave a single illustration of the theoretical need for a degeneracy resolution techniques in a pivoting method for the LCP. In this section, we shall document this need a bit further and present some standard cycling remedies that ensure the finiteness of the major pivoting algorithms studied in this book.

Examples of cycling

In this subsection, we give two examples of cycling—one for the symmetric PPM and the other for Lemke's method, Scheme I. In addition to providing evidence that cycling can occur, these examples verify the sharpness of certain lower bounds on problem size and cycle length.

4.9.1 Example. Consider the following LCP (q, M) in schematic form.

	1	z_1	z_2	z_3	z_4
w_1	-1	1	-0.3	-92108	173608
w_2	0	0.3	0.00001	0.5	-2
w_3	0	92108	-0.5	23840	-44932
w_4	0	-173608	2	-44932	84688

By breaking the matrix M into four blocks of order 2, it is not difficult to see that it is positive definite. Therefore, except for its nondegeneracy assumption, the symmetric PPM is applicable to this problem. Furthermore, it can be shown that the algorithm could execute the principal pivots

$$\langle w_4, z_4\rangle,\ \langle w_3, z_3\rangle,\ \langle w_2, z_2\rangle,\ \langle z_4, w_4\rangle,\ \langle z_3, w_3\rangle,\ \langle z_2, w_2\rangle,$$

which would bring one back to the original schema.

The entries of the matrix in this LCP are somewhat complicated, and perhaps a better example to illustrate cycling in the PPM can be found. If so, it will not be of order less than 4, for it is known that this is the minimum order for cycling to occur in the symmetric PPM. For further discussion of this result, see **4.12.28**.

A smaller—and much tamer—example illustrates cycling in Lemke's method.

4.9.2 Example. Consider the LCP in which

$$q = \begin{bmatrix} 0 \\ -2 \\ -3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

In this instance, we choose the covering vector $d = (0, 1, 1)$.

	1	z_0	z_1	z_2	z_3
w_1	0	0	−1	−1	1
w_2	−2	1	1	1	0
w_3	−3	1	1	1	1

Lemke's method will first execute the pivots $\langle w_3, z_0\rangle$ and $\langle w_2, z_3\rangle$ after which it will yield the schema

	1	w_3	z_1	z_2	w_2
w_1	1	1	−1	−1	−1
z_3	1	1	0	0	−1
z_0	2	0	−1	−1	1

(1)

At this schema, the algorithm executes the pivots $\langle w_1, z_2\rangle$, $\langle z_2, z_1\rangle$, $\langle z_1, w_2\rangle$, and $\langle w_2, w_1\rangle$ after which it returns to the schema (1). This is a cycle of length four. There are no shorter cycles for Lemke's method.

The next result shows that the order of the LCP in the numerical example above is as small as it can be.

4.9.3 Proposition. If Lemke's method **4.4.5** is applied to a linear complementarity problem (q, M) of order 2, it will terminate in a finite number of steps.

Proof. The statement is trivial if $q \geq 0$. There is no loss of generality in assuming that $q_1 < 0$. Let d denote the nonnegative covering vector used in Lemke's Scheme I. We may assume that $d = (d_1, d_2)$ where $d_1 > 0$ and $d_2 > 0$ if $q_2 < 0$. A crucial element of the proof will turn out to be that the pivotal transforms of d_2 are all nonnegative.

The initial schema is

	1	z_0	z_1	z_2
w_1	q_1	d_1	m_{11}	m_{12}
w_2	q_2	d_2	m_{21}	m_{22}

It is also not restrictive to assume that the first pivot of the procedure is $\langle w_1, z_0 \rangle$. After this pivot, the schema is

	1	w_1	z_1	z_2
z_0	q_1^1	d_1^1	m_{11}^1	m_{12}^1
w_2	q_2^1	d_2^1	m_{21}^1	m_{22}^1

in which $d_2^1 \geq 0$. The driving variable is now z_1. If z_0 blocks z_1, the pivot $\langle z_0, z_1 \rangle$ occurs and the algorithm terminates. It also terminates if z_1 is unblocked. Thus, we may assume that w_2 blocks z_1. After the pivot $\langle w_2, z_1 \rangle$, the schema is

	1	w_1	w_2	z_2
z_0	q_1^2	d_1^2	m_{11}^2	m_{12}^2
z_1	q_2^2	d_2^2	m_{21}^2	m_{22}^2

Again $d_2^2 \geq 0$. The next driving variable is z_2. If it is blocked by z_0, the method will pivot $\langle z_0, z_2 \rangle$ and terminate. It will also terminate if z_2

is unblocked. The remaining possibility is that z_1 blocks z_2. The pivot $\langle z_1, z_2 \rangle$ gives

	1	w_1	w_2	z_1
z_0	q_1^3	d_1^3	m_{11}^3	m_{12}^3
z_2	q_2^3	d_2^3	m_{21}^3	m_{22}^3

The driving variable for this schema is w_1. Since $d_2^3 \geq 0$, the variable z_2 cannot block w_1. So either w_1 is blocked by z_0 (in which case the procedure terminates after one last pivot) or else w_1 is unblocked and the algorithm terminates with a secondary ray. In any event, the algorithm cannot cycle and terminates in finitely many steps. □

Toward the prevention of cycling

The above examples reinforce the notion that—for the sake of theory if not practice—something must be done about "the degeneracy problem." It should be noted, however, that degeneracy *per se* is not a problem. As we have seen, degeneracy can (but need not) lead to a problem, namely the problem of cycling. Degeneracy resolution techniques are motivated by the need to eliminate this problem.

Like the simplex method for linear programming, pivoting methods for the LCP are intended to be (essentially) adjacent extreme point algorithms. They produce sequences of "adjacent" bases of a certain system of linear equations. These bases give rise to basic solutions of the equations. (In most cases, these basic solutions are extreme points of certain polyhedral sets). In the presence of degeneracy, there can be more than one basis corresponding to a particular (basic) solution of the linear system, and hence it is not necessarily the case that a change of basis leads to a different point. When an algorithm cycles at a basic solution, it generates a sequence of bases that repeats one of those already generated. This prevents the algorithm from terminating in a finite number of steps.

Several devices have been proposed for resolving degeneracy in the LCP. Two of them were used in Section 4.2, namely lexicographic ordering and least-index rules. Other techniques are mentioned in Section 4.12. In the remainder of this chapter, we first show how Lemke's Scheme I can be made finite by incorporating lexicographic ordering of vectors, then we show that a least-index pivot selection rule will make the symmetric principal pivoting method finite when applied to LCP's of the sufficient matrix type.

Lexicographic degeneracy resolution

In Section 4.2, we used lexicographic ordering in connection with Algorithm **4.2.2**. We shall now use this technique in conjunction with Lemke's Scheme I. To describe this version of the algorithm, we combine the tableau form used in Section 4.4 with the lexicographic machinery of Section 4.2. In particular, we consider the schema

	1	x	z_0	z
w	q	Q	d	M

(2)

In (2), d is the usual sort of nonnegative covering vector, and Q is chosen to be a nonsingular matrix with lexicographically positive rows. The identity matrix would do quite nicely.

The schema (2) is a way of representing the system of equations

$$Iw - Mz - dz_0 = q + Qx. \tag{3}$$

At present, we attach no meaning to the components of x, though we may think of them as being zero. We shall, however, be interested in their coefficients, the entries of Q and its pivotal transforms.

For the present purposes, it will be helpful to introduce the notations

$$\mathcal{Q} = [q,\, Q] \quad \text{and} \quad \mathcal{M} = [d,\, M].$$

Notationally, we shall regard q and d as the 0-th columns of $\mathcal{Q}$ and $\mathcal{M}$, respectively.

The effect we wish to achieve by using Q is to modify the adjacency structure of the set of almost complementary feasible bases. In particular, we want to prevent an almost complementary feasible basis from having more than two such neighbors.

Let A denote the $n \times (2n+1)$ matrix $[I,\, -M,\, -d]$. Invertible $n \times n$ submatrices of A are called *bases* in A. A basis B in A is called *almost* *complementary* if it contains $-d$ and not both of $I_{\bullet i}$ and $-M_{\bullet i}$ for any $i = 1, \ldots, n$. Relative to (3), a basis (almost complementary or otherwise) B is *feasible* (in the usual sense) if $B^{-1}q \geq 0$. What we need is the following notion.

4.9.4 Definition. Relative to the system (3) a basis B is *lexicographically feasible* if $B^{-1}Q \succ 0$, i.e., has lexicographically positive rows.

An initial almost complementary lexicographically feasible basis can be determined as follows. Let

$$r = \arg \text{ lexico } \max\{\frac{-1}{d_i}Q_{i\bullet} : Q_{i0} < 0\}. \tag{4}$$

This is precisely the analogue of the ratio test used in Algorithm **4.4.5** to obtain its first almost complementary feasible basis.

4.9.5 Proposition. If r is determined as in (4), then

$$B = [I_{\bullet 1}, \ldots, I_{\bullet r-1}, -d, I_{\bullet r+1}, \ldots, I_{\bullet n}]$$

is an almost complementary lexicographically feasible basis.

Proof. It is clear that B is an almost complementary basis; it is also a feasible basis in the ordinary sense. To see that B is lexicographically feasible, note that B^{-1} is the elementary matrix whose r-th column is $(1/d_r)(d_1, \ldots, d_{r-1}, -1, d_{r+1}, \ldots, d_n)$. Then, obviously,

$$\left(B^{-1}Q\right)_{r\bullet} = \frac{-1}{d_r}Q_{r\bullet} \succ 0;$$

moreover, for all $i \neq r$,

$$\left(B^{-1}Q\right)_{i\bullet} = Q_{i\bullet} - \frac{d_i}{d_r}Q_{r\bullet} \succ 0$$

by virtue of the definition of r and the fact that, by hypothesis, Q has lexicographically positive rows. (The latter stipulation takes care of the case where $d_i = q_i = 0$.) □

With r chosen as above, the driving variable for Lemke's method is z_r. In this case, and in general, the minimum ratio test used to determine a blocking variable (if any) is the lexicographic analogue of the usual one. For instance, at the ν-th iteration, if the k-th column of $\mathcal{M}^{\nu}$ is the column of the driving variable and this column contains at least one negative entry, then the lexicographic minimum ratio test is to find

$$t = \arg \text{ lexico } \min\{\frac{-1}{m_{ik}^{\nu}} \mathcal{Q}_{i\bullet}^{\nu} : m_{ik}^{\nu} < 0\}. \tag{5}$$

4.9.6 Proposition. Lexicographic feasibility is preserved when the lexicographic minimum ratio test (5) is used for the pivot selection rule in Lemke's method.

Proof. The argument is much the same as in **4.9.5**. □

The all-important property that the notion of lexicographic feasibility brings to bear on the cycling problem is given in the following result.

4.9.7 Proposition. An almost complementary basis can be adjacent to at most two lexicographically feasible almost complementary bases.

Proof. In the unique schema corresponding to an almost complementary basis there are at most two columns that are candidates for pivoting and hence for passing to an adjacent almost complementary basis. These are the column of the blocking variable and the column of the driving variable. (Only one of them would be indicated by Lemke's method. The other would correspond to reversing the almost complementary path.) In either of these columns, a pivot on any element other than the one determined by the lexicographic minimum ratio test would lead to a lexicographically *infeasible* basis. □

Least-index degeneracy resolution

In Section 4.3, we demonstrated that the principal pivoting method will process any linear complementarity problem (q, M) in which M is row sufficient. Here we shall show that by introducing a least-index pivot selection rule, the nondegeneracy assumption can be dropped provided that M is (row and column) sufficient. It will be helpful to adopt the notational apparatus developed in Section 4.3 for the PPM.

In applying the PPM to the linear complementarity problem (q, M), we break ties among the blocking variables according to the *least-index rule*

(A) If the distinguished variable is among the tied blocking variables, choose it as the blocking variable (and terminate the major cycle).

(B) Otherwise, choose the (basic) blocking variable with the smallest index as the exiting variable.

4.9.8 Algorithm. (Symmetric PPM with Least-Index Pivot Rule)

Step 0. Set $\nu = 0$; define $(\bar{w}^0, \bar{z}^0) = (q^0, 0)$. Let λ be any number less than $\min_i q_i^0$.

Step 1. If $q^\nu \geq 0$ or if $(\bar{w}^\nu, \bar{z}^\nu) \geq (0,0)$, stop; $(\bar{w}^\nu, \bar{z}^\nu) := (q^\nu, 0)$ is a solution. Otherwise[11], determine an index r such that $\bar{z}_r^\nu = \lambda$ or (if none such exist) an index r such that $\bar{w}_r^\nu < 0$.

Step 2. Let ζ_r^ν be the largest value of $z_r^\nu \geq \bar{z}_r^\nu$ satisfying the following conditions:

(i) $z_r^\nu \leq 0$ if $\bar{z}_r^\nu < 0$.

(ii) $W_r^\nu(\bar{z}_1^\nu, \ldots, \bar{z}_{r-1}^\nu, z_r^\nu, \bar{z}_{r+1}^\nu, \ldots, \bar{z}_n^\nu) \leq 0$ if $\bar{w}_r^\nu < 0$.

(iii) $W_i^\nu(\bar{z}_1^\nu, \ldots, \bar{z}_{r-1}^\nu, z_r^\nu, \bar{z}_{r+1}^\nu, \ldots, \bar{z}_n^\nu) \geq 0$ if $\bar{w}_i^\nu \geq 0$.

(iv) $W_i^\nu(\bar{z}_1^\nu, \ldots, \bar{z}_{r-1}^\nu, z_r^\nu, \bar{z}_{r+1}^\nu, \ldots, \bar{z}_n^\nu) \geq \lambda$ if $\bar{w}_i^\nu < 0$.

Step 3. If $\zeta_r^\nu = +\infty$, stop. No feasible solution exists. Otherwise, a basic variable, or the distinguished variable, and a condition in Step 2 are *associated* if the given variable appears in the condition's "if" clause. A variable is considered a *blocking variable* if making $z_r^\nu > \zeta_r^\nu$ would cause its associated Step 2 condition to be violated. If the driving variable is a blocking variable, then let $\bar{z}_r^{\nu+1} = 0$, $\bar{z}_r^{\nu+1} = \bar{z}_r^\nu$ for all $i \neq r$ and let $\bar{w}^{\nu+1} = W^{\nu+1}(\bar{z}^{\nu+1}) = W^\nu(\bar{z}^{\nu+1})$. Return to Step 1 with ν replaced by $\nu+1$. Otherwise, let s be the unique index determined by the least-index rule from among the blocking variables.

Step 4. If $m_{ss}^\nu > 0$, perform the principal pivot $\langle w_s^\nu, z_s^\nu \rangle$. Let

$$\bar{z}_s^{\nu+1} = W_s^\nu(\bar{z}_1^\nu, \ldots, \bar{z}_{r-1}^\nu, \zeta_r^\nu, \bar{z}_{r+1}^\nu, \ldots, \bar{z}_n^\nu), \bar{w}^{\nu+1} = W^{\nu+1}(\bar{z}^{\nu+1}).$$

[11] At the beginning of a major cycle, for each index r, at most one of w_r^ν, z_r^ν can be negative.

If $s = r$, return to Step 1 with ν replaced $\nu + 1$. If $s \neq r$, return to Step 2 with ν replaced $\nu + 1$. If $m^{\nu}_{ss} = 0$, perform the principal pivot $\{\langle w^{\nu}_s, z^{\nu}_r\rangle, \langle w^{\nu}_r, z^{\nu}_s\rangle\}$. Put $\bar{w}^{\nu+1}_r = \bar{z}^{\nu}_s$, $\bar{w}^{\nu+1}_s = \zeta^{\nu}_r$, $\bar{z}^{\nu+1}_i = \bar{z}^{\nu}_i$ for all $i \notin \{r, s\}$, and then $\bar{w}^{\nu+1}_i = W^{\nu+1}_i(\bar{z}^{\nu+1})$ for all $i \notin \{r, s\}$. Return to Step 2 with ν replaced by $\nu + 1$ and r replaced by s.

The finiteness argument

In the following, we show that the symmetric PPM with the least-index rule (stated above) will process any linear complementarity problem (q, M) in which M is sufficient. It is interesting to observe that the mechanics of the algorithm itself appears to require only the *row sufficiency* property. The finite termination of the algorithm (with or without the least-index rule) is assured if each major cycle is finite, for the total number of negative variables is nonincreasing during each major cycle and decreases strictly at the end of the major cycle. Such finiteness is realized when the problem is nondegenerate. We show here that, even for degenerate problems, the major cycles of the PPM are finite provided the least-index rule is enforced in the pivot selection criterion. As will be seen below, the finiteness of the PPM with the least-index rule hinges on the *column sufficiency* property. This is why we assume the matrix is both row and column sufficient.

Suppose cycling occurs in a major cycle of the algorithm. It is not restrictive to assume that it is one in which w_1 is the distinguished variable. It follows from the discussion following Algorithm **4.3.2** (just above **4.3.3**) that since w_1 and z_1 are monotonically increasing, both w_1 and z_1 are fixed during cycling. However, the algorithm tries to increase w_1 or z_1 in this major cycle. Hence stalling occurs during these steps. Accordingly, if we delete all the variables that are not involved during cycling, the PPM with the least-index rule merely looks for the index i such that

$$s = \min\{i : m^{\nu}_{i1} < 0\}$$

and then pivots on m^{ν}_{ss} (if $m^{\nu}_{ss} \neq 0$) or else on

$$\begin{bmatrix} m^{\nu}_{11} & m^{\nu}_{1s} \\ m^{\nu}_{s1} & m^{\nu}_{ss} \end{bmatrix} \qquad \text{if } m^{\nu}_{ss} = 0.$$

Without loss of generality, we may assume that all the variables are involved in the pivoting during cycling. Then, during cycling, the PPM with with least-index rule performs the same pivoting sequence as the following scheme does.

Step 0. Start with the system $w^\nu = q^\nu + M^\nu z^\nu$, $\nu = 0$, where $w^0 = q^0 + M^0 z^0$ is the initial system. (In the following, $M^\nu_{\bullet i}$ represents the column of M^ν corresponding to the nonbasic variable z^ν_i at iteration ν. Similarly, $M^\nu_{i\bullet}$ represents the row of M^ν corresponding to the basic variable w^ν_i.)

Step 1. If $M^\nu_{\bullet 1} \geq 0$, stop. The driving variable z^ν_1 can be increased strictly. Otherwise, let $s = \min\{i : m^\nu_{i1} < 0\}$.

Step 2. If $m^\nu_{ss} > 0$, perform a pivot on m^ν_{ss} and return to Step 1 with ν replaced by $\nu + 1$. Otherwise, perform a block pivot of order 2 on the principal submatrix

$$\begin{bmatrix} m^\nu_{11} & m^\nu_{1s} \\ m^\nu_{s1} & m^\nu_{ss} \end{bmatrix}$$

and return to Step 1 with ν replaced by $\nu + 1$.

If we can show that $M^\nu_{\bullet 1} \geq 0$ after a finite number of pivots in the above scheme, then, since the driving variable z^ν_1 can be increased strictly at this step, we obtain a contradiction to the assumption that cycling occurs in a major cycle (in which w_1 is the distinguished variable) of the PPM with the least-index rule.

Before proceeding, we present a small result on sufficient matrices. This result provides a mechanism for generating sufficient matrices of arbitrarily large order.

4.9.9 Lemma. Let $M \in R^{n\times n}$ be column (row) sufficient. Then for any real numbers a, b, c such that $ab < 0 \leq c$, the matrix

$$\hat{M} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} & a \\ m_{21} & m_{22} & \cdots & m_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} & 0 \\ b & 0 & \cdots & 0 & c \end{bmatrix}$$

is also column (row) sufficient.

Proof. It suffices to prove the assertion for column sufficient matrices. Let $\hat{x} = (x_1, x_2, \ldots, x_n, x_{n+1})^{\mathrm{T}}$ satisfy the inequalities $x_i(\hat{M}\hat{x})_i \leq 0$ for $i = 1, \ldots, n+1$. Then in particular,

$$x_1(m_{11}x_1 + \ldots + m_{1n}x_n) \leq -ax_1x_{n+1}$$

and

$$bx_1x_{n+1} \leq -cx_{n+1}^2 \leq 0.$$

Since $ab < 0$ it follows that $-ax_1x_{n+1} \leq 0$. Thus

$$x_i\Big(\sum_{i=1}^{n} m_{i1}x_i\Big) \leq 0 \quad i = 1, \ldots, n.$$

Since M is column sufficient,

$$x_i\Big(\sum_{i=1}^{n} m_{i1}x_i\Big) = 0 \quad i = 1, \ldots, n.$$

In particular, it follows that $x_1x_{n+1} = 0$. Hence $\hat{x} * (\hat{M}\hat{x}) = 0$. □

4.9.10 Lemma. In the above scheme, a pivot in row s, where $s \geq 2$, must be followed by a pivot in some row with a larger index before another pivot in row s can occur.

Proof. The proof is by induction. If the matrix M is of order 1 or 2, the lemma is trivial. Suppose the lemma holds when the order of M is less than n and now consider the case when M is of order n.

We shall examine the situation where two pivots occur in row s and $2 \leq s \leq n-1$. If, between these two pivots, there is no pivot in some row with a larger index, then by deleting $M_{\bullet n}$ and $M_{n \bullet}$, a contradiction to the inductive hypothesis can be derived. Therefore, it suffices to show that there is at most one pivot in row n.

Suppose a pivot occurs in row n at iteration ν_1. Let (T1) denote the corresponding tableau at this iteration. (For simplicity, we represent (T1) without using superscripts.)

	1	z_1	$\cdots$	z_n
w_1	q_1	m_{11}	$\cdots$	m_{1n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
w_n	q_n	m_{n1}	$\cdots$	m_{nn}

Tableau (T1)

By the choice of the pivot row, we have $m_{i1} \geq 0$ for all $i \leq n-1$ and $m_{n1} < 0$ in (T1).

Suppose the next occurrence of a pivot in row n is at iteration ν_2. When this occurs, z_n must be the exiting basic variable and w_1 is either basic (Case I) or nonbasic (Case II).

Case I. (w_1 is a basic variable at iteration ν_2.) Let σ be the set of indices i such that w_i is nonbasic at iteration ν_2. Note that $1 \notin \sigma$. Let $\bar{M}$ denote the principal transform of M at this iteration. Clearly, $\bar{M}$ can be obtained from M by performing a block pivot on the principal submatrix $M_{\sigma\sigma}$. Thus,

$$\bar{M}_{\sigma 1} = -M_{\sigma\sigma}^{-1} M_{\sigma 1}. \tag{6}$$

Now

$$\bar{M}_{\sigma 1} \simeq \begin{bmatrix} \oplus \\ \oplus \\ \vdots \\ \oplus \\ - \end{bmatrix} \quad \text{and} \quad M_{\sigma 1} \simeq \begin{bmatrix} \oplus \\ \oplus \\ \vdots \\ \oplus \\ - \end{bmatrix}. \tag{7}$$

Being a (nonsingular) principal submatrix of a sufficient matrix, $M_{\sigma\sigma}$ is a sufficient matrix. From (6) and (7), we have

$$M_{\sigma 1} * (M_{\sigma\sigma}^{-1} M_{\sigma 1}) \simeq \begin{bmatrix} \oplus \\ \oplus \\ \vdots \\ \oplus \\ - \end{bmatrix} * \begin{bmatrix} \ominus \\ \ominus \\ \vdots \\ \ominus \\ + \end{bmatrix} \simeq \begin{bmatrix} \ominus \\ \ominus \\ \vdots \\ \ominus \\ - \end{bmatrix}$$

which is impossible since $M_{\sigma\sigma}^{-1}$ is column sufficient.

Case II. (w_1 is a nonbasic variable at iteration ν_2.) Let the definition of σ be as in Case I, but note that now we have $1 \in \sigma$. Since $\bar{M}_{\sigma\sigma}$ is sufficient, the diagonal entry $\bar{m}_{11}$ is nonnegative. There are two cases.

Case II.1 ($\bar{m}_{11} > 0$.) The pivot on $\bar{m}_{11}$ would not change the sign configuration of $\bar{M}_{\sigma 1}$ namely

$$\bar{M}_{\sigma 1} \simeq \begin{bmatrix} \oplus \\ \oplus \\ \vdots \\ \oplus \\ - \end{bmatrix}.$$

Once this pivot is performed, we have Case I (with a different index set σ).

Case II.2 ($\bar{m}_{11} = 0$.) Here there are two more cases.

Case II.2.1 ($m_{11} > 0$.) By performing a pivot on m_{11} in tableau (T1), the variable w_1 becomes nonbasic and the sign configuration of $M_{\bullet 1}$ is unchanged. Therefore, as in Case I, a contradiction can be derived.

Case II.2.2 ($m_{11} = 0$.) Let (T2) denote the tableau at iteration ν_2.

	1	w_σ	$z_{\bar\sigma}$
z_σ	$\bar q_\sigma$	$\bar M_{\sigma\sigma}$	$\bar M_{\sigma\bar\sigma}$
$w_{\bar\sigma}$	$\bar q_{\bar\sigma}$	$\bar M_{\bar\sigma\sigma}$	$\bar M_{\bar\sigma\bar\sigma}$

Tableau (T2)

Now let $q_{n+1} \in R$ be arbitrary and enlarge (T1) to (T1*) as follows

	1	z_1	z_2	$\cdots$	z_n	z_{n+1}
w_1	q_1	m_{11}	m_{12}	$\cdots$	m_{1n}	-1
w_2	q_2	m_{21}	m_{22}	$\cdots$	m_{2n}	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
w_n	q_n	m_{n1}	m_{n2}	$\cdots$	m_{nn}	0
w_{n+1}	q_{n+1}	1	0	$\cdots$	0	1

Tableau (T1*)

By Lemma 1, the bordered matrix of tableau (T1*) is sufficient. The block pivot on the principal submatrix $M_{\sigma\sigma}$ in (T1*) produces a tableau (T2*) having (T2) as a subtableau.

	1	w_σ	$z_{\bar\sigma}$	z_{n+1}
z_σ	$\bar q_\sigma$	$\bar M_{\sigma\sigma}$	$\bar M_{\sigma\bar\sigma}$	$\bar M_{\sigma,n+1}$
$w_{\bar\sigma}$	$\bar q_{\bar\sigma}$	$\bar M_{\bar\sigma\sigma}$	$\bar M_{\bar\sigma\bar\sigma}$	$\bar M_{\bar\sigma,n+1}$
w_{n+1}	$\bar q_{n+1}$	$\bar M_{n+1,\sigma}$	$\bar M_{n+1,\bar\sigma}$	$\bar M_{n+1,n+1}$

Tableau (T2*)

Notice that (T2*) has the same basic z-variables as (T2), hence tableau (T2*) is the corresponding enlargement of (T2).

$$\begin{array}{ccccc} & & \text{Pivot on } M_{\sigma\sigma} & & \\ & (\text{T1}) & \rightarrow & (\text{T2}) & \\ \text{Enlargement} & \downarrow & & \downarrow & \text{Enlargement} \\ & (\text{T1*}) & \rightarrow & (\text{T2*}) & \\ & & \text{Pivot on } M_{\sigma\sigma} & & \end{array}$$

By pivotal algebra, we have

$$\begin{aligned} \bar{m}_{n+1,1} &= (M_{n+1,\sigma}\bar{M}_{\sigma\sigma})_1 \\ &= M_{n+1,\sigma}\bar{M}_{\sigma 1} \\ &= (1, 0, \ldots, 0)\bar{M}_{\sigma 1} \\ &= \bar{m}_{11} \\ &= 0. \end{aligned}$$

Now the matrix

$$\begin{bmatrix} m_{11} & m_{1,n+1} \\ m_{n+1,1} & m_{n+1,n+1} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

is nonsingular and can be used as a pivot block in (T1*). Denote the resulting tableau by (T2**). In it, w_1 and w_{n+1} are nonbasic while all the other w_i are basic.

	1	w_1	z_2	$\cdots$	z_n	w_{n+1}
z_1	$\bar{\bar{q}}_1$	$\bar{\bar{m}}_{11}$	$\bar{\bar{m}}_{12}$	$\cdots$	$\bar{\bar{m}}_{1n}$	$\bar{\bar{m}}_{1,n+1}$
w_2	$\bar{\bar{q}}_2$	$\bar{\bar{m}}_{21}$	$\bar{\bar{m}}_{22}$	$\cdots$	$\bar{\bar{m}}_{2n}$	$\bar{\bar{m}}_{2,n+1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
w_n	$\bar{\bar{q}}_n$	$\bar{\bar{m}}_{n1}$	$\bar{\bar{m}}_{n2}$	$\cdots$	$\bar{\bar{m}}_{nn}$	$\bar{\bar{m}}_{n,n+1}$
z_{n+1}	$\bar{\bar{q}}_{n+1}$	$\bar{\bar{m}}_{n+1,1}$	$\bar{\bar{m}}_{n+1,2}$	$\cdots$	$\bar{\bar{m}}_{n+1,n}$	$\bar{\bar{m}}_{n+1,n+1}$

Tableau (T2**)

In tableau (T1*), we have

$$m_{11} = 1,\ m_{i1} \geq 0 \quad i = 2, \ldots, n-1,\ m_{n1} < 0,\ \text{and}\ m_{n+1,1} = 1.$$

Hence

$$\bar{\bar{m}}_{11} = 0,\ \bar{\bar{m}}_{i1} \geq 0 \quad i = 2, \ldots, n-1,\ \bar{\bar{m}}_{n1} < 0,\ \text{and}\ \bar{\bar{m}}_{n+1,1} = -1.$$

Since both (T2*) and (T2**) are principal transforms of tableau (T1*), it follows that (T2**) is a principal transform of (T2*). In fact, if we define the index set $\rho = (\sigma \setminus \{1\}) \cup \{n+1\}$, then (T2**) can be obtained by performing a block pivot on the principal submatrix $\bar{M}_{\rho\rho}$ in (T2*). Therefore $\bar{\bar{M}}_{\rho 1} = -\bar{M}_{\rho\rho}^{-1}\bar{M}_{\rho 1}$. The indices n and $n+1$ belong to ρ and

$$\bar{m}_{n1} < 0,\ \bar{m}_{n+1,1} = 0,\ \bar{\bar{m}}_{n1} < 0,\ \bar{\bar{m}}_{n+1,1} = -1$$

while $\bar{m}_{i1} \geq 0$ and $\bar{\bar{m}}_{i1} \geq 0$ for all other $i \in \rho$. Accordingly, we obtain

$$\bar{M}_{\rho 1} * (\bar{M}_{\rho\rho}^{-1}\bar{M}_{\rho 1}) \simeq \begin{bmatrix} \oplus \\ \vdots \\ \oplus \\ - \\ 0 \end{bmatrix} * \begin{bmatrix} \ominus \\ \vdots \\ \ominus \\ + \\ + \end{bmatrix} \simeq \begin{bmatrix} \ominus \\ \vdots \\ \ominus \\ - \\ 0 \end{bmatrix}.$$

But this is impossible since $\bar{M}_{\rho\rho}^{-1}$ is (column) sufficient. □

4.9.11 Lemma. In tableau (T1), $M_{\bullet 1}^{\nu} \geq 0$ after a finite number of iterations.

Proof. For $j > 1$, let $\mu(j)$ be the number of pivots that occur in row j. In the proof of Lemma 2, we have shown that $\mu(n) \leq 1$. Furthermore, it follows from Lemma 2 that

$$\mu(j) \leq \sum_{i=j+1}^{n} \mu(i) + 1.$$

In other words,

$$\begin{aligned}\mu(n-1) &\le \mu(n)+1 \le 2\\ \mu(n-2) &\le 2^2\\ &\vdots\\ \mu(n-i) &\le 2^{i-1}+2^{i-2}+\cdots+2+2^0+1=2^i.\end{aligned}$$

Therefore, the scheme will terminate after a finite number of iterations. □

4.9.12 Theorem. In the case of a linear complementarity problem (q, M) with a sufficient matrix M, every major cycle of the PPM with the least-index rule consists of a finite number of pivots.

Proof. Suppose cycling occurs in a major cycle in which w_1 is the distinguished variable. Then, since w_1 and z_1 are monotonically increasing, both w_1 and z_1 are fixed during cycling. However, it follows from Lemma **4.9.11** that $M_{\bullet 1} \ge 0$ after a finite number of steps. Therefore, after a finite number of steps, we either end on a ray or we pivot outside the alleged cycle, thereby contradicting the assumption that cycling occurs. □

4.9.13 Corollary. In the sufficient matrix case, the PPM with least-index rule will process the LCP (q, M) in a finite number of steps.

Proof. Each major cycle of the algorithm reduces the number of negative components in (w, z) by at least one. The assertion now follows from the theorem. □

4.9.14 Remark. In implementing the least-index rule it is important to obey statement (A) which says that if the distinguished variable is among the tied blocking variables, then it is to be chosen as the blocking variable. Failure to do so can lead to the false impression that the problem is infeasible.

4.10 Computational Considerations

In this section we touch on two issues connected with computational aspects of the linear complementarity problem. The first of these relates to the implementation of pivoting methods, specifically to a practical alternative to pivoting in schemas. The second is concerned with numerical examples of linear complementarity problems on which certain pivoting methods necessarily require a large number of pivot steps. These are but two computational topics that could have been discussed.

Implementation

Thus far, every pivoting method covered in this chapter has been presented in schematic (tableau) form. Iterations were described in terms of pivotal transformations that affect all the data. In some cases, particularly those connected with certain matrix-theoretic properties, this explicit style of treating the algorithms helps to clarify why they work. This alone does not mean that these methods need be—or even should be—implemented that way.

Actually, at any given iteration, relatively little information is required to execute these algorithms. For most of them, deciding on termination or the next pivot element involves only the updated version of the constant column and that of the driving variable. If (q, M) is of order n, this decision involves at most $2n$ numbers, whereas the whole tableau may necessitate $n \times (n+1)$ or $n \times (n+2)$ pieces of data. The repeated updating of all these numbers is a computational burden; it may also lead to (and suffer from) roundoff error. In exact arithmetic, the data needed to proceed from one iteration to the next are uniquely determined by the original data and the current basis. We wish to describe, briefly, how to take advantage of this fact using techniques of numerical linear algebra (as commonly found in linear programming, for example). In doing so, we shall make reference to equations such as

$$Iw - Mz = q \tag{1}$$

and

$$Iw - dz_0 - Mz = q. \tag{2}$$

There are two main ways of carrying out this alternative to tableau-style pivoting. One of them uses the *full basis* which, in general, may be composed of some columns from the left-hand side of (1) or (2), depending on the algorithm. The other way, called the *compact* (or *reduced*) *basis* approach, works with certain submatrices of $-M$ or of $-[d, M]$. We shall discuss the compact basis approach as it applies to four algorithms and relegate discussion of the full basis approach to the Notes and References.

Let us consider Murty's least-index method **4.2.6** once again. Recall that it applies exclusively to $\boldsymbol{P}$-matrices. Accordingly, let the original problem be $(q^0, M^0) = (q, M)$ where $M \in \boldsymbol{P}$. At a typical iteration, say the ν-th, let α denote the index set of the currently basic z-variables. The corresponding index set for the basic w-variables is then given by $\bar{\alpha}$ the complement of α in $\{1, \ldots, n\}$. The next step of the algorithm requires updating the constant column, q^ν and checking it for nonnegativity. This task can be broken into two parts, each corresponding to one of the subvectors q^ν_α and $q^\nu_{\bar{\alpha}}$. The first subvector is given by the equation

$$q^\nu_\alpha = -(M^0_{\alpha\alpha})^{-1} q^0_\alpha,$$

but using the inverse matrix $(M^0_{\alpha\alpha})^{-1}$ is not really essential. Instead, the vector q^ν_α can be thought of as the unique solution of the equation

$$M^0_{\alpha\alpha} x = -q^0_\alpha.$$

Solving this equation is facilitated by having a suitable factorization of the nonsingular matrix $M^0_{\alpha\alpha}$. (This might be the LU factorization or the QR factorization, for instance.) Now according to the principal pivoting formulae, the other subvector, $q^\nu_{\bar{\alpha}}$, satisfies the equation

$$\begin{aligned} q^\nu_{\bar{\alpha}} &= q^0_\alpha - M^0_{\bar{\alpha}\alpha}(M^0_{\alpha\alpha})^{-1} q^0_\alpha \\ &= q^0_{\bar{\alpha}} + M^0_{\bar{\alpha}\alpha} q^\nu_\alpha. \end{aligned}$$

It should be noticed that the right-hand side is ultimately given in terms of the original data $q^0_{\bar{\alpha}}$, $M^0_{\bar{\alpha}\alpha}$ and the just-computed vector, q^ν_α.

Once these two subvectors are known, they can be tested for nonnegativity. If both are nonnegative, the algorithm terminates. If not, it finds the smallest index i such that $q^\nu_i < 0$. Let this be r. If $r \in \alpha$, then z_r is made nonbasic and r is transferred from α to $\bar{\alpha}$. If $r \in \bar{\alpha}$, then z_r is made basic and r is transferred from $\bar{\alpha}$ to α. In either case, the size of α (and hence the order of $M^0_{\alpha\alpha}$) changes by 1 from iteration ν to iteration $\nu+1$. At this stage, a factorization would be sought for the new matrix $M^0_{\alpha\alpha}$. This in turn, would call for the application of techniques for updating matrix factorizations. Some references on this subject are cited in **4.12.32**.

Like the above method, the n-step scheme presented in Section 4.8 uses only simple principal pivots, but is based on a parametric formulation that takes advantage of a special relationship between the matrix M and the positive direction vector d. The choices made in Algorithm **4.8.2** require only knowledge of the updated vectors q^ν and d^ν. Although the algorithm as expressed in **4.8.2** refers to $M_{\alpha\alpha}^{-1}$, the critical decisions depend on being able to solve equations with $M_{\alpha\alpha}$. Thus (4.8.2) could have given z_α as the solution to

$$M_{\alpha\alpha}^0 x = -q_\alpha^0.$$

Equation (4.8.4) could have been written in the form

$$M_{\alpha\alpha}^0 (q_\alpha^\nu, d_\alpha^\nu) = -(q_\alpha^0, d_\alpha^0). \tag{3}$$

Using the solution of this equation, it would then be possible to write (4.8.3) as

$$(q_{\bar\alpha}^\nu, d_{\bar\alpha}^\nu) = (q_{\bar\alpha}^0, d_{\bar\alpha}^0) + M_{\bar\alpha\alpha}^0 (q_\alpha^\nu, d_\alpha^\nu). \tag{4}$$

These observations mean that matrix factorizations can play an important role here too. There is, of course, a concomitant need for factorization updating techniques.

This approach can be extended to cover Algorithm **4.5.2**, the parametric principal pivoting method. In this procedure it is assumed that M is a sufficient matrix. For such LCPs, the algorithm cannot always rely on simple principal pivots, but it *can* be carried out by using block principal pivots of order at most 2. Herein lies an important difference. When handling an LCP of the $\boldsymbol{P}$-matrix type, the algorithm presents no need to compute any updated version of the matrix M, whereas this is not the case when the matrix is only known to be sufficient. In the latter circumstances, at the ν-th iteration of this algorithm, there is a distinguished basic variable w_r^ν whose value is zero; the algorithm needs to determine whether m_{rr}^ν is positive or zero. If the former holds, the algorithm proceeds with a simple principal pivot, exchanging the roles of w_r^ν and z_r^ν. In the latter case, a block principal pivot of order 2 is done. According to the specification of Algorithm **4.5.2**, z_r^ν is used as a driving variable; when the minimum ratio test indicates that it is blocked, say by w_s^ν, the aforementioned principal pivot of order 2 is accomplished via $\langle w_s^\nu, z_r^\nu \rangle$ and $\langle w_r^\nu, z_s^\nu \rangle$.

At a particular iteration $\nu \geq 1$ of this method, there will be a set of z-variables that are basic; again, let α denote the index set of all such variables. When $\nu = 1$, the matrix $M^{\nu}_{\alpha\alpha}$ is clearly nonsingular; indeed, it is just $(M^{0}_{\alpha\alpha})^{-1}$. In general, for $\nu > 1$, it is not difficult to show (inductively) that the same holds for $M^{\nu}_{\alpha\alpha}$. We ask the reader to do this in Exercise **4.11.23**.

The nonsingularity of $M^{\nu}_{\alpha\alpha}$ enables one to construct the information required for executing the algorithm under discussion here. At a typical iteration, say the ν-th, one first needs to update the constant column and the column of the parameter so as to find the new critical value of the parameter and the corresponding index, r. This is done as in (3) and (4). There are two sets of formulas for obtaining $M^{\nu}_{\bullet r}$. The one to be used depends on whether $r \in \alpha$ or $r \in \bar{\alpha}$. Suppose $r \in \alpha$. For the present purpose, let e_r denote the r-th column of an identity matrix of order $|\alpha|$. From the general block pivoting formulas, it follows that

$$M^{0}_{\alpha\alpha} M^{\nu}_{\alpha r} = e_r. \tag{5}$$

The solution of this equation (i.e., $M^{\nu}_{\alpha r}$), can then be used in

$$M^{\nu}_{\bar{\alpha} r} = M^{0}_{\bar{\alpha}\alpha} M^{\nu}_{\alpha r}. \tag{6}$$

The corresponding equations for the case where $r \in \bar{\alpha}$ are

$$M^{0}_{\alpha\alpha} M^{\nu}_{\alpha r} = -M^{0}_{\alpha r} \tag{7}$$

and

$$M^{\nu}_{\bar{\alpha} r} = M^{0}_{\bar{\alpha} r} + M^{0}_{\bar{\alpha}\alpha} M^{\nu}_{\alpha r}. \tag{8}$$

Thus, we see that to execute Algorithm **4.5.2**, it is enough to have a factorization of $M^{0}_{\alpha\alpha}$ for each iteration ν.

The reduced basis approach to Lemke's Scheme I (Algorithm **4.4.5**) runs along analogous lines but is somewhat more difficult to state because of the auxiliary column, d, and the almost complementary nature of (full) bases. Given the LCP (q, M) and the covering vector d, let

$$\bar{M} = [d, M].$$

Because d is the column originally associated with z_0, we may think of it as the 0-th column of $\bar{M}$. It could even be denoted $\bar{M}_{\bullet 0}$. As we have done above, let α denote the index set of the basic z-variables at iteration ν. Initially, α is empty. After the first iteration and prior to termination, the artificial variable z_0 will be basic, thus $0 \in \alpha$. Let γ denote the set of indices of the the *other* basic z-variables. Let β be the index set of the basic w-variables at iteration ν. There will be a unique index r such that $r \notin \alpha \cup \beta$ and both w_r and z_r are nonbasic. Let $\delta = \gamma \cup \{r\}$. Then the submatrix $\bar{M}_{\alpha\delta}$ corresponding to the basic z-variables and the nonbasic w-variables must be nonsingular. This matrix plays the role of the compact basis (just as $M_{\alpha\alpha}$ did in the previously discussed algorithms). In particular, it can be used to construct all the data needed by the algorithm. For instance, at iteration ν, the driving variable will be either w_r or z_r. In the former case, one uses systems analogous to (5) and (6), whereas in the latter, one uses analogues of (7) and (8). The updates of q can be gotten in a manner analogous to that in (3) and (4).

Worst case behavior

A common feature of the pivoting methods for the LCP presented in this chapter is their finiteness—at least when they are applied to nondegenerate problems or when fortified with degeneracy resolution techniques. When used appropriately, these algorithms will terminate after finitely many pivot steps, either with a solution or with a secondary ray (which in some cases can be interpreted to mean that the instance of the problem has no feasible solution).

Numerous computational studies of the pivoting algorithms have shown that processing LCPs of order n with these pivoting methods typically requires $O(n)$ pivot steps. Indeed, in Sections 4.7 and 4.8 we saw that there are algorithms for some special problems of order n that never require more than n pivots. Heartening as this information may be, it is also true that there exist pathological problems that will cause these methods to perform an exponential number of pivot steps. This serves as a reminder that a finite number is not necessarily a small number.

In this subsection we summarize some results on the computational complexity of pivoting methods for the LCP. We concentrate on just one side of this theory, namely the worst case analysis of pivoting methods for the LCP. In each case, the measure of interest is the number of pivot steps.

Murty's examples

We begin our discussion of the worst case behavior of pivoting methods for the LCP with two algorithms. The first of these is Murty's least-index method (Algorithm **4.2.6**); the second is the parametric version of Lemke's method (Algorithm **4.5.4**).

Let us consider the $n \times n$ upper triangular matrix $U(n)$ given by

$$u_{ij}(n) = \begin{cases} 0 & \text{if } i > j, \\ 1 & \text{if } i = j, \\ 2 & \text{if } i < j. \end{cases} \tag{9}$$

Being a triangular matrix with a positive diagonal, $U(n) \in \boldsymbol{P}$. Hence (q, M) has a unique solution for every $q \in R^n$. For the present purpose, we choose $q = -e$. To emphasize the dimension of this vector, we define

$$e(n) = e \in R^n. \tag{10}$$

Thus, we shall be concerned with the LCPs of the form $(-e(n), U(n))$ where $n \geq 2$.

Three observations about the LCP $(-e(n), U(n))$ can be made immediately. The first is that its unique solution is the vector

$$(z_1, \ldots, z_{n-1}, z_n) = (0, \ldots, 0, 1).$$

This is easy to verify directly. The second observation is that

$$-e(n) = \begin{bmatrix} -e(n-1) \\ -1 \end{bmatrix} \quad \text{and} \quad U(n) = \begin{bmatrix} U(n-1) & 2e(n-1) \\ 0 & 1 \end{bmatrix}.$$

From this representation, it is evident that the leading principal subproblem of order $n-1$ is just $(-e(n-1), U(n-1))$. This means that the leading subproblems are in a sense "nested." We state the third observation as a lemma.

4.10.1 Lemma. Algorithm **4.2.6** requires three pivots to solve the LCP $(-e(2), U(2))$.

Proof. This is readily checked. □

4.10.2 Theorem. For each integer $n \geq 2$, Algorithm **4.2.6** will require $2^n - 1$ pivots to solve the LCP $(-e(n), U(n))$.

Proof. The assertion is true for $n = 2$. Assume inductively that the theorem holds for the problem of order $n - 1$. The specification of the algorithm implies that z_n remains nonbasic (at value 0) until the leading principal subproblem of order $n-1$ is solved. By the inductive hypothesis, this will require $2^{n-1} - 1$ pivot steps of Algorithm **4.2.6**. Once this is done, the schema is

	1	z_1	z_2	$\cdots$	z_{n-2}	w_{n-1}	z_n
w_1	1	1	2	$\cdots$	2	2	-2
w_2	1	0	1	$\cdots$	2	2	-2
$\vdots$	$\vdots$	$\vdots$		$\ddots$		$\vdots$	$\vdots$
w_{n-2}	1	0	0		1	2	-2
z_{n-1}	1	0	0	$\cdots$	0	1	-2
w_n	-1	0	0	$\cdots$	0	0	1

The next pivot is then $\langle w_n, z_n \rangle$, and it yields the schema

	1	z_1	z_2	$\cdots$	z_{n-2}	w_{n-1}	w_n
w_1	-1	1	2	$\cdots$	2	2	-2
w_2	-1	0	1	$\cdots$	2	2	-2
$\vdots$	$\vdots$	$\vdots$		$\ddots$		$\vdots$	$\vdots$
w_{n-2}	-1	0	0		1	2	-2
z_{n-1}	-1	0	0	$\cdots$	0	1	-2
z_n	1	0	0	$\cdots$	0	0	1

Once again, the algorithm will solve the leading principal subproblem of order $n-1$ which (apart from an insignificant difference in row and column labels) has the same form as the original subproblem $(-e(n-1), U(n-1))$. The latter required $2^{n-1} - 1$ pivot steps. Thus, Algorithm **4.2.6** will require

$$(2^{n-1} - 1) + 1 + (2^{n-1} - 1) = 2^n - 1$$

pivot steps to solve $(-e(n), U(n))$. □

The preceding example $(-e(n), U(n))$ shows that Murty's least index algorithm can take an exponential number of pivot steps. A variant of this problem can be used to establish the computational complexity of another pivoting method. For this purpose, we use the parametric version of Lemke's method on the LCP $(q(n), M(n))$ where, for each integer $n \geq 2$, the matrix $M(n)$ is just the transpose of the $U(n)$ as defined in (9) and the vector $q(n)$ is defined by

$$q_i(n) \;=\; -\sum_{j=n+1-i}^{n} 2^j \quad i = 1, \ldots, n. \tag{11}$$

Using the observation that

$$\sum_{j=1}^{n} 2^j = 2^{n+1} - 2,$$

it is not hard to check that $(z_1, z_2, \ldots z_n) = (2^n, 0, \ldots, 0)$ is the unique solution of $(q(n), M(n))$. See Exercise **4.11.24**.

There is a natural greedy "decomposition scheme" for solving *any* LCP (such as this one) in which the matrix is a triangular $\boldsymbol{P}$-matrix, but that is not what the example is intended to illustrate. Indeed, using the pair $q(n)$ and $M(n)$, it is possible to construct a line that traverses all 2^n complementary cones relative to $M(n)$. (Recall that the complementary cones corresponding to an $n \times n$ $\boldsymbol{P}$-matrix always partition R^n.) When the generic point on the line

$$\{q(n) + \lambda e(n) : \lambda \in R^n\} \tag{12}$$

meets the boundary of a complementary cone, the algorithm calls for a pivot step. The data have the special property that the complementary cone to which the point $q(n)$ belongs is reached only after $2^n - 1$ principal pivots are performed. This becomes the foundation for demonstrating the fact that the parametric version of Lemke's method can require an exponential number of pivot steps before terminating.

The following example and accompanying figure should give some intuitive feeling for the kind of phenomenon under discussion here.

4.10.3 Example. For $n = 2$, the data for the LCP $(q(2), M(2))$ are given by

$$q(2) = \begin{bmatrix} -4 \\ -6 \end{bmatrix} \quad \text{and} \quad M(2) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

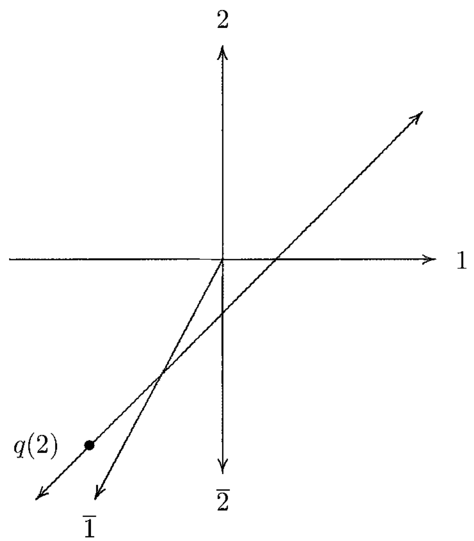

Figure 4.7

Consider the parametric LCP $(q(2) + \lambda e(2), M(2); \lambda \in R)$. The complementary cones and the parametric line are shown in Figure 4.7. Notice how the line given by $\{q(2) + \lambda e(2) : \lambda \in R\}$ traverses all four complementary cones relative to M. Notice also that if the slope were greater, say 3, then the line would *not* traverse all four complementary cones. Insofar as the LCP is concerned, the parametric version of Lemke's method would solve the problem with just one pivot.

Whether by means of this figure or by actually doing the calculations, it is elementary to show that the solution of $(q(2), M(2))$ by the parametric version of Lemke's method requires precisely $3 = 2^2 - 1$ pivot steps. Furthermore, the critical values of λ occur at 6, 4, and 2.

4.10.4 Theorem. For each integer $n \geq 2$, Algorithm **4.5.4** with direction vector $e(n)$ will require $2^n - 1$ pivots to solve the LCP $(q(n), M(n))$. The critical values of λ (at which pivots occur) are positive even integers in the interval $[0, 2^{n+1} - 2]$ of which there are $2^n - 1$.

Proof. The proof is by induction on n. We have already seen the case for $n = 2$. Assume that the theorem is true for the problem of order $n - 1$. Given $(q(n), M(n))$, Algorithm **4.5.4** with direction vector $e(n)$ will begin with the parameter λ equal to a very large number—conceptually, plus infinity. In particular,

$$[\,\lambda > 2^{n+1} - 2\,] \quad \Rightarrow \quad [\,q(n) + \lambda e(n) > 0\,].$$

Furthermore, the line $\{q(n) + \lambda e(n) : \lambda \in R_+\}$ meets the boundary of R^n_+ when $\lambda = 2^{n+1} - 2$. Because of the lower triangularity of $M(n)$ and the other special properties of the data, it is easy to see that for all $\lambda > 2^n$, the pivoting occurs in the last $n-1$ rows of the corresponding schema. For this range of values of λ it is as if the method were being applied to the smaller problem $(q(n-1), M(n-1))$. By the induction hypothesis, this requires $2^{n-1} - 1$ pivot steps corresponding to the critical values λ, namely $2^{n+1} - 2,\ 2^{n+1} - 4,\ \ldots,\ 2^n + 2$. When $\lambda = 2^n$, there occurs the pivot $\langle w_1, z_1 \rangle$. This has the effect of making the transforms of $q_1(n)$ and $e_1(n)$ become positive and negative, respectively. This, in turn, means that as λ is decreased, the value of the basic variable z_1 will increase since less is being subtracted from it than before. The triangularity of $M(n)$ prevents the subsequent principal pivots from destroying this property. Once λ is reduced below 2^n, the algorithm behaves as if it were solving the principal subproblem associated with the last $n - 1$ rows, except that the pivots occur in reverse order from the previous time through. Altogether, then, the process requires $(2^{n-1} - 1) + 1 + (2^{n-1} - 1) = 2^n - 1$ pivot steps, each one occurring at a positive even integer value between 0 and $2^{n+1} - 2$. □

It is useful to point out that $M(n)$ is a hidden $\boldsymbol{K}$-matrix; indeed, $M(n)$ belongs to the class of $\boldsymbol{H}$-matrices with positive diagonal entries. Hence, according to Exercise **4.11.18**, $M(n)$ possesses an n-step vector d that is given by $d_i = 3^{i-1}$, $i = 1, \ldots, n$. It is trivial to see that if Algorithm **4.5.4** is applied to $(q(n), M(n))$ with this d as the direction vector, then the algorithm terminates in just one pivot. Hence the exponential number of pivots asserted in **4.10.4** is a direct consequence of the covering vector $e(n)$ used in the algorithm. The point of this discussion is to stress the (already noted) fact that the covering vector in Lemke's method often has a dramatic influence on its computational performance.

The Birge-Gana example

We shall now show that for each integer $n \geq 2$ there exists an LCP $(\tilde{q}(n), \tilde{M}(n))$ of order n having the matrix $\tilde{M}(n) \in \boldsymbol{P}$ on which Van der Heyden's variable dimension scheme (Algorithm **4.6.3**) requires $2^n - 1$ pivot steps. Exponential worst case behavior of the asymmetric version of the principal pivoting method (Algorithm **4.3.5**) can be seen as a consequence of this result.

The data for this example are defined as follows:

$$\tilde{M}(n) = U(n),$$

where $U(n)$ is given by (9). The vector $\tilde{q}(n)$ is given by

$$\tilde{q}_i(n) = -\sum_{j=i}^{n} 2^j. \tag{13}$$

A useful fact—easily proved by induction—is that

$$\tilde{q}_i(n) = -(2^{n+1} - 2^i). \tag{14}$$

The vector $\tilde{q}(n)$ resembles $q(n)$ specified in (11). Indeed, if i and k are positive integers such that $i + k = n + 1$, then $\tilde{q}_i(n) = q_k(n)$. More importantly, it is useful to define the n-vector $\hat{q}(n)$ by

$$\hat{q}_i(n) = 2^i \quad i = 1, \ldots, n.$$

Then with $e(n) = e \in R^n$, (14) implies

$$\hat{q}(n) = \tilde{q}(n) + 2^{n+1} e(n). \tag{15}$$

4.10.5 Remarks. Part of the development rests on the following observations.

(1) Algorithm **4.6.3** involves (leading) subproblems of size k. Accordingly, when $M \in R^{n\times n}$, the last $n-k$ rows and columns can be ignored while solving the k-subproblem.

(2) When $M \in \boldsymbol{P}$ (as in the present case), each k-subproblem has a unique solution. For this reason, Steps 2(c) and 3 of Algorithm **4.6.3** will not be executed.

(3) Suppose $k \geq 2$ and $M \in \boldsymbol{P}$. In solving the k-subproblem, the algorithm's first pivot will make z_k basic. Thereafter, both w_k and z_k will be the basic pair until the final pivot whereupon w_k will become nonbasic. (This is a consequence of the preceding remark.) There will also be a nonbasic pair, w_i, z_i for some $i < k$.

(4) When solving the k-subproblem, the algorithm maintains the complementarity and nonnegativity of $(w_1, \ldots, w_{k-1})$ and $(z_1, \ldots, z_{k-1})$.

(5) As applied to the k-subproblem of $(\tilde{q}(n), U(n))$, Algorithms **4.6.3** and **4.3.5** perform the same steps. It can be shown (along the lines of **4.3.6**) that in the nondegenerate case, both members of the basic pair are strictly increasing functions of the driving variable (which is a member of the nonbasic pair). Moreover, the specification of $(\tilde{q}(n), U(n))$ implies that in each k-subproblem, $w_k = \tilde{q}_k(n) + z_k$ so that the members of the basic pair increase at the same rate.

A fact not known to the algorithm—but nevertheless true—is that for every integer $n \geq 1$, the unique solution of the n-problem $(\tilde{q}(n), U(n))$ can be expressed in closed form. Indeed, we have the following more general result.

4.10.6 Lemma. If $\tau > -2$, the LCP $(\tilde{q}(n) - \tau e(n), U(n))$ has the unique solution

$$(z_1, \ldots, z_{n-1}, z_n) = (0, \ldots, 0, 2^n + \tau).$$

Proof. Since $U(n) \in \boldsymbol{P}$, the problem has a unique solution, and since

$$w_n = \tilde{q}_n(n) - \tau + z_n = -2^n - \tau + 2^n + \tau = 0,$$

it suffices to verify that the other inequalities of the problem (namely $w_i \geq 0$ for $i = 1, \ldots, n-1$) are satisfied. Using (14) and the fact that $u_{in} = 2$ for $i < n$, we complete the proof by noting

$$w_i = \tilde{q}_i(n) - \tau + u_{in}z_n = -(2^{n+1} - 2^i) - \tau + 2^{n+1} + 2\tau = 2^i + \tau > 0. \quad \square$$

For the sake of motivation, and because our proof of the main result is inductive, we treat the case of $n = 2$ separately; we remark, in passing, that $3 = 2^2 - 1$.

4.10.7 Lemma. For every $\tau > -2$, Algorithm **4.6.3** requires 3 pivot steps to solve the LCP $(\tilde{q}(2) - \tau e(2), U(2))$.

Proof. This can be verified directly. The pivots are $\langle w_1, z_1 \rangle$, $\langle z_1, z_2 \rangle$, and $\langle w_2, w_1 \rangle$. □

4.10.8 Theorem. For every $\tau > -2$, Algorithm **4.6.3** requires $2^n - 1$ pivot steps to solve the LCP $(\tilde{q}(n) - \tau e(n), U(n))$.

Proof. The statement is true for $n = 2$. Assume, inductively, that it is true for the corresponding problem of order $n-1$. The algorithm will solve the leading $(n-1)$-problem before it executes a pivot in the last row of the schema. Moreover, due to the structure of $U(n)$, the last row is unaffected by the pivoting above it. Now for $i = 1, \ldots, n-1$, we have

$$\tilde{q}_i(n) - \tau = \tilde{q}_i(n-1) - (2^n + \tau).$$

Since $\tau' = 2^n + \tau > -2$, the inductive hypothesis implies that the algorithm will use precisely $2^{n-1} - 1$ pivot steps in solving the leading $(n-1)$-subproblem, $(\tilde{q}(n-1) - \tau' e(n-1), U(n-1))$. After this is done and the necessary principal rearrangement is performed, the schema is easily seen to be

	1	$z(n-2)$	w_{n-1}	z_n
$w(n-2)$	$\hat{q}(n-2) + \tau' e(n-2)$	$U(n-2)$	$2e(n-2)$	$-2e(n-2)$
z_{n-1}	$2^{n-1} + \tau'$	0	1	-2
w_n	$-\tau'$	0	0	1

The next pivot executed by the algorithm is $\langle w_1, z_n \rangle$. At this stage, z_n becomes basic at value $2^{n-1} + 1 + \tau/2$. As noted in **4.10.5**, both w_n and z_n will increase (strictly) when they are the basic pair. In this situation, z_n must increase by another $2^{n-1} - 1 + \tau/2$ units.

It now remains to show that the algorithm requires $2^{n-1} - 1$ more pivot steps to compute the solution of $(\tilde{q}(n) - \tau e(n), U(n))$. To do this, we shall resort to an equivalent parametric interpretation of what the last 2^{n-1} pivot steps of the algorithm are doing. In particular, (the nonbasic variable) z_n will first be viewed as the parameter in a parametric LCP of order $n-1$.

In this approach, the parametric LCP to be solved is of the form

$$(\hat{q}(n-1)+\tau' e(n-1), -2e(n-1), U(n-1)).$$

The parameter λ associated with the column $-2e(n-1)$ is actually the variable z_n, and it runs over the interval $\Lambda = [0, 2^n+\tau]$. By comparing this process with the parametric version of Lemke's method on a suitable related problem, it can be shown (see Exercise **4.11.28**) that this part of the process takes $2^{n-1}-1$ pivot steps.

After solving the aforementioned parametric problem, Van der Heyden's algorithm makes one more principal pivot, namely, $\langle w_n, z_n \rangle$. In all, then, it takes

$$(2^{n-1}-1)+(2^{n-1}-1)+1 = 2^n - 1$$

pivots to solve $(\tilde{q}(n), U(n))$ by Algorithm **4.6.3**. □

To see that Algorithm **4.3.5** can require an exponential number of pivot steps, it suffices to consider the LCP represented by the schema displayed in the proof of **4.10.8**. In that problem, only the last entry of the constant column is negative, and, of course, the matrix belongs to $\boldsymbol{P}$. Under such circumstances, **4.6.3** and **4.3.5** are the same. Accordingly, Algorithm **4.3.5** requires 2^{n-1} pivot steps to solve this LCP. This is the idea behind the proof of the following theorem.

4.10.9 Theorem. There exists an LCP of order n for which Algorithm **4.3.5** requires 2^n-1 pivot steps. □

4.11 Exercises

4.11.1 Adapt the proof of **4.1.1** to show that (4.1.1) holds when

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix},$$

$$B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is nonsingular, C denotes the matrix obtained from A by pivoting on B, and

$$D = \begin{bmatrix} C_{22} & C_{23} \\ C_{32} & C_{33} \end{bmatrix}.$$

4.11.2 This exercise concerns the invariance of matrix theoretic properties under principal pivoting.

(a) Show that the property of bisymmetry is invariant under principal pivoting.

(b) Prove or disprove that the following matrix classes are invariant under principal pivoting: $\boldsymbol{S}$, $\boldsymbol{E}_0$, $\boldsymbol{Q}_0$, $\boldsymbol{Q}$, hidden $\boldsymbol{Z}$, copositive, and adequate.

(c) Prove or disprove that the following matrix classes are invariant under principal pivoting: $\boldsymbol{S}_0$, and $\boldsymbol{H}$ with positive diagonals. [This part may be harder than (b).]

4.11.3 Let M be a given $n \times n$ matrix. Prove that M and all of its principal pivotal transforms have the same number of nonzero principal minors. Use this to show that the class $\boldsymbol{P}_1$ is invariant under principal pivoting.

4.11.4 This exercise identifies a large subclass of $\boldsymbol{P}_1$-matrices.

(a) Let $M \in \boldsymbol{P}_0 \cap \boldsymbol{Z} \cap R^{n\times n}$ with $n \geq 2$. Suppose M is singular and irreducible. Show that $M \in \boldsymbol{P}_1$ and that there exists a positive vector z such that $Mz = 0$.

(b) Let $M \in R^{n\times n}$ (with $n \geq 2$) be irreducible and singular. Suppose M has positive diagonal entries and the comparison matrix $\bar{M}$ is singular and belongs to $\boldsymbol{P}_0$. Deduce from part (a) and Theorem **3.3.15** that $M \in \boldsymbol{P}_1$.

4.11.5 Let $M \in R^{2\times 2}$ be given. Establish the following three characterizations.

(a) The matrix M is column sufficient if and only if (i) $M \in \boldsymbol{P_0}$, and (ii) no principal pivotal transform or principal rearrangement of M has the form

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \qquad b \neq 0.$$

(b) The matrix M is column sufficient if and only if every principal pivotal transform $\bar{M}$ of M has the properties (i) $\bar{m}_{ii} \geq 0$ for $i = 1, 2$, and (ii) for $i = 1, 2$, if $\bar{m}_{ii} = 0$ and $\bar{m}_{ij} = 0$ for $j \neq i$, then $\bar{m}_{ji} = 0$.

(c) The matrix M is (row and column) sufficient if and only if every principal pivotal transform $\bar{M}$ of M has the properties (i) $\bar{m}_{ii} \geq 0$ for $i = 1, 2$, and (ii) for $i = 1, 2$, if $\bar{m}_{ii} = 0$, then either $\bar{m}_{ij} = \bar{m}_{ji} = 0$ or $\bar{m}_{ij}\bar{m}_{ji} < 0$ for $j \neq i$.

4.11.6 A matrix $M \in R^{n \times n}$ is *sufficient of order k* if every $\ell \times \ell$ principal submatrix of M with $\ell \leq k$ is sufficient. For $n \geq 3$, establish

(a) The matrix M is sufficient if and only if every principal pivotal transform of M is sufficient of order $n - 1$.

(b) The matrix M is sufficient if and only if every principal pivotal transform of M is sufficient of order 2.

4.11.7 Let $M \in R^{n \times n}$ where $n \geq 3$. Prove or disprove that $M \in \boldsymbol{P_0}$ if and only if every principal pivotal transform of M is $\boldsymbol{P_0}$ of order 2.

4.11.8 This exercise shows that scaling can affect the behavior of an algorithm with respect to cycling.

(a) Verify that **4.2.2** applied to the LCP (q, M) with data

$$q = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \qquad M = \begin{bmatrix} .1 & 0 & .2 \\ .2 & .1 & 0 \\ 0 & .2 & .1 \end{bmatrix}$$

cycles in the manner described in Section 4.2.

(b) Let $\tilde{M} = 10M$ where M is the matrix given in (a). Show that **4.2.2** applied to $(q, \tilde{M})$ does not cycle.

4.11.9 Prove Proposition **4.2.8**.

4.11.10 This exercise concerns a large subclass of the matrix class $\boldsymbol{Z}$ and its connection to the pivoting methods. We say that a matrix $M \in R^{n\times n}$ is a *Stieltjes matrix* if M is symmetric and belongs to $\boldsymbol{K}$.

(a) An application of the LCP involving a Stieltjes matrix is the problem of finding the convex hull of a set of points in the plane. Show that the matrix M whose entries are given in (1.2.21) is a Stieltjes matrix.

(b) Another application of the LCP that yields a Stieltjes matrix occurs in the *isotone regression problem with a total order.* Mathematically, this problem can be formulated as a strictly convex quadratic program:

$$\begin{array}{ll} \text{minimize} & \sum_{i=0}^{n} d_i(x_i - a_i)^2 \\ \text{subject to} & x_0 \leq x_1 \leq \cdots \leq x_n \end{array}$$

where d_i's are positive weights and a_i's are arbitrary scalars. According to the discussion in Section 4.2, the above quadratic program can be converted into the LCP (q, M) where M is of order n. Show that this matrix M has precisely the same form as that in (1.2.21).

(c) The matrix M in the above two parts is a tridiagonal Stieltjes matrix; we denote it by $M(d)$ where $d \in R^{n+1}$ is a positive vector. Suppose a principal pivot is performed on the diagonal entry m_{ii} $(1 \leq i \leq n)$. Let $M' \in R^{(n-1)\times(n-1)}$ denote the Schur complement (M/m_{ii}), and let $q' \in R^{n-1}$ denote the transformed constant vector corresponding to M'. Show that M' is of the form $M'(d')$ where $d' \in R^n$ is given by

$$d'_j = \begin{cases} d_j & \text{if } 0 \leq j \leq i-2 \\ (d_{i-1}^{-1} + d_i^{-1})^{-1} & \text{if } j = i-1 \\ d_{j+1} & \text{if } i \leq j \leq n-1\ . \end{cases}$$

Show that the vector q' can be obtained very simply from the given q by updating no more than two entries.

(d) By using the results of part (c), describe a streamlined version of Algorithm **4.7.3** for solving an LCP (q, M) with a tridiagonal Stieltjes matrix $M \in R^{n\times n}$. The resulting algorithm should require no more than $O(n)$ comparisons and arithmetic operations.

4.11.11 This exercise concerns a specially structured LCP that arises from the *spatial price equilibrium problem.* The reader can find the description of this application in Section 5.1 under the heading "Network equilibrium problems". The LCP under consideration is of form $(A^{T}b, A^{T}DA)$ where A is the *node-arc incidence matrix* of a network with node set $\mathcal{N}$ and arc set $\mathcal{A}$, and D is a diagonal matrix with positive diagonal entries. The matrix $M = A^{T}DA$ is called a *weighted arc-arc adjacency matrix*; its rows and columns are labelled in terms of the arcs of the network, and the diagonal entries of D correspond to given weights on the nodes in $\mathcal{N}$.

(a) Give an explicit expression for the entries of the matrix M. Let $\alpha \subseteq \mathcal{A}$ be a subset of arcs. Show that the principal submatrix $M_{\alpha\alpha}$ is nonsingular if and only if the arcs in α contain no (undirected) cycle.

(b) Suppose that a principal pivot is performed on the diagonal entry m_{aa} where $a \in \mathcal{A}$. Let M' denote the Schur complement (M/m_{aa}). Show that M' is a weighted arc-arc adjacency matrix defined on the network with the same node set $\mathcal{N}$ but with the arc set $\mathcal{A} \setminus \{a\}$, provided that a is not contained in any cycle of the network. Identify the modified weights on the nodes.

(c) Generalize part (b) to the case of a principal block pivot on the submatrix $M_{\alpha\alpha}$ where $\alpha \subseteq \mathcal{A}$ and none of the arcs in α are contained in any cycle.

(d) Suppose that the network is a tree. Show that

$$\det M = \Big(\prod_{i \in \mathcal{N}} d_i \Big) \times \Big(\sum_{i \in \mathcal{N}} d_i^{-1} \Big).$$

Fix an arc a in the tree and let α be the set of arcs with a deleted. Use the above formula for $\det M$ and the Schur determinantal formula (2.3.14) to derive a simple expression for $(M/M_{\alpha\alpha})$.

4.11.12 Let a and b be real numbers such that $0 < a < b$ and define

$$M = \begin{bmatrix} -a & b \\ b & -a \end{bmatrix}.$$

(a) Show that $M \in \boldsymbol{Q}$.

(b) Show that if

$$q = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix},$$

then Lemke's method applied to (q, d, M) will terminate with a ray for any $d_1,\ d_2 > 0$.

4.11.13 Consider the LCP (q, M) in which

$$q = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 21 & 0 & 0 \\ 28 & 14 & 0 \\ 24 & 24 & 12 \end{bmatrix}.$$

Apply Lemke's method (Scheme I) to (q, d^1, M) and (q, d^2, M) with the covering vectors

$$d^1 = (21, 14, 12) \quad \text{and} \quad d^2 = (12, 14, 21),$$

respectively.

4.11.14 Set up the linear complementarity problem (q, M) corresponding to the KKT conditions of the quadratic program

$$\text{minimize } -4x_1 - x_2 - 4x_1x_2 \quad \text{subject to } x_1 + x_2 \leq 1,\ x_1 \geq 0,\ x_2 \geq 0.$$

Execute Lemke's algorithm (Scheme I) on a problem of the form (q, d, M), first with the covering vector $d^1 = (1, 1, 1)$ and then with the covering vector $d^2 = (1, 1, 0)$.

4.11.15 Consider an LCP (q, M) in which $-M \in \boldsymbol{Z}$ and M has nonpositive diagonal elements. Prove that when Lemke's method (Scheme I) is applied to such a problem, it terminates with a secondary ray immediately after the first pivot. Discuss the implications of this fact for the applicability of Lemke's method to the bimatrix game problem.

4.11.16 Let $M \in R^{n \times n}$ be row adequate and $q \in R^n$ be a vector in the range of M.

(a) Show that the conclusion of **4.2.1** holds.

(b) How is this assumption on the pair (q, M) related to that in **4.2.1**?

(c) Deduce from part (a) and **4.2.1** that if (q, M) satisfies either one of the two assumptions, and if the LCP (q, M) is nondegenerate, then Algorithm **4.5.2** always computes a solution of (q, M) using only simple diagonal pivots.

4.11.17 Consider the parametric LCP (4.5.1) with $d > 0$.

(a) Show that for $\lambda \neq \lambda'$, $\mathrm{SOL}(q + \lambda d, M) \cap \mathrm{SOL}(q + \lambda' d, M) \subseteq \{0\}$.

(b) Suppose $M \in \boldsymbol{P_0}$. Show that if there exist $x \in \mathrm{SOL}(q + \lambda d, M)$ and $x' \in \mathrm{SOL}(q + \lambda' d, M)$ satisfying $0 \neq x \leq x'$, then $\lambda \geq \lambda'$.

(c) Suppose that M is positive semi-definite. Show that if $\lambda > \lambda'$, then $d^{\mathrm{T}}x \leq d^{\mathrm{T}}x'$ for any $x \in \mathrm{SOL}(q + \lambda d, M)$ and $x' \in \mathrm{SOL}(q + \lambda' d, M)$.

4.11.18 Let $M \in \boldsymbol{H} \cap R^{n \times n}$ have positive diagonal entries, and let $\bar{M}$ denote the comparison matrix of M. Show that for any vector $\bar{d} \in R^n$ satisfying $\bar{M}\bar{d} > 0$, the vector $d = \frac{1}{2}(M + \bar{M})\bar{d}$ is an n-step vector for M.

4.11.19 Let M be as given in part (b) of **4.11.4** and q be arbitrary. Show that by solving two LCPs each of order $n - 1$ and each possessing an $(n-1)$-step vector, it is possible to decide whether the given problem (q, M) is solvable, and to compute a solution if it exists.

4.11.20 Let (q, M) be feasible with $M \in \boldsymbol{Z}$. Show that the solution of (q, M) computed by Algorithm **4.7.3** is the least element of $\mathrm{FEA}(q, M)$. Does the same conclusion hold for Lemke's method?

4.11.21 Consider the following property of a matrix $M \in R^{n \times n}$:

$$x \in \mathrm{SOL}(0, M) \quad \Rightarrow \quad (M + M^{\mathrm{T}})x \geq 0. \tag{1}$$

(a) Show that the class of matrices M satisfying the above implication includes the copositive matrices, the symmetric matrices and the pseudo-regular matrices. Show that the $\boldsymbol{P_0}$-matrix

$$M = \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

violates the implication (1).

(b) Let M be a matrix satisfying (1). Suppose that $d > 0$ is such that $\text{SOL}(d, M) = \{0\}$. Let $q \in (\text{SOL}(0, M))^*$ be given. Show that Algorithm **4.4.5** applied to (q, d, M) will compute a solution of (q, M) if the latter problem is nondegenerate.

See Exercise **7.6.6** for more properties of a matrix M satisfying the assumptions of part (b).

4.11.22 Consider the definitions of type I and type II solutions given in **4.6.1** and **4.6.2**, respectively. Show that a system (4.6.2) can possess infinitely many nonbasic solutions of these two types.

4.11.23 Let α denote the index set of the basic z-variables in Algorithm **4.5.2**, the symmetric parametric principal pivoting method. Show inductively that for all $\nu \geq 1$, the matrix $M^{\nu}_{\alpha\alpha}$ is nonsingular.

4.11.24 For the vector $q(n)$ given by (4.10.11) and the matrix $M(n)$ defined just above it, prove that $(q(n), M(n))$ has the unique solution

$$(z_1, z_2, \dots z_n) = (2^n, 0, \dots, 0).$$

4.11.25 Consider the strictly convex quadratic program:

$$\begin{array}{ll} \text{minimize} & x_1^2 - 2x_1x_2 + 2x_2^2 - x_1 - 3x_2 \\ \text{subject to} & -x_1 + 2x_2 \leq 1 \\ & x_1 + x_2 \leq 2 \\ & x_1,\ x_2 \geq 0. \end{array}$$

(a) Convert this program to an LCP with a symmetric positive semidefinite matrix, and apply Algorithm **4.2.11** to solve the resulting LCP.

(b) Apply Algorithms **4.3.5** and **4.4.5** directly to the LCP that corresponds to the Karush-Kuhn-Tucker conditions of the above program. For the latter algorithm, you may use your favorite covering vector.

4.11.26 Consider the application of Lemke's method, Scheme I (with e as the covering vector) to the LCP (q, M) that corresponds to the Karush-Kuhn-Tucker conditions of a convex quadratic program. What is the significance *vis-à-vis* the quadratic program when termination of the algorithm occurs with a secondary ray?

4.11.27 Let $(\tilde{q}(n), U(n))$ denote the LCP referred to in **4.10.8** and consider the case where $n = 4$. According to this result, Algorithm **4.6.3** takes 15 pivots to solve $(\tilde{q}(4), U(4))$. After 7 pivots, the corresponding schema is

	1	z_1	z_2	w_3	z_4
w_1	18	1	2	2	−2
w_2	20	0	1	2	−2
z_3	24	0	0	1	−2
w_4	−16	0	0	0	1

(a) Starting from the schema above, complete the solution of $(\tilde{q}(4), U(4))$ by Van der Heyden's method. [This will require 8 pivots.]

(b) Using the data given in the schema above, treat z_4 as a parameter and solve the parametric LCP $(\hat{q}(3), -2e(3), U(3))$ by Van der Heyden's method. Record the pivot locations as the parameter z_4 reaches the critical values at which basis changes are required.

(c) Record the pivot locations used when Murty's method **4.2.6** is used to solve $(-e(3), U(3))$. Compare these with the list developed in part (b).

4.11.28 Let $\tilde{q}(n)$, $e(n)$, and $U(n)$ be as defined in (4.10.13), (4.10.10), and (4.10.9), respectively.

(a) Show that for $\tau > -2$ and $d = e(n)$, Algorithm **4.5.4**, the parametric version of Lemke's Scheme I, takes $2^n - 1$ pivot steps to solve the LCP $(\tilde{q}(n) - \tau e(n), U(n))$.

(b) Show that after $2^{n-1} - 1$ pivot steps, the algorithm mentioned in part (a) produces the schema displayed in the proof of **4.10.8**, except that the latter has no z_0 column.

(c) Show that the algorithm used to solve the parametric problem considered in the proof of **4.10.8** generates the same pivots as as that considered in part (a) with its parameter run *backwards* from $2^{n+1} - 2 + \tau$ to $2^n + \tau$.

4.12 Notes and References

4.12.1 The symmetric difference formula (4.1.4) is due to A.W. Tucker (1960). The exposition given here is based on Parsons (1970). Theorem **4.1.3**, the invariance of $\boldsymbol{P}$ under principal pivoting, was also shown by Tucker (1963). The invariance theorem for positive definite and positive semi-definite matrices (see Theorem **4.1.5**) is from an unpublished paper of Tucker and Wolfe; it is cited in Cottle (1964a). The results on the invariance of the (row and column) sufficient matrices under principal pivoting come from Cottle (1990). For other references on principal pivoting, see Väliaho (1969) and Wendler (1971).

4.12.2 The class $\boldsymbol{P_1}$ was introduced by Cottle and Stone (1983). Theorem **4.1.13** was proved there by another method as well as the one given here. Theorem **4.1.10** and Corollary **4.1.11** are believed to be new.

4.12.3 Bard-type methods are named after Yonathan Bard, not to be confused with Jonathan F. Bard, who has also contributed to the LCP literature. (See Bard and Falk (1982).) Much of Y. Bard's work **4.2.2** was done in the 1960's, but was not published until 1972. For the claims made about the performance of the method, see Bard (1972, p. 120) and Bard (1974, p. 148).

4.12.4 The first simple principal pivoting methods are due to Zoutendijk (1960) and Bard (1972). These methods were used for solving special quadratic programs (nearest-point problems). We have identified **4.2.2** as the Zoutendijk/Bard method, but it should be noted that the form of the problem to which the steps are applied is slightly more general than that considered by either of the persons for whom the algorithm is named. See Zoutendijk (1960, p. 83) for remarks about the use of degeneracy-resolving techniques. Also see page 87 of the latter publication for proposals on other computational schemes.

4.12.5 R.L. Graves (1967) pointed out that the algorithm **4.2.2** is also executable for any LCP (q, M) where M has positive principal minors. In the same paper, Graves extended **4.2.2** to cover the case of LCPs induced by convex quadratic programs. His approach involved lexicographic pivot selection rules of the Zoutendijk type. The resulting method is not a simple principal pivoting method, though, as it may require block principal pivots on matrices of order 2.

4.12.6 Cycling in Bard's algorithm has been written about by Chang (1979), Murty (1974, 1988), and Kostreva (1979). The cycling example given in (4.2.14) and its elaboration in **4.11.8** are due to Stickney and Watson (1978). Algorithm **4.2.6** was originally described in Murty (1974). Murty (1988, pp. 258-259) observed that the pivot row can also be chosen according to a largest-index rule.

4.12.7 A study of (LCP) pivoting methods that reports computational superiority for Murty's Algorithm **4.2.6** was conducted by Kostreva (1989); see **5.12.17** for more discussion of this paper. Bard's findings relative to the earlier Algorithm **4.2.2** were similar.

4.12.8 In recent years, strong motivation for efficiently treating problems like (4.2.16) has been provided by the encouraging performance of the successive quadratic programming (SQP) approach to solving nonlinear programming problems. For a brief historical account of this development, see Gill, Murray and Wright (1981, pp. 245-247).

4.12.9 Algorithm **4.2.11** is a special case of a quadratic programming method due independently to Dantzig (1961, 1963) and to van de Panne and Whinston (1964a, 1964b). These and other algorithms were partially inspired by Wolfe's (1959) "simplex method for quadratic programming" and, in turn, the "critical line algorithm" of Markowitz (1956). Algorithm **4.2.11** enters into the paper of Goldfarb and Idnani (1983), although in the latter paper the emphasis is on a more sophisticated implementation.

4.12.10 Júdice and Pires (1988/89) have studied heuristic block principal pivoting methods modeled after Murty's least-index method. A similar idea is proposed in Kostreva (1976). For further discussion on the latter work, see **5.12.21**.

4.12.11 The "general" PPM appears in Cottle (1964a), Dantzig and Cottle (1967), Cottle and Dantzig (1968) and Cottle (1968a). To a great extent, the method was initially motivated by the aim of solving the "composite problems" that arise from symmetric dual quadratic programs as developed by Cottle (1963). The extension (of the symmetric PPM) given here for row sufficient matrices was worked out by Cottle (1990).

4.12.12 Proposition **4.3.6** is a variant of results given in Cottle (1968a) where the monotone behavior of the basic pair is also noted. This proposition is applicable to certain instances of the nonstreamlined version of Lemke's method in which there is a complement for the the artificial variable z_0.

4.12.13 The primary sources for the pivoting schemes covered in Section 4.4 are Lemke and Howson (1964) on bimatrix games and Lemke (1965) for the more general linear complementarity problem. These methods have received much attention in the literature. Among the more notable publications in this area are Lemke (1968), Cottle and Dantzig (1968), and Eaves (1971a). These and many other references will be cited below.

4.12.14 Algorithm **4.4.1** and its streamlined version Algorithm **4.4.5** are what is usually meant by the term "Lemke's method." Lemke, himself, called it "Scheme I." The method has been extended in a variety of ways. See Cottle and Dantzig (1970), Eaves (1971a), McCammon (1970), Mylander (1974), Todd (1976b), van de Panne (1974), and Werner and Wetzel (1985). For references on the effect of the covering vector d on the performance of Lemke's method see Mylander (1971, 1974), Todd (1986) and Krueger (1985).

4.12.15 In essence, Theorem **4.4.10** was proved by Cottle and Dantzig (1968). At that time, however, the class of strictly semi-monotone matrices had not been named and had not gotten the notation $\boldsymbol{E}$. Except for this theorem, the results in Section 4.4 under the heading of "More existence results" have not appeared exactly in the form given here. Theorem **4.4.11**, Corollary **4.4.12** and Theorem **4.4.13** are implicit in the work of Lemke (1965) and Cottle and Dantzig (1968). Although Theorem **4.4.15** has not appeared in the literature before, it is really just a refinement of the basic ray termination property; see Theorem **4.4.9**. One of the consequences of **4.4.15** is that Algorithm **4.4.5** will process an LCP (q, M) with M belonging to Eaves' class $\boldsymbol{L}$.

4.12.16 As shown by **4.4.17**, Lemke's method cannot always be counted on to produce a global minimum in a nonconvex quadratic programming problem—no matter what covering vector $d \geq 0$ is used. This interesting example is due to Mylander (1974).

4.12.17 We refer the reader back to **1.7.4** for some historical notes on the bimatrix game literature. Some other references on bimatrix games are Bastian (1974b, 1976b), Ben-Israel and Kirby (1969), Knuth, Papadimitriou and Tsitsiklis (1988), Lüthi (1976), Majthay (1972), Millham (1968), Mukhamediev (1978), Raghavan (1970), Shapley (1974), Vorob'ev (1977), Winkels (1978) and Ye (1988a). The numerical example of an elusive equilibrium point in a bimatrix game is due to Aggarwal (1973). See the related papers Todd (1976c, 1977) for further discussion.

4.12.18 Markowitz (1952, 1956, 1959) pioneered the portfolio selection problem. The elementary formulation of the problem given here leads to a parametric convex programming problem which is then converted into a parametric LCP. The extensive literature on portfolio selection includes Elton and Gruber (1979, 1987), Pang (1980), Perold (1984) and Sharpe (1963, 1970). For an application of the parametric linear complementarity problem to structural mechanics, see Maier (1970, 1972), Cottle (1972), and Kaneko (1975, 1977a). For a synthesis of several applications, see Pang, Kaneko and Hallman (1979).

4.12.19 The traffic equilibrium problem is a mathematical model for the prediction of traffic flow patterns in a congested transportation network. The monograph by Beckman, McGuire and Winsten (1956) had a great deal of influence on the early work in this area. The treatment of the traffic equilibrium problem by complementarity and variational inequality methods originated from a paper by Smith (1979), and much research has since been done with this approach. In particular, the paper by Asmuth, Eaves and Peterson (1979) studied the application of Lemke's algorithm to the affine case. The cornerstone of the complementarity/variational inequality approach to the traffic problem is the *user equilibrium principle* introduced by Wardrop (1952).

Many paradoxes arise in the study of the traffic equilibrium problem. Braess (1968) presented the first such example. The article by Steinberg and Stone (1988) is among the most recent contributions to this subject. Our treatment of these traffic paradoxes given at the end of Section 4.5 follows that in the latter reference.

4.12.20 The first algorithm for the parametric linear complementarity problem as such is due to Murty (1971b). From the standpoint of matrix classes, the method given here as Algorithm **4.5.2** is a generalization of the one given by Cottle (1972). The parametric form of Lemke's Scheme I was developed by Lemke (1978); see also McCammon (1970). For other approaches involving parametric complementarity problems, see Bank, Guddat, Klatte, Kummer, and Tammer (1983), Megiddo (1977), Meister (1979, 1983), Murty (1988), and van de Panne (1974, 1975).

4.12.21 Section 4.6 is based on two papers: Van der Heyden (1980) and Lemke (1978). Van der Heyden's paper did much to stimulate new results on the class $\boldsymbol{E}$; see for example Cottle (1980c).

4.12.22 Algorithm **4.7.3** first appeared in Chandrasekaran (1970). The greediness of Chandrasekaran's method was established by Saigal (1970). The fact that Lemke's Scheme I will processs any LCP with a $\boldsymbol{Z}$-matrix was proved by Saigal (1971a) whose argument we use here. The results **4.7.6** and **4.7.7** connecting Lemke's method for (q, M) and the simplex method of linear programming applied to (4.7.13) are due to Mohan (1976a). Chandrasekaran's algorithm for the LCP with a $\boldsymbol{Z}$-matrix was extended by Pang (1979a) to an LCP of the same type, but which allows upper bounds on the (primary) variables.

The vast amount of literature on the LCP of the $\boldsymbol{Z}$-type is evidenced by the references cited herein and in several of the notes in Chapter 3. This abundance of work is attributable to the numerous applications that this special class of LCP's possess, and to the fact that the $\boldsymbol{Z}$-property easily lends itself to some fruitful analysis.

4.12.23 In private conversation with J.S. Pang in 1977, I. Kaneko illustrated the fact that the covering vector d can have a drastic effect on the number of pivots in Lemke's Algorithm **4.4.5**, and more importantly, noted that a clever choice of d can render this algorithm highly efficient. Of course, the latter idea is the essence of the special pivot scheme, **4.8.2**. The presentation of this algorithm in Section 4.8 is based on the paper of Pang and Chandrasekaran (1985).

4.12.24 The condition (4.8.5), which is a weak form of that defining an n-step vector, first appeared in Cottle (1972) in the context of determining the "monotonicity" (which we call "isotonicity" here) of the function $\tilde{z}_b(\lambda)$ for $\lambda \geq 0$, see Proposition **4.8.6**. Cottle's study was the result of a question raised by Maier (1972) (see also De Donato and Maier (1972) and Maier (1970)) concerning certain problems in structural analysis. Cottle used the term *uniform mononotonicity property* to mean that this monotonicity property holds for all vectors $b \in R^n$ and all $d \in R^n_+$; he demonstrated that M has the uniform monotonicity property if and only if M is a $\boldsymbol{K}$-matrix. Refinements of this characterization were obtained by Kaneko (1977a, 1978d) and Megiddo (1977). A nonlinear version of Cottle's result was discussed in Megiddo (1978).

4.12.25 Theorem **4.8.7** was proved in Pang and Chandrasekaran (1985). Its converse, Theorem **4.8.10** was obtained by Morris and Lawrence (1988). The latter study was inspired by a geometric question raised in Kelly, Murty and Watson (1990) which asked whether the set of n-step vectors would form the interior of a simplicial cone. Morris and Lawrence provided an affirmative answer to this question which led to their demonstration of Theorem **4.8.10**. The ingenuity of their proof lies in its use of the matrices M^i in (4.8.8) whose significance in LCP theory had not been emphasized before.

4.12.26 Besides raising some interesting questions like the one mentioned above, the paper by Kelly, Murty and Watson (1990) attempted to analyze the set of n-step vectors from a geometric point of view. They introduced the notion of a *centrally projecting* point in a simplicial cone and related this concept to that of an n-step vector.

4.12.27 Before the proof of Theorem **4.8.7**, we discussed how this result is related to the issue of solving the LCP in terms of a linear program, and speculated that a "duality" relationship might exist to explain the phenomenon in question. As a matter of fact, this speculation is not based entirely on this one observation; a related phenomenon occurs in the results pertaining to the row and column sufficient matrices, see Section 3.5. There, the interesting thing is, of course, the fact that row sufficiency characterizes a certain property of the LCP, whereas column sufficiency characterizes another property; the two properties are seemingly quite different. This leads us to our main question: is there some kind of a unified framework that can be used to explain these results? Some progress in this direction has been made by Fukuda and Terlaky (1990), who propose what they call "the duality theorem of linear complementarity."

4.12.28 Degeneracy resolution in the LCP has been studied by Chang (1979). Much of the material in Section 4.9 is based on this source. Chang obtained results on the sizes of the smallest LCP's in which cycling can occur with different pivoting methods. (Chang's findings do not quite agree with those of Kostreva (1979) who used different ground rules.) A double least-index rule for resolving degeneracy in linear programming was pioneered by Bland (1977) and later extended to quadratic programming by Chang and Cottle (1980).

4.12.29 Our discussion of the lexicographic approach to Lemke's method is similar to that of Eaves (1971a) who was the first to carry out such an analysis. Eaves' treatment was somewhat more general, however.

4.12.30 As noted in Section 4.2, Murty (1974) introduced a least-index pivot selection rule in a Bard-type method. Later Chang (1979) extended the concept of a least-index rule to the principal pivoting method and to Lemke's method. In each case, however, the technique was restricted to instances in which the matrix M of the LCP was either positive semi-definite or a $\boldsymbol{P}$-matrix. Indeed, Chang gave an example showing that the least-index rule for resolving degeneracy need not work in Lemke's method applied to an LCP with a strictly copositive matrix. Algorithm **4.9.8** which extends Chang's result for the PPM to the case of (row and column) sufficient matrices is taken from Cottle and Chang (1992). As of this writing, a way to do the analogous thing relative to Lemke's method has not been found, as some of the devices that Chang (1979) used to handle the positive semi-definite and $\boldsymbol{P}$-matrix cases do not go through for sufficient matrices.

4.12.31 Several authors have presented ideas for the practical implementation of linear complementarity algorithms. Compact basis developments of the sort in Section 4.10 can be found, for example, in Sargent (1978) and Pang (1980). The full basis approach was advocated by Tomlin (1976, 1978) who dealt with Lemke's method. Tomlin's LCPL—an implementation of Lemke's method—was latter used by Rutherford (1986) in his work on equilibrium modeling. The nonlinear complementarity problems that arise in this subject are approximated by linear complementarity problems and solved by Lemke's method. This approach stimulates the desire to start Lemke's method from a "good" basis, rather than "from scratch." Ideas for doing this are discussed by Anstreicher, Lee and Rutherford (1991), and Kremers and Talman (1992).

4.12.32 The use of matrix factorizations in algorithms such as those presented in this chapter requires techniques for *updating* as well as for the underlying factorizations themselves. The reader is referred to **2.11.8** for more discussion and some references on these techniques.

4.12.33 The examples of exponential worst case behavior of Algorithms **4.2.6** and **4.5.4** presented in Section 4.10 are due to Murty (1978a). In a related paper, Fathi (1979) exhibits classes of LCP's with *symmetric* positive definite matrices for which these two algorithms take an exponential number of steps. The example used for demonstrating the possible exponential behavior of Algorithm **4.6.3** and implicitly of **4.3.5** comes from Birge and Gana (1983); our proof of the Birge-Gana Theorem **4.10.8** is somewhat different from theirs. For a broader discussion of the phenomenon involved in these examples, see Cottle (1980b).

4.12.34 Several pivoting methods for the LCP have not been discussed in this chapter. These include the enumerative methods of Garcia and Lemke (1970), Turnovec (1971), Jahanashahlou and Mitra (1979), Al-Khayyal (1987), De Moor (1988) 1989), De Moor, Vandenberghe, and Vandewalle (1992), the cutting-plane method of Jeroslow (1978), the n-cycle method of Watson (1974) and his hybrid method (1978), a scheme based on the solution of parametric linear programs by Wendler (1981), and the global optimization approaches of Pardalos and Rosen (1988) and Tuy, Thieu and Thai (1985). Some of these algorithms address the heroic task of finding *all* solutions to an LCP or an even more general problem.

Chapter 5

ITERATIVE METHODS

In the preceding chapter, we have discussed numerous pivoting methods for solving the linear complementarity problem. These methods are all finite and require the recursive solution of systems of linear equations. For problems of small to medium size (say, when n is no more than a few hundred), the pivoting methods are perhaps as good as methods of any other type. As the problem dimension increases, the efficiency of the pivoting methods tends to decrease due to two major difficulties: round-off errors and data storage. Round-off errors, if not handled properly, can cause severe numerical problems, such as incorrect pivots, erroneous solutions, or the breakdown of the method being used. What complicates the matter is the fact that these errors tend to accumulate very rapidly as the number of iterations increases. In order to (partially) control these errors, highly sophisticated numerical schemes are needed to ensure that the pivot steps are accurately and stably executed. Typically, such schemes are both time and storage consuming. The latter aspect raises the second difficulty associated with the pivoting methods for solving large-scale problems. The mere size of a problem can often cause the failure of a pivoting method.

Normally, problems of very large size tend to be sparse, i.e., the data are very likely to contain many zero elements. In a way, sparsity is an essential attribute of the problem that compensates for its large size. In fact, it would be difficult for any method to handle a large-scale, highly dense problem simply because of the complexity in the storage and management of the data. Pivoting methods are not particularly suitable for solving sparse problems because they can easily destroy the sparsity in just a few pivot steps. Advanced implementations of these methods can help in this regard, but they are not as effective as some other means.

As an alternative to the pivoting methods, *iterative schemes* have their advantages in solving large-scale linear complementarity problems. Typically, the iterative methods do not terminate finitely, but converge only in the limit. They are exempt from the two drawbacks that the pivoting methods have. Indeed, the iterative methods may be considered self-correcting and are much less sensitive to round-off errors. Furthermore, these methods tend to have all their iterations carried out on the original data, thus are able to maintain and exploit sparsity and any structure that the problem data might possess.

5.1 Applications

In order to motivate the development of the iterative methods, we discuss a few applications that can readily generate large-scale linear complementarity problems.

Contact problems

The analysis of elastic bodies in contact is a much-studied problem in mechanics. In this subsection, we shall discuss a simple case and show how it leads to a linear complementarity problem. The problem will be of large scale if the number n defined in the next paragraph is large.

Consider two elastic bodies (called Body 1 and Body 2) whose surfaces are "smooth." Suppose a *pairing* is established between a set of n points on the surface of Body 1 and a set of n points on the surface of Body 2. This association is not arbitrary or random; rather it is based on the idea that certain pairs of points on the two surfaces can come into contact with each

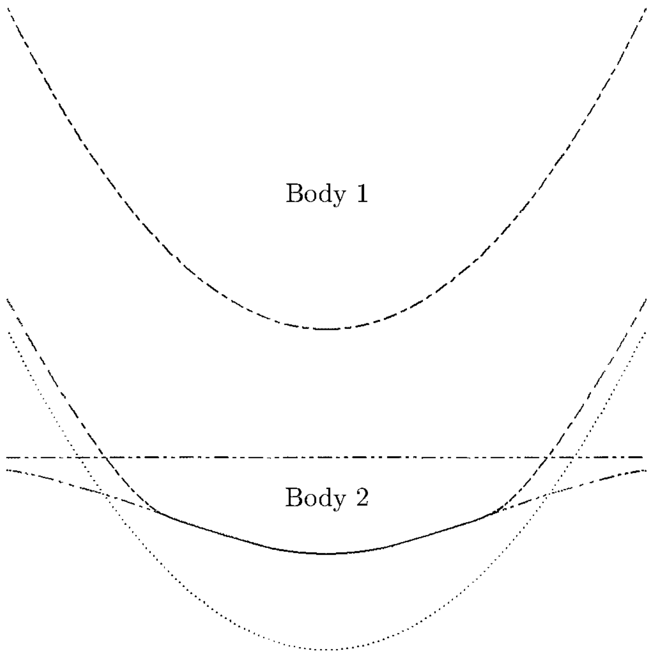

Figure 5.1: Contact problem

other in response to some externally applied loading scheme. It is assumed that the process leads to small deformations and that the two bodies obey the laws of linear elasticity.

Assume that *before loading*, the distance between the i-th point on the two bodies is d_i, $i = 1, \ldots, n$. Likewise, let z_i denote the contact stress (force) at the i-th point. As the bodies come into contact, elastic deformations (in the vertical direction) are produced at the n pairs of points. Let v_i^1 and v_i^2 denote the deformations for the i-th pair of points and the corresponding elastic bodies. If free penetration of one body by the other could occur, the result of the externally applied load p would be a uniform reduction of the distances between pairs of points by an amount α, known as the *rigid-body approach.*

The problem is to determine the rigid-body approach α and the contact stresses, z_i. These variables must satisfy the following conditions.

- *Compatibility of deformation.* In fact, the bodies do not penetrate each other, and the distance between the i-th pair of points is nonnegative. In light of how this distance is measured, the deformations and the rigid-body approach must satisfy

$$v_i^1 + v_i^2 + d_i - \alpha \geq 0 \quad i = 1, \ldots, n. \tag{1}$$

  As will be seen shortly, the quantities v_i^1 and v_i^2 are given by linear functions of the contact stresses. This makes (1) a system of linear inequalities in the unknowns of the problem.

- *Equilibrium of forces.* The forces at the n candidate contact points must balance the applied load p. Thus,

$$\sum_{i=1}^{n} z_i = p. \tag{2}$$

- *Contact criterion.* For each $i = 1, \ldots, n$, let w_i denote the quantity on the left-hand side of (1). The number w_i represents the clearance between the i-th pair of points on the two elastic bodies. For each i, when the clearance is positive (i.e., the points of the i-th pair are not in contact), the corresponding contact stress must be zero; when the the i-th contact stress is positive, the corresponding clearance must be zero. Thus, for $i = 1, \ldots, n$, the contact conditions are

$$\begin{aligned} w_i > 0 \quad &\Rightarrow \quad z_i = 0, \\ z_i > 0 \quad &\Rightarrow \quad w_i = 0. \end{aligned} \tag{3}$$

  Treating the w_i and z_i as coordinates of two nonnegative n-vectors w and z, respectively, we can write (3) as

$$w \geq 0, \quad z \geq 0, \quad \text{and} \quad z^{\mathrm{T}} w = 0.$$

It is assumed that the vectors of elastic deformations $v^1 = (v_1^1, \ldots, v_n^1)$ and $v^2 = (v_1^2, \ldots, v_n^2)$ are given by

$$v^1 = D^1 z \quad \text{and} \quad v^2 = D^2 z \tag{4}$$

where D^1 and D^2 are symmetric matrices of influence coefficients. The matrix $D = D^1 + D^2$ is symmetric and (for physical reasons) positive definite. In light of (4), equation (1) can be rewritten as

$$w = d + Dz - e\alpha \geq 0.$$

In preparation for rewriting equation (2), define

$$\zeta = -p + e^{\mathrm{T}}z.$$

We can now express the conditions above as

$$\begin{bmatrix} w \\ \zeta \end{bmatrix} = \begin{bmatrix} d \\ -p \end{bmatrix} + \begin{bmatrix} D & -e \\ e^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} z \\ \alpha \end{bmatrix}$$

$$w \geq 0, \quad z \geq 0, \quad z^{\mathrm{T}}w = 0, \quad \alpha \geq 0, \quad \zeta = 0.$$

This is clearly a mixed linear complementarity problem.

The problem takes a simpler, more transparent form when the rigid-body approach is known and the equilibrium conditions are satisfied. It is then a matter of solving the LCP $(d - \alpha e, D)$. Notice, however, that when α is not known but is instead regarded as a nonnegative parameter, then $(d - \alpha e, D)$ is a parametric LCP, and since D is symmetric and positive definite, each individual LCP $(d - \alpha e, D)$ has a unique solution $z(\alpha)$. Solving the original contact problem can be interpreted as a search for the (least) value of α for which $\zeta = -p + e^{\mathrm{T}}z(\alpha) = 0$. In the special case where D is also a $\boldsymbol{K}$-matrix, the components of $z(\alpha)$ are nondecreasing functions of α (see Proposition **3.11.9**) and hence ζ is too.

Free-boundary problem for journal bearings

A *journal bearing* consists of a rotating shaft (the journal) separated from a surface (the bearing) by a thin film of lubricating fluid. The longitudinal axes of the journal and the bearing are parallel. Figure 5.2 depicts a simple journal bearing of the kind discussed here. Figure 5.3 shows a region Ω, the planar unfolding of the bearing surface. Figure 5.4 represents a cross-section perpendicular to the longitudinal axis of a journal bearing.

The problem is to find the distribution of pressure p in the lubricant. An important underlying assumption of the model is that the lubricating film is so thin that there is no variation in pressure in the direction perpendicular to the axis of the journal. In terms of Figure 5.4, this means that for each value of θ, the pressure is constant on the line from the journal to the bearing. Accordingly, one can view the problem as the determination of the pressure distribution of the lubricant on the bearing surface. In polar coordinates, the thickness (i.e., depth) of the film is denoted by the function $h(x, \theta)$. In the case of a full cylindrical bearing (one which completely encloses the journal as assumed here and shown in Figure 5.2), the thickness of the lubricant depends only on the θ-coordinate and hence is denoted $h(\theta)$. With $\theta_{\min} = \arg\min_{\theta\in[0,2\pi]} h(\theta)$, this function satisfies the conditions

$$h(\theta) > 0 \qquad \theta \in [0, 2\pi], \tag{5}$$

$$\frac{dh(\theta)}{d\theta} < 0 \qquad \theta \in (0, \theta_{\min}), \tag{6}$$

$$\frac{dh(\theta)}{d\theta} > 0 \qquad \theta \in (\theta_{\min}, \theta_f). \tag{7}$$

It is assumed that when $\theta = \theta_f$, the pressure in the lubricant becomes so low that it vaporizes, and that at $\theta = 0$ this vapor condenses into its liquid state. The resulting interface between the liquid and gaseous phases of the lubricant is called the *free boundary.*

In the finite-length journal bearing indicated in Figure 5.2, the location of the free boundary is a function of the axial coordinate x and is denoted $\theta_f(\cdot)$. It is convenient to consider a planar unfolding of the journal bearing. See Figure 5.3. The pressure is zero (i.e., atmospheric) along and beyond the free boundary (the location of which is unknown in advance). In the region where $p > 0$, it satisfies a Reynolds equation

$$\nabla(h^3 \nabla p) - \frac{dh}{d\theta} = 0.$$

The boundary conditions require that the pressure be zero along the four edges of the developed bearing and that the derivative of the pressure in the direction, n, normal to the tangent of the free boundary at $(x, \theta_f(x))$ be zero. This is expressed in the equation

$$p = \partial p/\partial \mathrm{n} = 0 \quad \text{on the free boundary.}$$

No analytic solution to the problem is known, so we seek a numerical solution using a five-point finite-difference approximation scheme. It turns

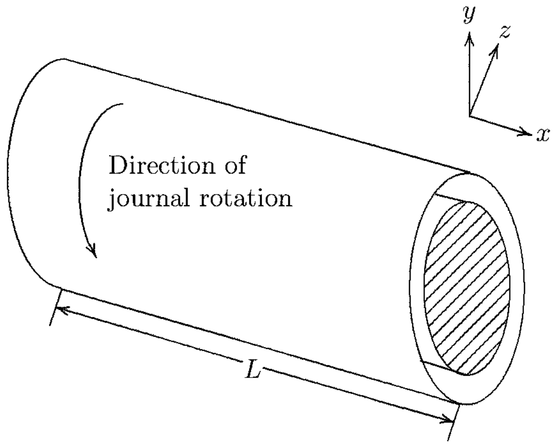

Figure 5.2: Side view of journal bearing

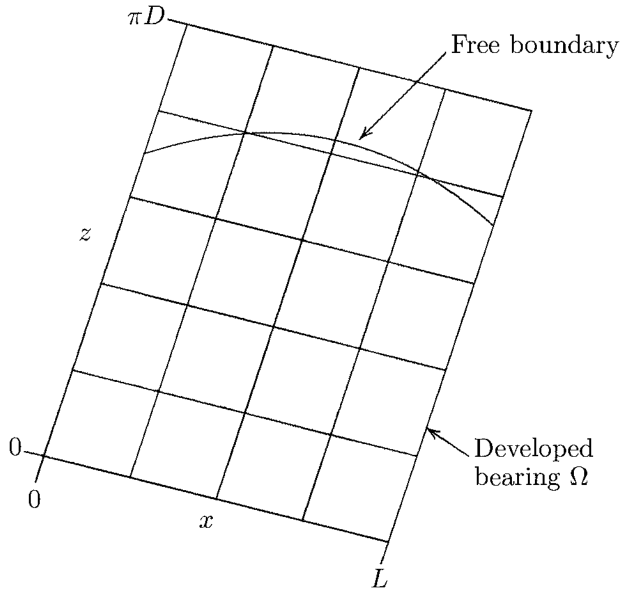

Figure 5.3: Planar unfolding of journal bearing

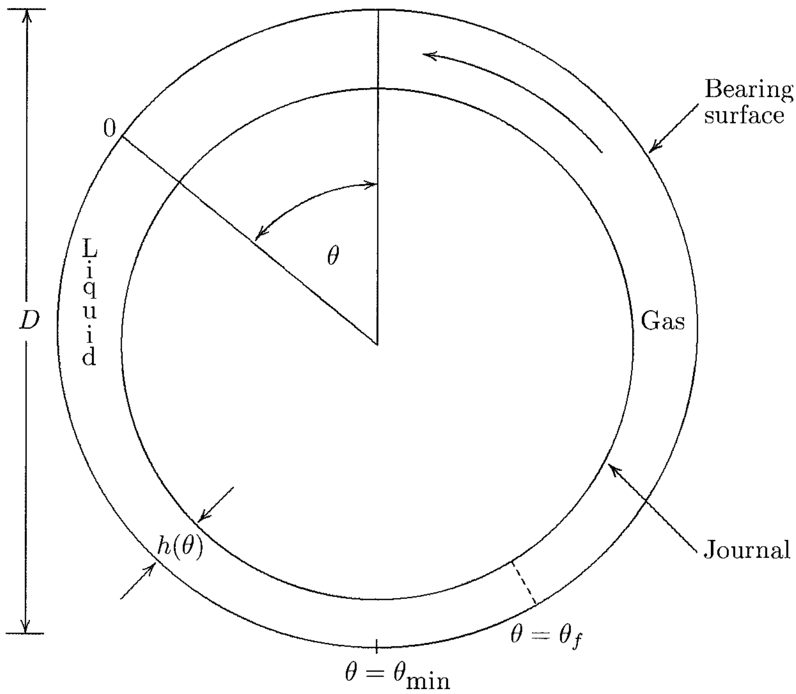

Figure 5.4: Cross-section of journal bearing

out that the resulting discrete system has an equivalent formulation given by the linear complementarity problem (q, M) described below.

After transforming from polar to rectangular coordinates, we place a grid on the developed bearing with grid sizes Δx and Δz in the x and z directions, respectively. (See Figure 5.3.) Lattice points on the grid representing the journal bearing will have coordinates (i, j) where $i = 0, 1, \ldots, m, m+1$ and $j = 0, 1, \ldots, n, n+1$. We denote the (unknown) pressure at grid point (i, j) by p_{ij}. The boundary conditions require that $p_{ij} = 0$ for $i = 0$ or $m+1$ and $j = 0$ or $n+1$. Expressions such as $h_{i-1/2,j}$ with fractional subscripts denote values of the corresponding function (in this case h) at points in the plane of the grid that lie midway between the obvious corresponding lattice points. The scheme for this is indicated in Figure 5.5 .

As suggested in Figures 5.2–5.4, the bearing has length L and diameter D. It can be shown from the mathematical formulation above that we get the following data. For $i = 1, \dots, m,\ \ j = 1, \dots, n,$ and $k = (i-1)n + j$,

$$q_{i,j} = 6\pi \frac{(h_{i,j+1/2} - h_{i,j-1/2})}{\Delta z}$$

$$m_{k,k-n} = -\left(\frac{D}{L}\right)^2 h^3_{i-1/2,j} \left(\frac{1}{\Delta x}\right)^2$$

$$m_{k,k-1} = -h^3_{i,j-1/2} \left(\frac{1}{\Delta z}\right)^2, \qquad \text{if } j > 1$$

$$m_{k,k} = \left(\frac{D}{L}\right)^2 \frac{(h^3_{i+1/2,j} + h^3_{i-1/2,j})}{(\Delta x)^2} + \frac{(h^3_{i,j+1/2} + h^3_{i,j-1/2})}{(\Delta z)^2}$$

$$m_{k,k+1} = -h^3_{i,j+1/2} \left(\frac{1}{\Delta z}\right)^2, \qquad \text{if } j < n$$

$$m_{k,k+n} = -\left(\frac{D}{L}\right)^2 h^3_{i+1/2,j} \left(\frac{1}{\Delta x}\right)^2$$

$$m_{k,l} = 0 \quad \text{otherwise.}$$

The m_{rs}, defined above, for which either $s \le 0$ or $s \ge mn$ are ignored. (See property (i) below.) We let $q = (q^1, \dots, q^m)$ where $q^i = (q_{i,1}, \dots, q_{i,n})$. In the case depicted above, $h(x, \theta) = h(\theta)$ for all x, so $h_{i,j-1/2}$ and $h_{i,j+1/2}$ are independent of i. In the resulting LCP (q, M), the matrix M has the following properties:

(i) $M \in R^{mn \times mn}$ is a symmetric block tridiagonal $\boldsymbol{K}$-matrix with blocks
$$M_{i,i'} \in R^{n \times n} \quad i, i' = 1, \dots, m;$$

(ii) There is a negative diagonal matrix Λ such that $M_{i+1,i} = M_{i,i+1} = \Lambda$ for all $i = 1, \dots, m-1$;

(iii) There is a symmetric tridiagonal matrix T such that $M_{i,i} = T$ for all $i = 1, \dots, m$.

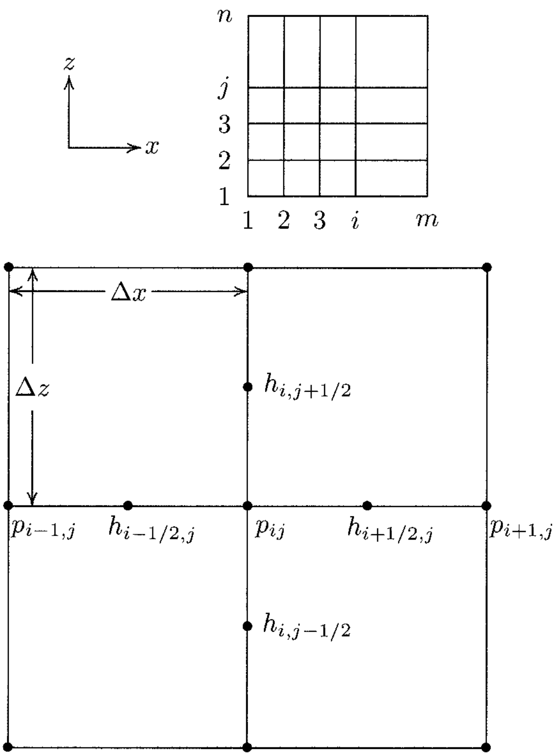

Figure 5.5: Five-point finite-difference approximation

Network equilibrium problems

We discuss another source of applications for large-scale linear complementarity problems. These applications arise from the computation of network equilibria. Typically, the latter computational problem can be formulated as a variational inequality and/or nonlinear complementarity problem which after linearization yields a sequence of linear complementarity problems. The high dimensionality of these LCPs is due to the large size of the network. We explain a simplified application of this type.

Consider an equilibrium model of international or interregional trade in a single commodity. Let n be the number of regions under consideration. In each region, there is a market characterized by classical supply and demand curves. In the absence of imports or exports, the equilibrium price and quantity produced and consumed will be determined by the intersection of these curves. If imports are introduced, consumption will exceed production but at a lower equilibrium price.

To simplify the discussion, we assume that the regional supply and demand functions are linear and given by the expression

$$p_i = a_i - b_i y_i \tag{8}$$

where

p_i is the equilibrium price in the i-th region,

y_i is the net import of the i-th region,

a_i is the equilibrium price in the absence of imports (and exports) and is positive,

b_i is related to the elasticity of supply and demand and is also positive.

We introduce the nonnegative flow variables x_{ij} which represent the (net) exports from region i to region j, and transportation costs c_{ij} which represent the unit cost of shipment from i to j. The additional interregional trade equilibrium conditions are

$$p_i + c_{ij} - p_j \geq 0 \quad \text{for all } \; i, j \tag{9}$$

$$x_{ij}(p_i + c_{ij} - p_j) = 0 \quad \text{for all } \; i, j. \tag{10}$$

The rationale behind these conditions is that if the inequality (9) fails to hold, exporters will buy in market i at price p_i, transport to market j at unit cost c_{ij} and sell at price p_j thus making a profit. Exports from i to j will increase until the elasticity effects in markets i and j raise (and lower) these prices so that additional profit to exporters is no longer possible. Thus, if $x_{ij} > 0$, (9) must be satisfied as an equality, and we have the complementarity condition (10). The model is completed by the flow conservation equations (a definition of the net imports) :

$$y_i = \sum_{j=1}^{n} x_{ji} - \sum_{j=1}^{n} x_{ij}, \qquad \text{for all } i. \tag{11}$$

It is not difficult to convert the above equilibrium model into an LCP. For this purpose, let A denote the node-arc incidence matrix of a complete network with n nodes and let $B = \text{diag}(b_i)$. Using the flow equation (11) and the price function (8), we may eliminate the import variables y_i and the price variables p_i. By substituting these expressions into the remaining equilibrium conditions (9) and (10), the model becomes the problem of finding a flow vector $x \in R^{n(n-1)}$ which solves the LCP (q, M) with

$$q = c + A^{\mathrm{T}}a, \qquad M = A^{\mathrm{T}}BA.$$

This LCP is of the order $n(n-1)$ which is already quite large when n reaches, say 100.

There are many generalizations of the above model. For example, the price-quantity relation could be given by a non-diagonal affine function

$$p = a - By$$

where B is an arbitrary positive definite matrix, or there could be more than one commodity. In the multi-commodity model, the size of the resulting LCP becomes even larger. The nonlinear version of the model is also common in applications. At the present time, one of the most effective solution approaches for the general network equilibrium problem is the linearization procedure briefly outlined in Section 1.2.

In summary, we have discussed in this section several application areas which require the solution of large-scale linear complementarity problems. The common attribute of the defining matrices in the resulting LCPs is that they are sparse and/or specially structured. The iterative methods presented in this chapter offer a highly effective approach for solving these and many related problems.

5.2 A General Splitting Scheme

A large number of iterative methods for solving the LCP have their origin in the solution of systems of linear equations. Historically, these methods for the LCP were developed to solve nonnegatively constrained, strictly convex quadratic programs. They have been applied very successfully to solve some large LCPs arising from such applications as the ones described in the last section. These methods also provide the basis for more sophisticated ones that we shall introduce in some later sections.

As mentioned in Chapter 2, many iterative methods for solving systems of linear equations can be described by means of a matrix splitting. In what follows, we extend these methods to the context of the LCP (q, M). To accomplish this, we *split* the matrix M as the sum of two matrices B and C, i.e., let

$$M = B + C$$

where B and C are real matrices of the same order as M. Such a representation of M is called a *splitting*. We shall denote this splitting by the pair (B, C). Given the splitting (B, C) of M, the LCP (q, M) can be transformed into a fixed-point problem; indeed, for an arbitrary vector z, we may consider the LCP (q^z, B) where

$$q^z = q + Cz,$$

and the (multivalued) mapping which associates with this vector z the solution set of the LCP (q^z, B). Clearly, a vector z solves the LCP (q, M) if and only if it is a fixed point of this LCP mapping, i.e., if z is itself a solution of (q^z, B). (When B is the identity matrix, the corresponding LCP mapping becomes the function defined by (1.4.4).) In terms of this fixed-point formulation, we introduce the following iterative method for solving the LCP (q, M).

5.2.1 Algorithm. (The Basic Splitting Method)

Step 0. *Initialization.* Let z^0 be an arbitrary nonnegative vector, set $\nu = 0$.

Step 1. *General iteration.* Given $z^\nu \geq 0$, solve the LCP (q^ν, B) where

$$q^\nu = q + Cz^\nu,$$

and let $z^{\nu+1}$ be an arbitrary solution.

Step 2. *Test for termination.* If $z^{\nu+1}$ satisfies a prescribed stopping rule, terminate. Otherwise, return to Step 1 with ν replaced by $\nu + 1$.

In essence, the above method is just a straightforward fixed-point iteration on the aforementioned LCP mapping; in particular, it follows that if $z^\nu = z^{\nu+1}$, then z^ν solves the LCP (q, M) and the method terminates there. More generally, it is trivial to show that if a sequence $\{z^\nu\}$ produced by **5.2.1** converges to a vector z^*, then z^* must be a solution of (q, M).

In general, in order for the method **5.2.1** to be well-defined, each subproblem (q^ν, B) must have at least one solution. For this reason, we shall assume throughout the discussion that (B, C) is a *Q-splitting*, i.e., that B is a $\boldsymbol{Q}$-matrix. Note that the subproblem (q^ν, B) is not required to have a unique solution; if multiple solutions exist, any one can be picked as $z^{\nu+1}$. Furthermore, in order for the method to be practical, each subproblem (q^ν, B) must be relatively easy to solve.

The choice of a suitable *stopping rule* is intimately related to the notion of the *residue*; the latter subject will be discussed in detail in Section 5.10.

Choices of B

Different choices of the splitting (B, C) lead to different algorithms for solving the LCP (q, M); the simplest choice of all is probably the one with B being the identity matrix. In this case, each iterate $z^{\nu+1}$ is given by the explicit expression

$$z^{\nu+1} = \max(0, -q + (I - M)z^\nu)$$

(see the fixed-point function h in (1.4.4)). A slight generalization of the preceding choice is to pick B as an arbitrary positive diagonal matrix D. This leads to the expression

$$z^{\nu+1} = \max(0, z^\nu - D^{-1}(q + Mz^\nu)).$$

In particular, if D is equal to the diagonal part of M (which is assumed to be positive), the resulting method is commonly known as the *projected Jacobi* method. The word "projected" refers to the fact that $z^{\nu+1}$ is the projection of the vector $u^{\nu+1} = z^\nu - D^{-1}(q + Mz^\nu)$ onto the nonnegative orthant; indeed the vector $u^{\nu+1}$ is the iterate obtained from the well-known Jacobi iterative method applied to the system of linear equations

$$q + Mz = 0.$$

Generalizing the diagonal choice, we may take B to be a triangular matrix with positive diagonal entries. With this choice, each subproblem (q^ν, B) is trivially solvable by either a forward or a backward substitution scheme, depending on whether B is lower or upper triangular. In particular, if we choose B to be the lower triangular matrix

$$B = L + \omega^{-1} D, \tag{1}$$

where L and D are, respectively, the strictly lower triangular and diagonal parts of M, and where $\omega \in (0, 2)$ is a prescribed relaxation parameter, we are led to the *projected successive overrelaxation* (abbreviated as the PSOR) method. In this case, the components of the iterate $z^{\nu+1}$ are given recursively by

$$z_i^{\nu+1} = \max(0, z_i^\nu - \omega m_{ii}^{-1}(q_i + \sum_{j<i} m_{ij} z_j^{\nu+1} + \sum_{j \geq i} m_{ij} z_j^\nu)), \quad i = 1, \dots, n$$

where we have assumed as before, that the diagonal elements of M are positive. When $\omega = 1$, the PSOR method reduces to the *projected Gauss-Seidel* method. Typically, the scalar ω has an important effect on the efficiency of the SOR method; its range $(0, 2)$ is derived from the context of solving systems of linear equations; an explanation will be given later (see Corollary **5.3.6**).

Another interesting choice of B is a block diagonal matrix. In this case, each subproblem (q^ν, B) decomposes into a finite number (which is equal to the number of blocks in B) of individual sub-subproblems each of which can be solved independently of the others and by any suitable algorithm, either a pivotal method or even a different iterative scheme. This choice of B leads to various iterative methods that are particularly effective on parallel computers which—because of their special architecture—can take advantage of the total separation of the sub-subproblems most profitably. To give an example of a block diagonal B, let us assume that the matrix

$$M = \begin{bmatrix} M_{11} & M_{12} & \dots & M_{1N} \\ M_{21} & M_{22} & \dots & M_{2N} \\ \vdots & \vdots & & \vdots \\ M_{N1} & M_{N2} & \dots & M_{NN} \end{bmatrix}$$

is partitioned into N^2 submatrices M_{ij} where each submatrix M_{ij} is of the order $N_i \times N_j$. Let B be the block diagonal matrix with the M_{ii} as the diagonal blocks:

$$B = \begin{bmatrix} M_{11} & & & \\ & M_{22} & & \\ & & \ddots & \\ & & & M_{NN} \end{bmatrix}. \tag{2}$$

With this choice of B, the subproblem (q^ν, B) decomposes into N smaller LCPs, the i-th one of which is defined by the principal submatrix M_{ii} and has order N_i.

Related to the block diagonal choice is a block (lower or upper) triangular B matrix. In this case, each subproblem (q^ν, B) does not necessarily decouple into separate sub-subproblems, but can be solved by sequentially solving a finite number of sub-subproblems each of which is of smaller size than the original LCP (q, M). An example of such a block triangular choice is the family of BSOR (i.e., *block SOR*) methods. (The PSOR method discussed previously is sometimes called a *point* method.) As before, let M be partitioned into N^2 submatrices M_{ij}. Let D, L and U be the corresponding block diagonal, strictly block lower triangular and strictly block upper triangular parts of M respectively. Then the block SOR splitting (B, C) is obtained by setting

$$B = L + \omega^{-1} D$$

where $\omega \in (0, 2)$ is a given parameter. The display below illustrates this choice:

$$B = \begin{bmatrix} \omega^{-1} M_{11} & & & \\ M_{21} & \omega^{-1} M_{22} & & \\ \vdots & \vdots & \ddots & \\ M_{N1} & M_{N2} & \dots & \omega^{-1} M_{NN} \end{bmatrix}. \tag{3}$$

An important distinction between a block diagonal and a block triangular matrix B is the manner in which each subproblem (q^ν, B) is solved; the former choice leads to the family of *parallel methods*—a term derived from the fact that the sub-subproblems can be solved in parallel, whereas the latter choice yields the family of *sequential methods*. The practical efficiency of these methods depends very much on the architecture of the computer on which they are implemented. On a traditional sequential computer, the parallel methods are typically not as effective as their sequential counterparts. But on some of the more recently available highly parallel computers, the effectiveness of the parallel methods becomes increasingly evident.

Besides those mentioned above, there are many other possible choices for B that lead to interesting iterative schemes. For example, we may choose B to be a $\boldsymbol{Z}$-matrix. In this case, each subproblem (q^ν, B) can be solved efficiently by the specialized pivoting methods described in Section 4.7. Another choice for B would be the transpose of a hidden $\boldsymbol{K}$-matrix for which the n-step method described in Section 4.8 can be applied to solve the subproblems.

5.3 Convergence Theory

In this section, we study the convergence properties of the sequence $\{z^\nu\}$ produced by the iterative scheme introduced in **5.2.1**. In general, there are three basic approaches under which convergence results can be established; namely,

1. the symmetry approach,
2. the contraction approach, and
3. the monotonicity approach.

Each of these is based on a different argument and depends on different assumptions on the matrix M in the LCP (q, M). The symmetry approach produces results that are particularly pertinent to applications of the iterative methods to strictly convex quadratic programs; in general, these results can be established under rather mild assumptions. The contraction approach is useful for asymmetric LCPs, i.e., for problems where the matrix M is asymmetric; here the classical contraction principle is the main tool (see Section 2.5 under the heading "Nonlinear equations"). The monotonicity approach is built on the least-element theory of hidden $\boldsymbol{Z}$-matrices and provides conditions under which a certain (nonsingular) transformation of the sequence $\{z^\nu\}$ will be monotonically convergent.

The symmetry approach

As the term "symmetry" suggests, this approach depends on the blanket assumption that the matrix M in the LCP (q, M) is symmetric. Under this symmetry assumption, the LCP (q, M) is intimately related to the quadratic program (cf. (1.4.1))

$$\begin{array}{ll} \text{minimize} & f(z) = q^{\mathrm{T}}z + \tfrac{1}{2}z^{\mathrm{T}}Mz \\ \text{subject to} & z \geq 0. \end{array} \tag{1}$$

The objective function $f(z)$ plays a central role throughout the convergence proofs; it is used as a merit function for monitoring the progress of the method. Indeed, since the objective of the quadratic program (1) is to minimize $f(z)$, we hope that the sequence $\{f(z^\nu)\}$ will be at least monotonically decreasing. In order for this to hold, we introduce a key property of the splitting (B, C) .

5.3.1 Definition. The splitting (B, C) is said to be *weakly regular* if $B-C$ is positive semi-definite, and *regular* if $B - C$ is positive definite.

Using the above definition, we establish a basic lemma which is the key to the convergence proof under the symmetry argument.

5.3.2 Lemma. Let M be a symmetric matrix, and let (B, C) be a weakly regular Q-splitting of M. Then,

$$f(z^\nu) - f(z^{\nu+1}) \geq \tfrac{1}{2}(z^\nu - z^{\nu+1})^{\mathrm{T}}(B - C)(z^\nu - z^{\nu+1}) \geq 0. \tag{2}$$

Moreover, if (B, C) is regular, then $f(z^\nu) = f(z^{\nu+1})$ if and only if $z^\nu = z^{\nu+1}$.

Proof. By an easy calculation, we have

$$\begin{aligned} & f(z^\nu) - f(z^{\nu+1}) = \\ & (z^\nu - z^{\nu+1})^{\mathrm{T}}(q + Mz^{\nu+1}) + \tfrac{1}{2}(z^\nu - z^{\nu+1})^{\mathrm{T}}M(z^\nu - z^{\nu+1}) = \\ & (z^\nu - z^{\nu+1})^{\mathrm{T}}(q + Cz^\nu + Bz^{\nu+1}) + \tfrac{1}{2}(z^\nu - z^{\nu+1})^{\mathrm{T}}(B - C)(z^\nu - z^{\nu+1}) \\ & \geq \tfrac{1}{2}(z^\nu - z^{\nu+1})^{\mathrm{T}}(B - C)(z^\nu - z^{\nu+1}) \end{aligned}$$

where the last inequality follows because $z^\nu \geq 0$ and $z^{\nu+1}$ is a solution of the LCP (q^ν, B). This establishes the expression (2). The last assertion of the lemma is obvious. □

As noted in the preceding section, if $z^\nu = z^{\nu+1}$, then z^ν solves the LCP (q, M). Thus, an implication of **5.3.2** is that if the iterative method has not yet terminated at z^ν, then there must be a strict decrease in the objective value of the program (1); consequently, as the algorithm proceeds, the sequence $\{f(z^k)\}$ is strictly decreasing. Using this descent property, we derive the following lemma which establishes a convergence property of the sequence $\{z^\nu\}$ under the sole assumption of symmetry of M.

5.3.3 Theorem. Let (B, C) be a regular Q-splitting of the symmetric matrix M. Then, every accumulation point of any sequence $\{z^\nu\}$ produced by **5.2.1** is a solution of the LCP (q, M).

Proof. Let $\tilde{z}$ be an accumulation point of a sequence $\{z^\nu\}$ produced by **5.2.1**. Suppose that $\{z^{\nu_i}\}$ is a subsequence converging to $\tilde{z}$. Then $\{f(z^{\nu_i})\}$ converges to $f(\tilde{z})$. Moreover, the entire sequence $\{f(z^\nu)\}$ is bounded below; this is because the sequence $\{f(z^\nu)\}$ is nonincreasing (by **5.3.2**) and the subsequence $\{f(z^{\nu_i})\}$ converges. Consequently, the sequence $\{f(z^\nu)\}$ converges. Since the splitting (B, C) is regular, condition (2) implies that $\{z^\nu - z^{\nu+1}\}$ converges to zero; thus, $\{z^{\nu_i+1}\}$ also converges to $\tilde{z}$. By its definition, the iterate z^{ν_i+1} satisfies the conditions

$$\begin{aligned} q + Cz^{\nu_i} + Bz^{\nu_i+1} &\geq 0 \\ z^{\nu_i+1} &\geq 0 \\ (z^{\nu_i+1})^{\mathrm{T}}(q + Cz^{\nu_i} + Bz^{\nu_i+1}) &= 0. \end{aligned}$$

Passing to the limit $\nu_i \to \infty$, we deduce that $\tilde{z}$ solves the LCP (q, M). □

Theorem **5.3.3** establishes a fundamental property of the *subsequential limits* of the sequence $\{z^\nu\}$ produced by Algorithm **5.2.1**. This result does not assert the existence of such subsequential limits, nor does it ensure the convergence of the entire sequence of iterates. As a matter of fact, most of the convergence results obtained in this section are concerned with the accumulation points of the sequence $\{z^\nu\}$. A complete characterization of the convergence of the entire sequence $\{z^\nu\}$ under just the assumptions of Theorem **5.3.3** is postponed until Chapter 7 where the result is derived based on some sensitivity properties of the LCP (see Theorem **7.2.10**).

In order for an accumulation point of the sequence $\{z^\nu\}$ to exist, it is sufficient for the sequence $\{z^\nu\}$ to be bounded; in turn, this is satisfied if the level set

$$\{z \geq 0 : f(z) \leq f(z^0)\}$$

is bounded. In general, if M is a symmetric copositive matrix, then the latter set is bounded if the following implication holds

$$[0 \neq z \geq 0, \ z^{\mathrm{T}}Mz = 0] \quad \Rightarrow \quad q^{\mathrm{T}}z > 0. \tag{3}$$

(Indeed, if there exists an unbounded sequence of nonnegative nonzero vectors $\{x^\nu\}$ such that $f(x^\nu) \leq f(z^0)$ for all ν, then any accumulation point of the normalized sequence $\{x^\nu/\|x^\nu\|\}$ can be shown to violate the above implication.) In particular, the sequence $\{z^\nu\}$ must be bounded if M is strictly copositive.

The next result is a refinement of the foregoing analysis and gives a specialized set of sufficient conditions for a sequence produced by Algorithm **5.2.1** to be bounded.

5.3.4 Lemma. Let M be a symmetric matrix, and let (B, C) be a regular Q-splitting of M. Suppose that

(a) the quadratic function $f(z) = q^{\mathrm{T}}z + \frac{1}{2}z^{\mathrm{T}}Mz$ is bounded below for $z \geq 0$;

(b) the following implication holds:

$$[0 \neq z \geq 0, \ Mz \geq 0, \ z^{\mathrm{T}}Mz = 0] \quad \Rightarrow \quad q^{\mathrm{T}}z > 0. \tag{4}$$

Then any sequence $\{z^\nu\}$ generated by **5.2.1** is bounded.

Proof. By assumption (a), the sequence $\{f(z^\nu)\}$ is bounded below; by Lemma **5.3.2**, the same sequence is nonincreasing. Thus, $\{f(z^\nu)\}$ converges. By (2), it follows that $\{z^\nu - z^{\nu+1}\}$ converges to zero. Suppose the sequence $\{z^\nu\}$ is unbounded. Without loss of generality, we may assume that $\|z^\nu\| \to \infty$. Consider the normalized sequence $\{z^\nu/\|z^\nu\|\}$. This latter sequence is bounded and thus has an accumulation point $\tilde{z}$ which must be nonzero and nonnegative. Let $\{z^{\nu_i+1}/\|z^{\nu_i+1}\|\}$ be a subsequence converging to $\tilde{z}$. Since z^{ν_i+1} solves the LCP (q^{ν_i}, B) for each ν_i, we have

$$q + C(z^{\nu_i} - z^{\nu_i+1}) + Mz^{\nu_i+1} \geq 0 \tag{5}$$

$$z^{\nu_i+1} \geq 0 \tag{6}$$

$$(z^{\nu_i+1})^{\mathrm{T}} (q + C(z^{\nu_i} - z^{\nu_i+1}) + Mz^{\nu_i+1}) = 0. \tag{7}$$

Dividing the two inequalities (5) and (6) by $\|z^{\nu_i+1}\|$ and the equation (7) by $\|z^{\nu_i+1}\|^2$, and then passing to the limit $\nu_i \to \infty$, we easily deduce that $\tilde{z} \in \mathrm{SOL}(0, M)$.

Assumption (a) implies that the matrix M is copositive (by Proposition **3.7.14**). Thus, we have

$$\begin{aligned} 0 &= (z^{\nu_i+1})^{\mathrm{T}} (q + C(z^{\nu_i} - z^{\nu_i+1}) + Mz^{\nu_i+1}) \\ &\geq (z^{\nu_i+1})^{\mathrm{T}} (q + C(z^{\nu_i} - z^{\nu_i+1})). \end{aligned}$$

Dividing the last inequality by $\|z^{\nu_i+1}\|$ and passing to the limit $\nu_i \to \infty$, we deduce that $q^{\mathrm{T}}\tilde{z} \leq 0$. But this contradicts the assumption (b). Consequently, the sequence $\{z^\nu\}$ must be bounded. This establishes the lemma. □

The implication (4) is a stronger version of (3.8.2) in 3.8 but weaker than (3). Geometrically, (4) assumes that the vector q is in $\operatorname{int} S^*$ where $S = \mathrm{SOL}(0, M)$; this assumption is vacuously satisfied if $M \in \boldsymbol{R_0}$. The two assumptions (a) and (b) in **5.3.4** are related to each other but neither one implies the another. Indeed, according to **3.7.14**, the fact that the quadratic function $f(z)$ is bounded below on R^n_+ is equivalent to the copositivity of the matrix M and the validity of the implication

$$[z \geq 0,\ z^{\mathrm{T}}Mz = 0] \quad \Rightarrow \quad q^{\mathrm{T}}z \geq 0.$$

As pointed out in **3.7.13**, the quadratic function $z^{\mathrm{T}}(q+Mz)$ is not bounded below on the nonnegative orthant for

$$q = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

thus, the same is true for the function $f(z)$ by **3.7.14**. It is easy to verify that the above matrix M is in $\boldsymbol{R_0}$, hence the implication (4) is satisfied by default. Conversely, if M is any symmetric positive semi-definite matrix which does not belong to $\boldsymbol{R_0}$, and q is any vector in the range space of M, then the pair (q, M) satisfies (a) but fails (b) in **5.3.4**.

By combining **5.3.3** and **5.3.4**, it follows that if M is symmetric and satisfies assumptions (a) and (b) of **5.3.4**, then any sequence $\{z^\nu\}$ produced by Algorithm **5.2.1** with a regular Q-splitting (B, C) must have at least one accumulation point and any such point solves the LCP (q, M). Note that according to **3.7.12** and **3.7.14**, assumption (a) alone is enough to yield the existence of a solution to the LCP (q, M); however, the convergence of the sequence $\{z^\nu\}$ does not follow from these previous results.

If M is symmetric and strictly copositive, then the aforementioned convergence must hold for all vectors q and arbitrary starting vectors $z^0 \geq 0$. It turns out that the strict copositivity of M is also a necessary condition for this convergence result to hold. Before we formally state this characterization of convergence, it is useful to note that if M is a strictly copositive matrix and if (B, C) is any weakly regular splitting of M, then B must be strictly copositive. This follows easily from the identity

$$B = \tfrac{1}{2}(B - C + M); \tag{8}$$

in particular, (B, C) is a Q-splitting of M. The following result summarizes the relationship between the strict copositivity of M and the convergence of Algorithm **5.2.1**.

5.3.5 Theorem. Let M be a symmetric matrix. If M is strictly copositive, then for all vectors q and all initial vectors $z^0 \geq 0$, any sequence $\{z^\nu\}$ produced by **5.2.1** with a regular splitting is bounded with at least one accumulation point; moreover, any such point is a solution of the LCP (q, M). Conversely, if M has a regular Q-splitting with this (global, subsequential) convergence property, then M must be strictly copositive.

Proof. In view of the above remarks, it suffices to prove the converse part. We first show that $M \in \boldsymbol{R_0}$. Assume the contrary. Let $\tilde{z}$ be a nonzero solution of $(0, M)$. Take $q = -B\tilde{z}$ and $z^\nu = \nu\tilde{z}$ for each ν. It is then easy to see that $z^{\nu+1}$ is a solution of the LCP (q^ν, B). Thus, by assumption, this sequence $\{z^\nu\}$ is bounded which is clearly impossible. Consequently, $M \in \boldsymbol{R_0}$.

Suppose now that M is not strictly copositive. Let $\hat{z} \geq 0$ be a nonzero vector such that $\hat{z}^{\mathrm{T}} M \hat{z} \leq 0$. Since $\hat{z}$ is nonzero, it is not a solution of $(0, M)$. Take $q = 0$ and consider a sequence $\{z^\nu\}$ produced by the algorithm **5.2.1** with $z^0 = \hat{z}$ as the initial vector. By **5.3.2**, we have

$$f(z^1) < f(z^0) \leq 0$$

where $f(z) = \frac{1}{2} z^{\mathrm{T}} M z$. By assumption, the sequence $\{z^\nu\}$ is bounded and any one of its accumulation points is a solution of $(0, M)$. Since zero is the only solution of the latter LCP, it follows that $\{z^\nu\}$ converges to zero. Consequently, by **5.3.2** again, we deduce

$$0 = \lim_{\nu \to \infty} f(z^\nu) \leq f(z^1) < f(z^0) \leq 0$$

which is a contradiction. This establishes the theorem. □.

By specializing Theorem **5.3.5** to specific splittings (B, C) of M, we may derive convergence results for various iterative schemes. The following corollary concerns the family of (sequential) block SOR-methods (5.2.3).

5.3.6 Corollary. Let M be a symmetric matrix partitioned into submatrices M_{ij} with each diagonal submatrix M_{ii} being positive definite. Let D, L and U be the corresponding block diagonal, strictly block lower triangular, and strictly block upper triangular parts of M, respectively. Let ω be any given positive scalar. Then,

(a) for any initial vector $z^0 \geq 0$, the sequence $\{z^\nu\}$ produced by the resulting block SOR-method is well-defined;

(b) the block SOR-splitting (B, C) where

$$B = L + \omega^{-1} D$$

is regular if and only if $\omega < 2$;

(c) if $0 < \omega < 2$, then M is strictly copositive if and only if for any initial vector $z^0 \geq 0$ and any q, the sequence $\{z^\nu\}$ produced by the block SOR-method is bounded, and each of its accumulation points solves the LCP (q, M).

Proof. Obviously, if $\omega > 0$, then B is a $\boldsymbol{P}$-matrix. Thus, each LCP (q^ν, B) has a unique solution $z^{\nu+1}$. This proves (a). To establish the regularity of the splitting, we note that

$$B - C = L - U + \frac{2-\omega}{\omega} D.$$

Since M is symmetric, it follows that $L - U$ is skew-symmetric. Moreover, the matrix D is positive definite. Consequently, $B - C$ is positive definite if and only if $\omega < 2$. This establishes (b). The last part follows from **5.3.5**. □

All the convergence results established so far are of the *subsequential* type, they make no claim as to whether or not the entire sequence converges. In what follows, we establish some necessary and sufficient conditions for the latter kind of convergence to hold. We need a lemma which is an easy consequence of Proposition **3.7.14**.

5.3.7 Lemma. Let M be a symmetric and nondegenerate $n \times n$ matrix. The following statements are equivalent.

(a) M is strictly copositive.

(b) M is copositive.

(c) The quadratic function $f(z) = q^{\mathrm{T}}z + \frac{1}{2}z^{\mathrm{T}}Mz$ is bounded below on R^n_+ for some $q \in R^n$.

(d) The quadratic function $f(z) = q^{\mathrm{T}}z + \frac{1}{2}z^{\mathrm{T}}Mz$ is bounded below on R^n_+ for all $q \in R^n$.

Proof. In view of **3.7.14**, it suffices to prove [(b) $\Rightarrow$ (a)]. This is accomplished by induction on n. Suppose that M is not strictly copositive. Then there exists a nonzero vector $x \in R^n_+$ such that $x^{\mathrm{T}}Mx = 0$. By an induction hypothesis, we may assume that $x > 0$. It follows from Exercise **3.12.9** that M is positive semi-definite. Since M is nondegenerate, it must be positive definite and hence strictly copositive. This establishes the lemma. □

The following theorem gives a set of necessary and sufficient conditions for the sequence $\{z^\nu\}$ produced by Algorithm **5.2.1** to converge under the assumption that M is symmetric and nondegenerate.

5.3.8 Theorem. Let M be a symmetric and nondegenerate $n \times n$ matrix. Let (B, C) be a regular Q-splitting of M. The following statements are equivalent.

(a) For some vector $q \in R^n$ and any initial vector $z^0 \geq 0$, any sequence $\{z^\nu\}$ produced by **5.2.1** is bounded and has at least one accumulation point; moreover, any such point solves the LCP (q, M).

(b) For some vector $q \in R^n$, the quadratic function $f(z) = q^{\mathrm{T}}z + \frac{1}{2}z^{\mathrm{T}}Mz$ is bounded below for $z \geq 0$.

(c) For some vector $q \in R^n$ and any initial vector $z^0 \geq 0$, any sequence $\{z^\nu\}$ produced by **5.2.1** converges to a solution of the LCP (q, M).

(a′) For any vector $q \in R^n$ and any initial vector $z^0 \geq 0$, the conclusion of (a) holds.

(b′) For any vector $q \in R^n$, the conclusion of (b) holds.

(c′) For any vector $q \in R^n$ and any initial vector $z^0 \geq 0$, the conclusion of (c) holds.

(d) M is strictly copositive.

(e) M is copositive.

Proof. (a) $\Rightarrow$ (b). Using the given $z^0 \geq 0$ as the initial iterate, generate a sequence $\{z^\nu\}$ by **5.2.1**. By (a), some subsequence converges to some solution $\tilde{z}$ of the LCP (q, M). By Lemma **5.3.2**, we have

$$f(z^0) \geq f(\tilde{z}).$$

Since M is nondegenerate, the LCP (q, M) has a finite number of solutions by Theorem **3.6.3**. Consequently, for any $z^0 \geq 0$, $f(z^0)$ is bounded below by the minimum of the quadratic function values $f(\tilde{z})$ generated by a finite set of $\tilde{z}$ vectors. Thus, (b) follows.

(b) $\Rightarrow$ (c). Since any nondegenerate matrix belongs to the class $\boldsymbol{R_0}$, we may conclude by combining **5.3.3** and **5.3.4** that any sequence $\{z^\nu\}$ produced by **5.2.1** is bounded with at least one accumulation point, and any such point solves the LCP (q, M). Since (q, M) has only a finite number of solutions (because M is nondegenerate), the sequence $\{z^\nu\}$ has a finite number of accumulation points. Thus, by Theorem **2.1.10**, the sequence $\{z^\nu\}$ must converge. This establishes (c).

(c) $\Rightarrow$ (a). This is obvious.

Finally, the equivalence of all eight statements (a), (b), (c), (a′), (b′), (c′), (d) and (e) follows from Lemma **5.3.7**. □

The positive semi-definite case

Theorem **5.3.8** is in general not applicable to a positive semi-definite LCP (q, M) unless the matrix M is positive definite. The following theorem summarizes the convergence properties of Algorithm **5.2.1** applied to an LCP of the positive semi-definite type.

5.3.9 Theorem. Let M be an $n \times n$ symmetric positive semi-definite matrix. Let (B, C) be a regular splitting of M. Then,

(a) for any initial vector $z^0 \geq 0$, the sequence $\{z^\nu\}$ generated by **5.2.1** is uniquely defined;

(b) if there exists a vector z such that $q + Mz > 0$, then the sequence $\{z^\nu\}$ is bounded, and each of its accumulation points is a solution of the LCP (q, M);

(c) if the LCP (q, M) has a solution, then the sequence $\{Mz^\nu\}$ converges to some vector $M\hat{z}$ and $\hat{z}$ solves the LCP (q, M).

Notice that the convergence of the sequence $\{Mz^\nu\}$ is equivalent to that of the sequence $\{w^\nu\} = \{q + Mz^\nu\}$. Thus, part (c) of the theorem asserts the convergence of the associated w-sequence; it does not assert even the boundedness of the sequence $\{z^\nu\}$, let alone its convergence. Clearly, the converse of part (c) is trivially valid in the sense that if the sequence $\{Mz^\nu\}$ converges to some $M\hat{z}$ with $\hat{z}$ solving the LCP (q, M), then the latter problem must have a solution. The significance of this part of the theorem lies in the fact that under only the assumption of the solvability of the LCP (q, M), it is possible to demonstrate the convergence of the w-sequence associated with the z-sequence produced by Algorithm **5.2.1**. In Section 5.4, this result will be used to establish the convergence of the entire sequence of iterates $\{z^\nu\}$.

The assumption in part (b) is a Slater-type condition which appears fairly often in nonlinear programming; the vector z satisfying $q + Mz > 0$ is not required to be nonnegative. In general, if M is a symmetric positive semi-definite matrix and if there exists a vector z satisfying $q + Mz > 0$, then the LCP (q, M) must have a solution. There are several ways to see this; one way is to show that the implication (3.8.2) in **3.8.6** holds, thus, that theorem is applicable. Hence, the assumption in (b) is stronger than the one in (c), therefore, so is the respective conclusion.

The proofs of parts (a) and (b) of **5.3.9** are fairly straightforward. For (a), it suffices to note that if M is a positive semi-definite matrix and if (B, C) is a regular splitting of M, then B must be positive definite by the representation (8). For (b), it suffices to note that if M is symmetric and positive semi-definite and if there is a vector z such that $q + Mz > 0$, then the implication (3) must hold. The proof of part (c) relies on the following lemma which identifies a special property of a symmetric positive semi-definite matrix.

5.3.10 Lemma. Let M be an $n \times n$ symmetric positive semi-definite matrix, and α be a nonempty subset of $\{1, \ldots, n\}$. Then,

(a) the sequence $\{M_{\bullet\alpha} y_\alpha^\nu\}$ is bounded if and only if $\{M_{\alpha\alpha} y_\alpha^\nu\}$ is so;

(b) the sequence $\{M_{\bullet\alpha} y_\alpha^\nu\}$ converges to the vector $M_{\bullet\alpha} \bar{y}_\alpha$ if and only if $\{M_{\alpha\alpha} y_\alpha^\nu\}$ converges to $M_{\alpha\alpha} \bar{y}_\alpha$.

Proof. In either case, we need to prove only the "if" part.

Suppose that the sequence $\{M_{\alpha\alpha} y_\alpha^\nu\}$ is bounded but $\{M_{\bullet\alpha} y_\alpha^\nu\}$ is unbounded. We may assume without loss of generality that $\{\|M_{\bullet\alpha} y_\alpha^\nu\|\} \to \infty$. The normalized sequence $\{M_{\bullet\alpha} y_\alpha^\nu / \|M_{\bullet\alpha} y_\alpha^\nu\|\}$ is bounded; moreover, every one of its accumulation points must be nonzero, and by Theorem **2.6.24**, of the form $M_{\bullet\alpha} \hat{y}_\alpha$ for some vector $\hat{y}_\alpha$. Consider any one accumulation point $M_{\bullet\alpha} \hat{y}_\alpha$, and assume (without loss of generality) that it is the limit of the entire sequence $\{M_{\bullet\alpha} y_\alpha^\nu / \|M_{\bullet\alpha} y_\alpha^\nu\|\}$. So, we have

$$M_{\alpha\alpha} \hat{y}_\alpha = \lim_{\nu \to \infty} \frac{M_{\alpha\alpha} y_\alpha^\nu}{\|M_{\bullet\alpha} y_\alpha^\nu\|} = 0$$

where the last equality follows because the numerator is bounded and the denominator approaches ∞. Since a symmetric positive semi-definite matrix is column adequate (see Section 3.4), Exercise **3.12.12** implies that $M_{\bullet\alpha} \hat{y}_\alpha = 0$ which is a contradiction. This establishes part (a).

To prove part (b), suppose that $\{M_{\alpha\alpha} y_\alpha^\nu\}$ converges to $M_{\alpha\alpha} \bar{y}_\alpha$. Part (a) then implies that the sequence $\{M_{\bullet\alpha} y_\alpha^\nu\}$ is bounded. By Theorem

2.6.24, any accumulation point of $\{M_{\bullet\alpha} y_\alpha^\nu\}$ must be of the form $M_{\bullet\alpha}\tilde{y}_\alpha$ for some vector $\tilde{y}_\alpha$. It follows that $M_{\alpha\alpha}\bar{y}_\alpha = M_{\alpha\alpha}\tilde{y}_\alpha$. Consequently, if $M_{\bullet\alpha}\hat{y}_\alpha$ is another accumulation point of the sequence $\{M_{\bullet\alpha} y_\alpha^\nu\}$, we must have $M_{\alpha\alpha}\hat{y}_\alpha = M_{\alpha\alpha}\tilde{y}_\alpha$. The symmetry and positive semi-definiteness of M then imply that $M_{\bullet\alpha}\hat{y}_\alpha = M_{\bullet\alpha}\tilde{y}_\alpha$, which in turn implies that the sequence $\{M_{\bullet\alpha} y_\alpha^\nu\}$ has only one accumulation point. Hence, $\{M_{\bullet\alpha} y_\alpha^\nu\}$ converges, and part (b) is established. □

Proof of 5.3.9(c). Since M is symmetric positive semi-definite and the LCP (q, M) has a solution, it follows that the quadratic function

$$f(z) = q^{\mathrm{T}}z + \tfrac{1}{2}z^{\mathrm{T}}Mz$$

is bounded below for $z \geq 0$ (by the equivalence between the LCP (q, M) and the quadratic program (1)). By the same argument used in Lemma **5.3.4**, we may deduce that the sequence $\{z^{\nu+1} - z^\nu\}$ converges to zero. We show that the sequence $\{Mz^\nu\}$ is bounded. Assume the contrary. Then, a subsequence $\{\|Mz^\nu\| : \nu \in K\} \to \infty$. This implies that $\{z^\nu : \nu \in K\}$ is unbounded. There exist a nonempty index set $\alpha \subseteq \{1, \ldots, n\}$ and a subsequence $\{z^\nu : \nu \in K'\}$ with $K' \subseteq K$ so that $\{z_j^\nu : \nu \in K'\} \to \infty$ if $j \in \alpha$ and $\{z_j^\nu : \nu \in K'\}$ is bounded if $j \notin \alpha$. Thus, for all $\nu \in K'$ sufficiently large, we have

$$0 = (q + Cz^{\nu-1} + Bz^\nu)_\alpha$$

which implies

$$M_{\alpha\alpha}z_\alpha^\nu = -((q + C(z^{\nu-1} - z^\nu))_\alpha + M_{\alpha\bar{\alpha}}z_{\bar{\alpha}}^\nu). \tag{9}$$

This last equation shows that the sequence $\{M_{\alpha\alpha}z_\alpha^\nu : \nu \in K'\}$ is bounded, thus so is $\{M_{\bullet\alpha}z_\alpha^\nu : \nu \in K'\}$ by **5.3.10**. Since

$$Mz^\nu = M_{\bullet\alpha}z_\alpha^\nu + M_{\bullet\bar{\alpha}}z_{\bar{\alpha}}^\nu,$$

it follows that $\{Mz^\nu : \nu \in K'\}$ is bounded. This contradicts the fact that $\{\|Mz^\nu\| : \nu \in K\} \to \infty$, and establishes the boundedness of the sequence $\{Mz^\nu\}$.

Let y be any accumulation point of $\{Mz^\nu\}$, and let $\{Mz^\nu : \nu \in K\}$ be a subsequence converging to y. There exist a (possibly empty) index set $\alpha \subseteq \{1, \ldots, n\}$ and a subsequence $\{Mz^\nu : \nu \in K'\}$ with $K' \subseteq K$ such that $\{z_j^\nu : \nu \in K'\} \to \infty$ if $j \in \alpha$ and $\{z_j^\nu : \nu \in K'\}$ is bounded if $j \notin \alpha$. As before, equation (9) holds for all $\nu \in K'$ sufficiently large. Thus, the sequence $\{M_{\alpha\alpha} z_\alpha^\nu : \nu \in K'\}$ is bounded, and by Lemma **5.3.10**, so is $\{M_{\bullet\alpha} z_\alpha^\nu : \nu \in K'\}$. Without loss of generality, we may assume that $\{M_{\bullet\alpha} z_\alpha^\nu : \nu \in K'\}$ converges to some vector which must be of the form $M_{\bullet\alpha}\hat{z}_\alpha$ for some $\hat{z}_\alpha \geq 0$. Since the sequence $\{z_{\bar{\alpha}}^\nu : \nu \in K'\}$ is bounded, we may further assume that it converges to some vector $\hat{z}_{\bar{\alpha}} \geq 0$. It then follows $y = M\hat{z}$. We verify that $\hat{z}$ solves the LCP (q, M). We have already noted that $\hat{z}$ is nonnegative. Passing to the limit $\nu \to \infty$ in the equation (9), we obtain

$$(q + M\hat{z})_\alpha = 0$$

by the fact that $z^{\nu-1} - z^\nu \to 0$. Moreover, for each $\nu \in K'$, we have

$$\begin{aligned} 0 &\leq (q + Cz^{\nu-1} + Bz^\nu)_{\bar{\alpha}} \\ &= (q + C(z^{\nu-1} - z^\nu) + Mz^\nu)_{\bar{\alpha}} \end{aligned}$$

and

$$0 = (z_{\bar{\alpha}}^\nu)^{\mathrm{T}}(q + C(z^{\nu-1} - z^\nu) + Mz^\nu)_{\bar{\alpha}}.$$

Passing to the limit $\nu \to \infty$, $\nu \in K'$, and noting

$$z_{\bar{\alpha}}^\nu \to \hat{z}_{\bar{\alpha}}, \; Mz^\nu \to M\hat{z},$$

we conclude that

$$(q + M\hat{z})_{\bar{\alpha}} \geq 0, \; (\hat{z}_{\bar{\alpha}})^{\mathrm{T}}(q + M\hat{z})_{\bar{\alpha}} = 0.$$

Consequently, $\hat{z} \in \mathrm{SOL}(q, M)$.

Summarizing, we have proved that if y is any accumulation point of the sequence $\{Mz^\nu\}$, then there exists a solution $\hat{z}$ of the LCP (q, M) such that $y = M\hat{z}$. By Theorem **3.1.7(d)**, there is only one such value for $M\hat{z}$. Consequently, the sequence $\{Mz^\nu\}$ converges to $M\hat{z}$ where $\hat{z}$ solves the LCP (q, M). This establishes the theorem. □

Application to quadratic programs

The convergence results established under the symmetry assumption are most pertinent to the application of the iterative method **5.2.1** to solve a strictly convex quadratic program. To illustrate such an application, consider the quadratic program (1.2.1) where the matrix Q is symmetric positive definite. The Karush-Kuhn-Tucker conditions define the LCP (q, M) with the vector q and matrix M given by (1.2.3). Since the matrix M there is not symmetric, we need to transform the LCP (1.2.2) into one which has a symmetric matrix. To accomplish this, we perform a block pivot on the matrix Q in the system (1.2.2), obtaining the equivalent system

$$\begin{aligned} x &= -Q^{-1}c + Q^{-1}u + Q^{-1}A^{\mathrm{T}}y \geq 0, & x \geq 0, \quad x^{\mathrm{T}}u = 0 \\ v &= -b - AQ^{-1}c + AQ^{-1}u + AQ^{-1}A^{\mathrm{T}}y \geq 0, & y \geq 0, \quad y^{\mathrm{T}}v = 0 \end{aligned} \tag{10}$$

which defines the alternate LCP (q, M) with

$$q = -\begin{bmatrix} Q^{-1}c \\ b + AQ^{-1}c \end{bmatrix}, \quad M = \begin{bmatrix} I \\ A \end{bmatrix} Q^{-1}[\, I \;\; A^{\mathrm{T}} \,]. \tag{11}$$

The above matrix M differs from the one in (1.2.3) in two major respects (although both of them are positive semi-definite). First, the former matrix is symmetric whereas the latter is not. Second, the latter matrix contains a principal submatrix (the lower right block) which is zero; on the other hand, if the matrix A contains no zero rows, then the matrix M in (11) has all its diagonal entries positive. Due to the presence of a zero principal submatrix in the matrix M of (1.2.3), the PSOR method fails to be applicable for solving the LCP (1.2.2).

In order to apply Algorithm **5.2.1** to solve the alternate LCP (10), let (B, C) be an arbitrary regular splitting of M in (11). Then, for any initial (u^0, y^0), the uniquely defined sequence $\{(u^\nu, y^\nu)\}$ generated by the algorithm induces a corresponding sequence $\{(x^\nu, v^\nu)\}$ defined by

$$\begin{aligned} x^\nu &= -Q^{-1}c + Q^{-1}u^\nu + Q^{-1}A^{\mathrm{T}}y^\nu \\ v^\nu &= -b - AQ^{-1}c + AQ^{-1}u^\nu + AQ^{-1}A^{\mathrm{T}}y^\nu. \end{aligned}$$

Note that vector x^ν refers to the original variables of the program (1.2.1) and v^ν the slack vector. Borrowing terminology from quadratic programming theory, we call $\{(x^\nu, v^\nu)\}$ the *primal* sequence, and $\{(u^\nu, y^\nu)\}$ the *dual* sequence. The following result summarizes the convergence properties of these two sequences.

5.3.11 Corollary. In the above setting, the following three statements hold:

(a) if the quadratic program (1.2.1) is feasible, then the sequence $\{x^\nu\}$ converges to the unique optimal solution of (1.2.1) and $\{v^\nu\}$ converges to an optimal slack vector;

(b) if (1.2.1) has a strictly feasible vector, i.e., if there exists a vector $x > 0$ such that $Ax > b$, then the dual sequence $\{(u^\nu, y^\nu)\}$ is bounded, and any of its accumulation points solves the LCP (q, M) with q and M given by (11);

(c) conversely, if for any vectors b and c and any initial nonnegative iterate (u^0, y^0), the sequence $\{(u^\nu, y^\nu)\}$ is bounded and any one of its accumulation points solves (10), then there must exist a vector $x > 0$ satisfying $Ax > 0$.

Proof. The first two parts follow from Theorem **5.3.9** and the fact that the optimal solution to the quadratic program (1.2.1) is unique. To prove part (c), we apply the converse part in Theorem **5.3.5**. Indeed according to that result, we may conclude that if the assumption in part (c) holds, then the matrix M given in (11) must be strictly copositive. By the symmetry and positive definiteness of the matrix Q, it is easy to establish that such a matrix M is strictly copositive if and only if

$$[u \geq 0, \; v \geq 0, \; u + A^{\mathrm{T}}v = 0] \quad \Rightarrow \quad (u, v) = (0, 0).$$

By Ville's theorem of the alternative, **2.7.11**, the latter implication is equivalent to the existence of a vector $x > 0$ such that $Ax > 0$. This completes the proof. □

One may generalize the above discussion to include equality constraints in the QP (1.2.1) and/or to the situation where not all the variables are restricted to be nonnegative. Unfortunately, this entire process of derivation of a symmetric LCP depends crucially on the invertibility of the matrix Q and breaks down if the objective function is merely convex but not strictly convex (as in the case of a linear program). For such problems, special care needs to be taken; one approach is described in Section 5.6 where a convex quadratic program is converted into a sequence of strictly convex ones by a strong convexification procedure.

Related to the requirement of strict convexity, the approach described above has another potential weakness in solving large sparse, non-separable quadratic programs. For these problems, the matrix Q is non-diagonal and the presence of Q^{-1} in the matrix M in (11)—especially if Q^{-1} is computed explicitly—could easily destroy the sparsity and/or other nice properties of the data. In light of this consideration, the alternate LCP (10) is typically useful only for solving separable problems; for non-separable problems, the idea of *diagonalization* offers a way of transforming the problems into a sequence of separable ones to which the above symmetrization scheme can be applied (see the subsection of Section 5.5 on a symmetric variational inequality approach).

The foregoing discussion points out an important issue concerning the implementation of an iterative scheme applied to solve the symmetric LCP (10). This is the fact that one should avoid forming the matrix M in (11) explicitly even in the case of a diagonal Q; the reason is that any potentially nice property of the matrix A could easily be destroyed in the explicit computation of M. Fortunately, in an iterative scheme such as the PSOR method, it is not difficult to implement the method with M represented in the product form as given in (11), see Exercise **5.11.4**.

The contraction approach

Unlike the symmetry approach, the contraction approach relies on no objective function but makes use of a contraction argument to establish the convergence of a sequence produced by Algorithm **5.2.1**. In turn, there are two types of contraction: norm contraction and vector contraction, each requiring a different property on the splitting (B, C) and thereby applicable to different realizations of the basic algorithm. Throughout the contraction approach, the matrix B in the splitting (B, C) is always a $\boldsymbol{P}$-matrix; thus each iterate $z^{\nu+1}$ is uniquely defined.

The basic convergence results derived under the contraction approach are of a universal nature in the sense that they apply to all vectors q with the same matrix M. Consequently, they constitute the analog of Theorem **5.3.5** established under the symmetry approach. These results require (implicitly or explicitly) that the matrix M be in class $\boldsymbol{P}$. As a consequence of this $\boldsymbol{P}$-property, the LCP (q, M) must have a unique solution to which all sequences of iterates $\{z^\nu\}$ converge for arbitrary starting z^0. Generalizations of the convergence results presented below that deal with a specific LCP with a fixed vector q (and under relaxation of the $\boldsymbol{P}$-property of M) can be obtained by refining the proof technique; see Theorem **5.6.1** and Exercise **5.11.8**.

The following is the basic convergence result under the norm contraction argument.

5.3.12 Theorem. Suppose that B is a positive definite matrix. Let $\tilde{B}$ be any nonsingular matrix such that $\tilde{B}^{\mathrm{T}}\tilde{B} = \frac{1}{2}(B + B^{\mathrm{T}})$. Suppose also

$$\rho(\tilde{B}, C) := \|(\tilde{B}^{-1})^{\mathrm{T}} C \tilde{B}^{-1}\|_2 < 1. \tag{12}$$

Then, the matrix $M = B + C$ is positive definite. Moreover, for an arbitrary vector q and any starting vector $z^0 \geq 0$, the uniquely defined sequence of iterates $\{z^\nu\}$ produced by Algorithm **5.2.1** satisfies

$$\|\tilde{B}(z^{\nu+1} - z^\nu)\|_2 \leq \rho(\tilde{B}, C)\|\tilde{B}(z^\nu - z^{\nu-1})\|_2, \tag{13}$$

and converges to the unique solution of the LCP (q, M).

Proof. We observe that the matrix M is positive definite if and only if the matrix $\frac{1}{2}(B + B^{\mathrm{T}}) + C$ is so. Since $\tilde{B}$ is nonsingular and

$$\tfrac{1}{2}(B + B^{\mathrm{T}}) + C = \tilde{B}^{\mathrm{T}}(I + (\tilde{B}^{-1})^{\mathrm{T}} C \tilde{B}^{-1})\tilde{B},$$

it follows that M is positive definite if and only if $I + (\tilde{B}^{-1})^{\mathrm{T}} C \tilde{B}^{-1}$ is so. It is not difficult to verify that the latter matrix must be positive definite if the quantity $\rho(\tilde{B}, C)$ is less than one as we have assumed. This establishes the positive definiteness of M.

To establish the contraction (13), we note that for each ν,

$$w^{\nu+1} = q + Cz^\nu + Bz^{\nu+1} \geq 0, \quad z^{\nu+1} \geq 0, \quad (w^{\nu+1})^{\mathrm{T}} z^{\nu+1} = 0.$$

Thus, we have

$$\begin{aligned} 0 &\geq (z^{\nu+1} - z^\nu)^{\mathrm{T}}(w^{\nu+1} - w^\nu) \\ &= (z^{\nu+1} - z^\nu)^{\mathrm{T}} C (z^\nu - z^{\nu-1}) + (z^{\nu+1} - z^\nu)^{\mathrm{T}} B (z^{\nu+1} - z^\nu). \end{aligned}$$

Rearranging the terms and noting the identity $x^{\mathrm{T}}Bx = \|\tilde{B}x\|_2^2$, we derive

$$\begin{aligned} \|\tilde{B}(z^{\nu+1} - z^\nu)\|_2^2 &\leq -(\tilde{B}(z^{\nu+1} - z^\nu))^{\mathrm{T}}((\tilde{B}^{-1})^{\mathrm{T}} C \tilde{B}^{-1})(\tilde{B}(z^\nu - z^{\nu-1})) \\ &\leq \|\tilde{B}(z^{\nu+1} - z^\nu)\|_2 \; \|(\tilde{B}^{-1})^{\mathrm{T}} C \tilde{B}^{-1}\|_2 \; \|\tilde{B}(z^\nu - z^{\nu-1})\|_2 \end{aligned}$$

where the last inequality follows from the Cauchy-Schwartz inequality. Cancelling one factor $\|\tilde{B}(z^{\nu+1} - z^{\nu})\|_2$, we obtain the desired inequality (13). It follows from this inequality and the contraction principle that the sequence $\{z^{\nu}\}$ converges; it is then a simple matter to verify that the limit vector solves the LCP (q, M). The uniqueness of the solution follows by the positive definiteness of the matrix M. This completes the proof. □

Part of the assertion in Theorem **5.3.12** is that if the quantity $\rho(\tilde{B}, C)$ is less than unity, then the matrix M must be positive definite. As a matter of fact, a partial converse of this statement holds in the sense that if M is positive definite, then for any symmetric positive definite matrix G, one can always choose B to be a suitable positive multiple of G and ensure that the quantity $\rho(\tilde{B}, C)$ is less than one. This converse is the second conclusion in the following result.

5.3.13 Proposition. Let M and B be positive definite with $M = B + C$ and B is symmetric. Let $\tilde{B}$ be any matrix such that $\tilde{B}^{\mathrm{T}}\tilde{B} = B$. Then,

(a) $\rho(\tilde{B}, C) < 1$ if and only if $2M - M^{\mathrm{T}}B^{-1}M$ is positive definite;

(b) if G is any symmetric positive definite matrix and if $B = \lambda G$ with

$$\lambda > \frac{\|M\|_2^2}{2\mu\gamma}$$

where μ and γ denote, respectively, the smallest eigenvalue of the matrices $\frac{1}{2}(M + M^{\mathrm{T}})$ and G, then $\rho(\tilde{B}, C) < 1$.

Proof. The quantity $\rho(\tilde{B}, C) < 1$ if and only if for any nonzero vector x,

$$\|(\tilde{B}^{-1})^{\mathrm{T}}C\tilde{B}^{-1}x\|_2^2 < x^{\mathrm{T}}x.$$

Let $y = \tilde{B}^{-1}x$. Then,

$$x^{\mathrm{T}}x = y^{\mathrm{T}}(\tilde{B}^{\mathrm{T}}\tilde{B})y = y^{\mathrm{T}}By$$

and by the symmetry of B,

$$\begin{aligned}\|(\tilde{B}^{-1})^{\mathrm{T}}C\tilde{B}^{-1}x\|_2^2 &= y^{\mathrm{T}}C^{\mathrm{T}}(\tilde{B}^{\mathrm{T}}\tilde{B})^{-1}Cy \\ &= y^{\mathrm{T}}(M^{\mathrm{T}} - B)B^{-1}(M - B)y \\ &= y^{\mathrm{T}}M^{\mathrm{T}}B^{-1}My - 2y^{\mathrm{T}}My + y^{\mathrm{T}}By.\end{aligned}$$

The desired equivalence in (a) now follows easily.

To prove part (b), we first point out that the scalars μ and γ are positive by the positive definiteness of M and G respectively. By a straightforward manipulation, it is not difficult to verify that with the given choice of λ and B, the matrix $2M - M^{\mathrm{T}}B^{-1}M$ is indeed positive definite. Thus, the desired conclusion follows from part (a). □

A simple choice of G in **5.3.13** is the identity matrix. With such a choice, the above results establish the convergence of the following trivial iteration:

$$z^{\nu+1} = \max(0, z^{\nu} - (q + Mz^{\nu})/\lambda). \tag{14}$$

Specifically, for $\lambda > \|M\|_2^2/(2\mu)$, this sequence $\{z^{\nu}\}$ converges to the unique solution of the LCP (q, M) if M is positive definite. More generally, **5.3.12** and **5.3.13** together provide the convergence for a broad class of *symmetrization* methods for solving a positive definite LCP by a sequence of symmetric linear complementarity subproblems. The potential advantage of such a symmetrization scheme is that each of the symmetric subproblems can in turn be solved by an iterative method (like an SOR scheme) whose convergence can be established under the (less restrictive) symmetry approach.

The vector contraction argument requires a set of properties on the splitting (B, C) which are somewhat different from those in the norm contraction approach. Under the vector contraction approach, the sequence $\{z^{\nu}\}$ is shown to contract in the vector sense; such contraction is a stronger property than the norm contraction used previously and implies the latter. The following lemma identifies a characterizing property of a $\boldsymbol{K}$-matrix relevant to the vector contraction argument.

5.3.14 Lemma. Let M be a $\boldsymbol{K}$-matrix and N a nonnegative matrix. Then $\rho(M^{-1}N) < 1$ if and only if $M - N$ is a $\boldsymbol{K}$-matrix.

Proof. Consider first the case where N is a positive matrix. Since M is a $\boldsymbol{K}$-matrix, M^{-1} is a nonnegative matrix by **3.11.10**(c). Thus, $M^{-1}N$ is a positive matrix. By Theorem **2.2.21**, there exists a vector $x > 0$ such that $M^{-1}Nx = \rho(M^{-1}N)x > 0$. Then, $Nx = \rho(M^{-1}N)Mx$ and thus,

$$(M - N)x = (\rho(M^{-1}N)^{-1} - 1)Nx.$$

Suppose $\rho(M^{-1}N) < 1$. Then, $(M - N)x > 0$. Since $M - N$ is obviously a $\boldsymbol{Z}$-matrix, it follows from **3.11.10**(d) that $M - N$ is a $\boldsymbol{K}$-matrix. Conversely, if $M - N$ is in the class $\boldsymbol{K}$, then $(M - N)^{-1}$ exists and is nonnegative. So, we have

$$0 < x = (\rho(M^{-1}N)^{-1} - 1)(M - N)^{-1}Nx$$

which implies that $\rho(M^{-1}N) < 1$. Consequently, the lemma is proved for a positive matrix N.

In general, suppose N is a nonnegative matrix. If $\rho(M^{-1}N) < 1$, then the same is true with N replaced by $N + \varepsilon E$ where $\varepsilon > 0$ is a sufficiently small scalar and E is the matrix of all ones. Thus, the above proof shows that the matrix $M - (N + \varepsilon E)$ is a $\boldsymbol{K}$-matrix. Since

$$M - N \geq M - (N + \varepsilon E),$$

it follows that $M - N$ is also a $\boldsymbol{K}$-matrix. Conversely, if $M - N$ is in class $\boldsymbol{K}$, then so is $M - (N + \varepsilon E)$ for all $\varepsilon > 0$ sufficiently small. Consequently, $\rho(M^{-1}(N + \varepsilon E)) < 1$ by the previous argument. Since $\rho(M^{-1}N) \leq \rho(M^{-1}(N + \varepsilon E))$, the desired conclusion $\rho(M^{-1}N) < 1$ follows readily. □

The following is the main convergence result using the vector contraction argument.

5.3.15 Theorem. Suppose that B is an $\boldsymbol{H}$-matrix with positive diagonals. Let $\bar{B}$ denote the comparison matrix of B. Suppose

$$\| \bar{B}^{-1}|C| \, \| < 1 \tag{15}$$

for some monotone norm $\| \cdot \|$. Then, the matrix $M = B + C$ is itself an $\boldsymbol{H}$-matrix with positive diagonals. Moreover, for an arbitrary vector q and any starting vector $z^0 \geq 0$, the uniquely defined sequence of iterates $\{z^\nu\}$ produced by Algorithm **5.2.1** satisfies

$$\bar{B}|z^{\nu+1} - z^\nu| \leq |C||z^\nu - z^{\nu-1}|, \tag{16}$$

and it converges to the unique solution of the LCP (q, M).

Proof. The norm condition (15) implies that $\rho(\bar{B}^{-1}|C|) < 1$. Thus, it follows from Lemma **5.3.14** that the matrix $\bar{B} - |C|$ is in class $\boldsymbol{K}$. This implies in particular that all the diagonal entries of M must be positive. Moreover, it is easy to see that $\bar{M} \geq \bar{B} - |C|$ where $\bar{M}$ is the comparison matrix of M. Consequently, $\bar{M}$ is also in class $\boldsymbol{K}$, and M is an $\boldsymbol{H}$-matrix with positive diagonals.

To establish the expression (16), we first remark that each iterate $z^{\nu+1}$ is uniquely defined because any $\boldsymbol{H}$-matrix with positive diagonals must be in class $\boldsymbol{P}$ (see Section 3.3). We verify (16) component by component. Consider an arbitrary index i and assume that

$$|z^{\nu+1} - z^{\nu}|_i = (z^{\nu+1} - z^{\nu})_i.$$

Under this assumption, the inequality (16) clearly holds if $z_i^{\nu+1} = 0$ because the i-th component of the left-hand vector in (16) is then nonpositive and the right-hand component is always nonnegative. Suppose $z_i^{\nu+1} > 0$. Then, we have

$$(q + Cz^{\nu} + Bz^{\nu+1})_i = 0.$$

On the other hand, we also have

$$(q + Cz^{\nu-1} + Bz^{\nu})_i \geq 0.$$

Subtracting the last two expressions and rearranging terms, we deduce

$$(B(z^{\nu+1} - z^{\nu}))_i \leq -(C(z^{\nu} - z^{\nu-1}))_i$$

which implies

$$(\bar{B}|z^{\nu+1} - z^{\nu}|)_i \leq (|C||z^{\nu} - z^{\nu-1}|)_i \tag{17}$$

because $b_{ii} > 0$ and $|z^{\nu+1} - z^{\nu}|_i = (z^{\nu+1} - z^{\nu})_i$. In a similar fashion, we may establish the same inequality (17) if $|z^{\nu+1} - z^{\nu}|_i = (z^{\nu} - z^{\nu+1})_i$. Consequently, the inequality (16) must hold. Since $\bar{B}$ has a nonnegative inverse by the $\boldsymbol{H}$-property of the matrix B, it follows that

$$|z^{\nu+1} - z^{\nu}| \leq \bar{B}^{-1}|C||z^{\nu} - z^{\nu-1}|.$$

Since the norm $\|\cdot\|$ is monotone, we obtain

$$\|z^{\nu+1} - z^{\nu}\| \leq \|\,\bar{B}^{-1}|C|\,\|\;\|z^{\nu} - z^{\nu-1}\|$$

which establishes the contraction property of the sequence $\{z^\nu\}$ and also the theorem. □

Part of the assertion of Theorem **5.3.15** is that the matrix M must itself be an $\boldsymbol{H}$-matrix with positive diagonals if the splitting (B, C) satisfies the assumed properties. The following result shows that conversely, if M is a given $\boldsymbol{H}$-matrix with positive diagonals, then the PSOR splitting given by (5.2.1), with the relaxation parameter ω suitably restricted, must satisfy the norm condition (15) and thus the convergence conclusion in **5.3.15** holds for the corresponding SOR method.

5.3.16 Corollary. Let M be an $\boldsymbol{H}$-matrix with positive diagonals. Let $M = D+L+U$ be the decomposition of M into its diagonal, strictly lower and strictly upper triangular parts respectively. Let $B = L + \omega^{-1}D$ where $\omega > 0$. Then, there exists an $\bar{\omega} \in (1, 2]$ such that for all $\omega \in (0, \bar{\omega})$, the convergence conclusion in **5.3.15** holds for the PSOR splitting (B, C) .

Proof. By the $\boldsymbol{H}$-property of M, there exists a positive vector d such that $\bar{M}d > 0$ where $\bar{M}$ is the comparison matrix of M. Define the vector norm

$$\|z\|_d = \max_i d_i^{-1}|z_i|$$

which is clearly monotone. Let

$$\bar{\omega} = 2 \min_i \frac{m_{ii}d_i}{\sum_j |m_{ij}|d_j}.$$

It is then easy to verify that $\bar{\omega} \in (1, 2]$. Let $\omega \in (0, \bar{\omega})$ be given. Clearly, the matrix B has all diagonal entries positive. We show that B is an $\boldsymbol{H}$-matrix. It suffices to verify that

$$(D - \omega|L|)d > 0. \tag{18}$$

If $\omega \leq 1$, (18) holds because $(D - \omega|L|)d \geq \bar{M}d > 0$. On the other hand, if $\omega \in (1, \bar{\omega})$, then by the choice of $\bar{\omega}$, we have

$$\begin{aligned}(D - \omega|L|)d &\geq ((1 + \omega)D - \omega(D + |L| + |U|))d \\ &\geq (\omega - 1)Dd > 0.\end{aligned}$$

We now verify that the condition (15) holds for the induced matrix norm $\|\cdot\|_d$. Let z be any vector such that $\|z\|_d = 1$. It suffices to show that

$\|y\|_d < 1$ where

$$y = (D - \omega|L|)^{-1}(|1 - \omega|D + \omega|U|)z.$$

We show by induction on i that $d_i^{-1}|y_i| < 1$ for all i. Let $\tilde{z}_i = d_i^{-1}|z_i|$ and $\tilde{y}_i = d_i^{-1}|y_i|$. Then, $\tilde{z}_i \leq 1$ for each i. Since

$$(D - \omega|L|)y = (|1 - \omega|D + \omega|U|)z, \tag{19}$$

we have

$$\tilde{y}_1 \leq (|1 - \omega|m_{11}d_1\tilde{z}_1 + \omega \sum_{j\neq 1} |m_{1j}|d_j\tilde{z}_j)/(m_{11}d_1)$$

$$\leq (|1 - \omega|m_{11}d_1 + \omega \sum_{j\neq 1} |m_{1j}|d_j)/(m_{11}d_1).$$

If $\omega \leq 1$, then

$$\tilde{y}_1 \leq ((1 - \omega)m_{11}d_1 + \omega \sum_{j\neq 1} |m_{1j}|d_j)/(m_{11}d_1) < 1$$

because $(\bar{M}d)_1 > 0$. On the other hand, if $\omega \in (1, \bar{\omega})$, then

$$\tilde{y}_1 \leq ((\omega - 1)m_{11}d_1 + \omega \sum_{j\neq 1} |m_{1j}|d_j)/(m_{11}d_1) < 1$$

by the choice of $\bar{\omega}$. Now, suppose that $\max_{1\leq i\leq k-1} \tilde{y}_i < 1$. Then from (19), we obtain

$$\tilde{y}_k \leq (|1 - \omega|m_{kk}d_k\tilde{z}_k + \omega \sum_{j>k} |m_{kj}|d_j\tilde{z}_j + \omega \sum_{j<k} |m_{kj}|d_j\tilde{y}_j)/(m_{kk}d_k)$$

$$\leq (|1 - \omega|m_{kk}d_k + \omega \sum_{j\neq k} |m_{kj}|d_j)/(m_{kk}d_k).$$

By the same argument as for the case $k = 1$, we deduce that $\tilde{y}_k < 1$ completing the induction and the proof. □

Besides the basic differences between a typical contraction result and a symmetry result, Corollary **5.3.16** differs from **5.3.6** in two additional respects. First, **5.3.6** deals with the family of block SOR methods, whereas **5.3.16** applies only to the point methods. Second, the range of permissible

ω values is $(0, 2)$ in the former result, whereas that in the latter result is $(0, \bar{\omega})$ with $\bar{\omega}$ generally less than 2.

The opening discussion in this subsection pointed out the general limitations of the contraction approach, and these are verified by the various results that follow. In summary, the analysis suggests that as far as the solution by iterative methods is concerned, the symmetric LCP tends to be somewhat easier to handle than the asymmetric LCP in the sense that the convergence of the methods can be derived under less restrictive assumptions when the problem is defined by a symmetric matrix. Such restriction is an important motivation for introducing the class of symmetrization methods in which an asymmetric LCP is solved by a sequence of symmetric subproblems. It remains an unresolved issue as to whether more general convergence results can be derived by the contraction arguments.

The monotonicity approach

In the monotone approach for convergence, a certain transformation of the sequence $\{z^\nu\}$ is shown to possess a monotonicity property. The tool to derive the main result is the least-element theory presented in Section 3.11.

5.3.17 Theorem. Let (B, C) be a splitting of M. Suppose that there exists a $\boldsymbol{K}$-matrix X such that the matrix $BX = Y$ is in class $\boldsymbol{Z}$ and that $CX \leq 0$. Suppose also that the LCP (q, M) is feasible; let $z^0 \in \text{FEA}(q, M)$ be given. Then,

(a) both $M = B + C$ and B are hidden $\boldsymbol{Z}$-matrices;

(b) for each ν, the LCP (q^ν, B) has a solution $z^{\nu+1}$ which is feasible to (q, M) and satisfies

$$0 \leq X^{-1} z^{\nu+1} \leq X^{-1} z^\nu; \tag{20}$$

(c) the sequence $\{z^\nu\}$ in (b) converges to a solution of the LCP (q, M).

One distinction between Theorem **5.3.17** and the results obtained from the symmetry and the contraction approaches lies in the sequence of iterates involved. In the symmetry approach, the convergence holds for any sequence generated by Algorithm **5.2.1**; in the contraction approach, the property of the splitting (B, C) implies the uniqueness of the sequence produced; in **5.3.17**, the conclusion applies to a particular sequence as specified in part (b) of the above theorem. According to the theorem, each vector in this sequence is feasible to the given LCP (q, M), provided that the starting vector is so chosen. While this feasibility property is not required in the previous approaches, it is essential in the proof that follows.

Proof of 5.3.17. That B is a hidden $\boldsymbol{Z}$-matrix is clear. Note that $MX = Y + CX$; since $CX \leq 0$, the matrix $Y + CX$ belongs to $\boldsymbol{Z}$. So, M is a hidden $\boldsymbol{Z}$- matrix. This proves part (a).

To prove part (b), suppose that $z^\nu \in \text{FEA}(q, M)$. Then, it is easy to see that z^ν is also feasible for the LCP (q^ν, B). By the necessity part of Theorem **3.11.18** and its proof, it follows that the LCP (q^ν, B) has a least-element solution $z^{\nu+1}$ satisfying

$$X^{-1} z^{\nu+1} \leq X^{-1} z$$

for any vector z which is feasible to (q^ν, B). In particular, we deduce $X^{-1}z^{\nu+1} \leq X^{-1}z^\nu$. Since $X \in \boldsymbol{K}$, it follows that X^{-1} is nonnegative; thus, so is $X^{-1}z^{\nu+1}$. Since CX is nonpositive, it follows that $Cz^{\nu+1} = CX(X^{-1}z^{\nu+1}) \geq CX(X^{-1}z^\nu) = Cz^\nu$, and so $z^{\nu+1}$ is also feasible for the LCP (q, M). Part (b) now follows by a simple inductive argument. The monotonicity property (20) implies that the sequence $\{X^{-1}z^\nu\}$ converges; thus, so does $\{z^\nu\}$. As we have seen in several instances before, the limit of the sequence $\{z^\nu\}$ must solve the LCP (q, M). $\square$

According to the above proof and the least-element theory of Section 3.11, the vector $z^{\nu+1}$ in part (b) of **5.3.17** can be obtained by solving the linear program

$$\begin{array}{ll} \text{minimize} & p^{\mathrm{T}} z \\ \text{subject to} & q^\nu + Bz \geq 0 \\ & z \geq 0 \end{array}$$

for any vector p satisfying $p^{\mathrm{T}}X > 0$. This provides a way of constructing the sequence $\{z^\nu\}$. If B is itself a $\boldsymbol{Z}$-matrix, then an alternative way to compute $z^{\nu+1}$ is by means of the special methods described in Section 4.7.

Theorem **5.3.17** admits a slight simplification when both M and B are $\boldsymbol{Z}$-matrices and $M \leq B$. In this case, the matrix X may be chosen as the identity and the condition $CX \leq 0$ obviously holds because $C = M - B \leq 0$. The further specialization of B as a block diagonal matrix is of interest because this choice leads to a parallel iterative method for solving the LCP (q, M). In particular, if M is partitioned into submatrices M_{ij} with each diagonal block M_{ii} being square, then one choice for B is the block diagonal matrix whose diagonal blocks are those of M, see (5.2.2). By specializing **5.3.17** to this setting, we derive the following consequence.

5.3.18 Corollary. Let M be a $\boldsymbol{Z}$-matrix partitioned into submatrices M_{ij} such that each diagonal submatrix M_{ii} is square. Let B be the block diagonal matrix consisting of the diagonal blocks M_{ii}. Suppose that the LCP (q, M) is feasible. Then, provided that the initial vector z^0 is chosen feasible to (q, M), the sequence of vectors $\{z^\nu\}$, with each $z^{\nu+1}$ being the least-element solution of the LCP (q^ν, B), is well defined and converges monotonically to some solution of the LCP (q, M). □

5.4 Convergence of Iterates: Symmetric LCP

Employing three different approaches for convergence, we have discussed the limiting properties of a sequence $\{z^\nu\}$ produced by the basic splitting method **5.2.1**. Under the contraction and monotone approaches, the convergence of the whole sequence $\{z^\nu\}$ is established; under the symmetry approach, only the subsequential convergence property of $\{z^\nu\}$ has been analyzed. In the sequel, we extend the results for the symmetric positive semi-definite problem and establish the convergence of the iterates in this case.

The positive semi-definite case

We consider the iterative method **5.2.1** for solving the LCP (q, M) where the matrix M is assumed to be symmetric positive semi-definite and (q, M) is assumed solvable. Our goal here is to show that any sequence $\{z^\nu\}$ produced by **5.2.1** with a regular splitting (B, C) of M converges to a solution of (q, M) under the assumed properties of (q, M). The proof of this conclusion relies on the convergence of $\{w^\nu\} = \{q + Mz^\nu\}$ established in Theorem **5.3.9** and on the convergence of an auxiliary sequence as well as on the rate of convergence of the latter sequence.

Since by assumption, M is symmetric positive semi-definite, we may write

$$M = AA^{\mathrm{T}}$$

for some matrix $A \in R^{n \times n}$. With this factorization of M, we define the auxiliary sequence $\{y^\nu\}$ where

$$y^\nu = A^{\mathrm{T}} z^\nu.$$

The proof of the convergence of the sequence $\{z^\nu\}$ consists of two stages; the first stage is to establish the convergence of the sequence $\{y^\nu\}$ to a vector, say $\bar{y}$, and the second stage is to derive an inequality of the type: for sufficiently large ν,

$$\sigma \|y^{\nu+1} - \bar{y}\|^2 \leq \eta \|z^{\nu+1} - z^\nu\|^2 \leq \|y^\nu - \bar{y}\|^2 - \|y^{\nu+1} - \bar{y}\|^2 \tag{1}$$

where σ and η are certain positive constants and where $\|\cdot\|$ denotes the l_2-norm of vectors. In turn, the convergence of $\{y^\nu\}$ and the right-hand inequality in (1) are not difficult to establish. The most involved part of the whole argument is the derivation of the left-hand inequality in (1). We first summarize the convergence properties of the sequence $\{y^\nu\}$ in the result below. In this and the subsequent results, the setting is as just described, and the statement of the assumptions is not repeated.

5.4.1 Proposition. In the above setting, the two statements below hold.

(a) The sequence $\{y^\nu\}$ converges to a vector $\bar{y}$ of the form $A^{\mathrm{T}}\bar{z}$ where $\bar{z}$ solves the LCP (q, M).

(b) There exists a ν_1 such that for every $\nu \geq \nu_1$, the limit $\bar{y}$ is equal to $A^{\mathrm{T}} z^*$ for some $z^* \in R^n$ (which depends on ν) such that $z_i^* = 0$ for every i satisfying $z_i^{\nu+1} = 0$.

Proof. By Theorem **5.3.9**, the sequence $\{Mz^\nu\}$ converges to some vector $M\bar{z}$ where $\bar{z}$ solves (q, M). Hence, the sequence $\{AA^{\mathrm{T}}(z^\nu - \bar{z})\} \to 0$. By an argument analogous to the proof of Lemma **5.3.10**, we may deduce $\{A^{\mathrm{T}}(z^\nu - \bar{z})\} \to 0$. Hence, part (a) follows.

The proof of (b) is by contradiction. Suppose the assertion is false. Then there exists an infinite index set κ such that for every index $\nu \in \kappa$,

$$\bar{y} = A^{\mathrm{T}} z^* \quad \Rightarrow \quad z^*_{J(\nu)} \neq 0 \tag{2}$$

where

$$J(\nu) = \{i : z_i^{\nu+1} = 0\}.$$

Since the set of possible $J(\nu)$ is finite, there exists an infinite subset $\kappa' \subseteq \kappa$ such that

$$J(\nu) = J^* \quad \text{for all } \ \nu \in \kappa'.$$

Since $\{z^{\nu+1} - z^{\nu}\} \to 0$, $\bar{y}$ is the limit of $\{A^{\mathrm{T}} z^{\nu+1} : \nu \in \kappa'\}$. By Theorem **2.6.24** again, it follows that $\bar{y}$ is equal to $A^{\mathrm{T}} z$ for some vector $z \in R^n$ satisfying $z_{J^*} = 0$. But this contradicts the implication (2). Consequently, part (b) is proved. □

The next result formally asserts the validity of the right-hand inequality in (1) for large ν.

5.4.2 Lemma. There exists a constant $\eta > 0$ such that for all ν large enough,

$$\eta \| z^{\nu+1} - z^{\nu} \|^2 \leq \| y^{\nu} - \bar{y} \|^2 - \| y^{\nu+1} - \bar{y} \|^2.$$

Proof. Since $w^{\nu} = q + M z^{\nu}$ converges to $\bar{w} = q + M \bar{z}$, it follows that for all ν sufficiently large,

$$(z^{\nu})^{\mathrm{T}} \bar{w} = 0. \tag{3}$$

Let

$$f(z) = q^{\mathrm{T}} z + \tfrac{1}{2} z^{\mathrm{T}} M z.$$

We have

$$\begin{aligned}
&\tfrac{1}{2}(\| y^{\nu} - \bar{y} \|^2 - \| y^{\nu+1} - \bar{y} \|^2) \\
&= \tfrac{1}{2}(z^{\nu} - \bar{z})^{\mathrm{T}} M (z^{\nu} - \bar{z}) - \tfrac{1}{2}(z^{\nu+1} - \bar{z})^{\mathrm{T}} M (z^{\nu+1} - \bar{z}) \\
&= \tfrac{1}{2}((z^{\nu})^{\mathrm{T}} M z^{\nu} - (z^{\nu+1})^{\mathrm{T}} M z^{\nu+1}) + (z^{\nu+1} - z^{\nu})^{\mathrm{T}} M \bar{z} \\
&= f(z^{\nu}) - f(z^{\nu+1}) + (z^{\nu+1} - z^{\nu})^{\mathrm{T}} (q + M \bar{z}) \\
&\geq \tfrac{1}{2}(z^{\nu} - z^{\nu+1})^{\mathrm{T}} (B - C)(z^{\nu} - z^{\nu+1})
\end{aligned}$$

where the last inequality follows by (3) and Lemma **5.3.2**. Now, it suffices to take η to be the smallest eigenvalue of the symmetric part of the matrix $B - C$ which is positive definite by the regular property of the splitting (B, C) . □

Next, we turn our attention to the other inequality in the expression (1). We first establish an immediate consequence of part (b) in **5.4.1** which asserts a certain proximity property of the the limit vector $\bar{y}$ in relation to the iterate $y^{\nu+1}$ for all ν sufficiently large. In order to state this property, let $I(\nu)$ denote the complement of the index set $J(\nu)$ in $\{1, \ldots, n\}$, i.e.,

$$I(\nu) = \{i : z_i^{\nu+1} > 0\}.$$

Define the affine subspace

$$S_\nu = \{y \in R^n : (q + Ay)_i = 0 \text{ for all } i \in I(\nu)\}.$$

The following is the asserted proximity property of $\bar{y}$.

5.4.3 Proposition. There exists ν_2 such that for all $\nu \geq \nu_2$, $\bar{y}$ is the projection of $y^{\nu+1}$ onto the affine subspace S_ν under the l_2-norm.

Proof. Since $w^{\nu+1} \to \bar{w}$ as $\nu \to \infty$, it follows that there exists ν' such that for all $\nu \geq \nu'$,

$$i \in I(\nu) \quad \Rightarrow \quad \bar{w}_i = (q + A\bar{y})_i = 0,$$

(cf. equation (3)). Hence, $\bar{y} \in S_\nu$ for all $\nu \geq \nu'$. Let $\nu_2 = \max(\nu_1, \nu')$ where ν_1 is as given by **5.4.1**. Then, for each $\nu \geq \nu_2$, let z^* be the corresponding vector obtained in part (b) of this last proposition. We have

$$\bar{y} = y^{\nu+1} + (\bar{y} - y^{\nu+1}) = y^{\nu+1} + \sum_{i \in I(\nu)} (z^* - z^{\nu+1})_i (A_{i\cdot})^{\mathrm{T}}.$$

This is enough to establish the desired projection property of $\bar{y}$. □

Having established the special projection property of the limit $\bar{y}$, we proceed to derive an upper bound of the error $\|\bar{y} - y^{\nu+1}\|$. This derivation is based on the next result which will later be generalized in Chapter 7.

5.4.4 Proposition. Let $H \in R^{m\times n}$ and $b \in R^m$ be given, and let

$$V = \{x \in R^n : Hx + b = 0\}.$$

Then, there exists a constant $L > 0$ such that for any vector $y \in R^n$

$$\|y - y^*\| \leq L\|b + Hy\|$$

where y^* denotes the orthogonal projection of y onto V.

Proof. Exercise **5.11.6** contains a sketch of the proof for this result. □

We apply the last proposition to the affine subspaces S_ν. Since there is only a finite number of possible index sets $I(\nu)$, we deduce the existence of a constant $L' > 0$ such that for all ν sufficiently large,

$$\|\bar{y} - y^{\nu+1}\| \leq L'\|(q + Ay^{\nu+1})_{I(\nu)}\|.$$

For an index $i \in I(\nu)$, we have

$$0 = (q + Cz^\nu + Bz^{\nu+1})_i = (q + C(z^\nu - z^{\nu+1}) + Ay^{\nu+1})_i.$$

Hence,

$$\|(q + Ay^{\nu+1})_{I(\nu)}\| \leq \|C\|\ \|z^\nu - z^{\nu+1}\|.$$

By combining the last two inequalities, we have proven the following result.

5.4.5 Corollary. There exists a constant $\sigma' > 0$ such that for all ν sufficiently large,

$$\|\bar{y} - y^{\nu+1}\| \leq \sigma'\|z^{\nu+1} - z^\nu\|.$$

In particular, the left-hand inequality in (1) holds. □

With the above results, we may now formally state and prove the desired convergence of the entire sequence of iterates $\{z^\nu\}$ produced by the splitting method **5.2.1** for solving a feasible LCP (q, M) with a regular splitting (B, C) of the symmetric positive semi-definite matrix M.

5.4.6 Theorem. Let $M \in R^{n\times n}$ be symmetric positive semi-definite, and let (q, M) be solvable. Let (B, C) be a regular splitting of M, and $\{z^\nu\}$ be a sequence produced by **5.2.1** with an arbitrary starting vector $z^0 \in R^n$. Then, $\{z^\nu\}$ converges to a solution of (q, M).

Proof. The inequality (1) implies

$$\|y^{\nu+1} - \bar{y}\| \leq \rho \|y^\nu - \bar{y}\| \tag{4}$$

where $\rho = (1+\sigma)^{-\frac{1}{2}}$. Hence, for all ν large enough,

$$\|y^{\nu+1} - \bar{y}\| \leq \rho^\nu \|y^0 - \bar{y}\|.$$

Since $\rho < 1$, the infinite series $\sum_{\nu=0}^\infty \|y^\nu - \bar{y}\|$ converges. For each index i, we have

$$z_i^{\nu+1} = z_i^0 + \sum_{\iota=0}^{\nu} (z_i^{\iota+1} - z_i^\iota).$$

The series $\sum_{\iota=1}^\infty (z_i^{\iota+1} - z_i^\iota)$ is absolutely convergent because

$$\sum_{\iota=0}^{\nu} |z_i^{\iota+1} - z_i^\iota| \leq (\eta)^{-\frac{1}{2}} \sum_{\iota=0}^{\nu} \|y^\iota - \bar{y}\|$$

by Lemma **5.4.2**. Hence, the sequence $\{z^\nu\}$ converges. By Theorem **5.3.3**, the limit of $\{z^\nu\}$ belongs to SOL(q, M). □

In the preceding analysis, the inequality (1) has played a major role. When the matrix M is not positive semi-definite, this derivation breaks down because the auxiliary sequence $\{y^\nu\}$ no longer can be defined. Yet, it is still possible to derive a related inequality in terms of the sequence $\{f(z^\nu)\}$. We postpone this extension until Chapter 7 where the extended analysis will be given as a consequence of some sensitivity properties of the LCP.

The inequality (4) shows that the auxiliary sequence $\{y^\nu\}$ converges to $\bar{y}$ at a geometric rate. From this, a similar rate result can be obtained for the primary sequence $\{z^\nu\}$. A more general form of this latter assertion is established in Corollary **7.2.12**.

5.5 Splitting Methods With Line Search

The basic splitting algorithm **5.2.1** admits diverse realizations to which the convergence results established in the previous two sections are applicable. Despite such wide applicability, when these results are specialized to certain iterative methods, they tend to require some strong conditions on either the splitting (B, C) or the matrix M; as a result, rather restrictive conclusions are obtained. We illustrate this point with the projected Jacobi method which has the matrix B equal to the diagonal part (assumed positive) of M. In order for the symmetry results to be applicable to this method, the Jacobi splitting is required to be regular, in other words, the matrix $2B - M$ is required to be positive definite. This, of course, means that the matrix M must satisfy some additional properties besides being symmetric and having positive diagonal entries. Similarly, some special $\boldsymbol{P}$- (or $\boldsymbol{Z}$-) property on M is needed for the contraction (or the monotone) results to be applicable. Ideally, it would be desirable for this simple method (or its modification) to converge under some weaker assumptions on M.

The above illustration points to a general limitation of the convergence results of the last section; this is the fact that they are not applicable to the LCP (q, M) where the matrix M is an asymmetric positive semi-definite matrix or a $\boldsymbol{P}$-matrix. In this and the next section, we describe various ways to overcome the aforementioned restrictions of the basic splitting algorithm and its convergence theory.

The symmetric case

One way to enlarge the domain of applicability of the splitting algorithm **5.2.1** is to place the LCP (q, M) in the context of an optimization problem. With this in place, the next step is to consider the subproblem of solving the LCP (q^ν, B) as a direction-finding routine. This is then followed by a one-dimensional linesearch on a certain objective function. In order for this conceptual approach to be practically successful, it is essential that a solution of (q^ν, B) yields a search direction along which a suitable merit function can be decreased. In the terminology of **2.5.10**, we hope to identify a real-valued function for which a solution of the latter LCP would be a descent direction at the iterate z^ν. The question, of course, is: what is such a function? One candidate is

$$f(z) = q^{\mathrm{T}}z + \tfrac{1}{2}z^{\mathrm{T}}Mz,$$

which appears as the objective in the quadratic program (5.3.1). In order for this function to play the role of a merit function for the splitting method, we need to rely on the close connection between the LCP (q, M) and the quadratic program (5.3.1). For this reason, the approach described in this subsection is applicable only to the symmetric LCP, and the matrix M is so assumed.

Let $z^{\nu+1/2}$ denote an arbitrary solution of the subproblem (q^ν, B). We define the search direction as $d^\nu = z^{\nu+1/2} - z^\nu$. The lemma below identifies a property on the splitting (B, C) to ensure the satisfaction of the descent condition $(d^\nu)^{\mathrm{T}}(q + Mz^\nu) < 0$. (Note: $\nabla f(z) = q + Mz$.)

5.5.1 Lemma. Let M be a symmetric matrix and B be a positive semi-definite matrix. Let $z^{\nu+1/2}$ be a solution of the LCP (q^ν, B). Then the vector $d^\nu = z^{\nu+1/2} - z^\nu$ satisfies

$$(d^\nu)^{\mathrm{T}}(q + Mz^\nu) \leq -(d^\nu)^{\mathrm{T}} B d^\nu \leq 0. \tag{1}$$

If B is either symmetric or positive definite, and if $(d^\nu)^{\mathrm{T}}(q + Mz^\nu) = 0$, then z^ν solves the LCP (q, M).

Proof. Since $z^{\nu+1/2}$ solves (q^ν, B), we have

$$0 \geq (d^\nu)^{\mathrm{T}}(q + Cz^\nu + Bz^{\nu+1/2}) = (d^\nu)^{\mathrm{T}}(q + Mz^\nu + Bd^\nu) \tag{2}$$

from which the desired inequality (1) follows easily.

Suppose that B is positive definite. If $(d^\nu)^{\mathrm{T}}(q + Mz^\nu) = 0$, then (1) and the positive definiteness of B imply $d^\nu = 0$ which yields $z^{\nu+1/2} = z^\nu$. It follows that $z^\nu \in \mathrm{SOL}(q, M)$. If B is symmetric positive semi-definite and $(d^\nu)^{\mathrm{T}}(q + Mz^\nu) = 0$, then (1) implies $Bd^\nu = 0$. Thus, $Bz^\nu = Bz^{\nu+1/2}$ and (2) yields

$$0 = (d^\nu)^{\mathrm{T}}(q + Cz^\nu + Bz^{\nu+1/2}).$$

Consequently, $0 = (z^\nu)^{\mathrm{T}}(q + Cz^\nu + Bz^\nu)$. Since $z^{\nu+1/2} \in \mathrm{SOL}(q^\nu, B)$ and $Bz^\nu = Bz^{\nu+1/2}$, it follows that $z^\nu \in \mathrm{FEA}(q, M)$. Consequently, z^ν solves (q, M). □

The positive (semi-) definiteness of B required in the above lemma is different from the regularity of the splitting (B, C) in the previous symmetry approach (cf. **5.3.2**). If M is positive semi-definite, then regularity of the splitting (B, C) implies that B must be positive definite; clearly, the converse need not be true.

With Lemma **5.5.1** on hand, we introduce the following algorithm.

5.5.2 Algorithm. (A Splitting Method With Line Search)

Step 0. *Initialization.* Let (B, C) be a splitting of the matrix M and z^0 be an arbitrary nonnegative vector. Set $\nu = 0$.

Step 1. *General iteration: compute direction.* Given $z^\nu \geq 0$, solve the the LCP (q^ν, B) and let $z^{\nu+1/2}$ be an arbitrary solution. Let $d^\nu = z^{\nu+1/2} - z^\nu$.

Step 2. *General iteration: compute stepsize.* Define the stepsize τ_ν as follows: if $(d^\nu)^{\mathrm{T}} M d^\nu \leq 0$, set $\tau_\nu = 1$; otherwise, let τ_ν be a nonnegative number satisfying

$$f(z^\nu + \tau_\nu d^\nu) = \min\{f(z^\nu + \tau d^\nu) : z^\nu + \tau d^\nu \geq 0, \tau \geq 0\}.$$

Step 3. *Test for termination.* Set $z^{\nu+1} = z^\nu + \tau_\nu d^\nu$ and test $z^{\nu+1}$ for termination. If termination fails, return to Step 1 with ν replaced by $\nu + 1$.

The distinction between the above algorithm and **5.2.1** lies of course in the determination of the stepsize τ_ν. In the previous case, τ_ν was set equal to unity for each ν. (Clearly, a unit stepsize means that one simply takes a solution of the LCP (q^ν, B) to be the next iterate.) That such a unit steplength could be taken was due to the regularity of the splitting (B, C); indeed, according to **5.3.2**, this property of the splitting ensures a suffcient decrease of the quadratic function $f(z)$ (in the sense that (5.3.2) holds). In the absence of the regularity assumption, such decrease is no longer guaranteed without an extra linesearch.

In order to better understand the choice of the stepsize τ_ν in **5.5.2**, suppose that the direction d^ν satisfies $(d^\nu)^{\mathrm{T}}(q + Mz^\nu) < 0$. (This descent condition is valid under the assumptions of Lemma **5.5.1**.) We may write

$$f(z^\nu) - f(z^\nu + \tau_\nu d^\nu) = -\tau_\nu (d^\nu)^{\mathrm{T}}(q + Mz^\nu) - \frac{\tau_\nu^2}{2}(d^\nu)^{\mathrm{T}} M d^\nu.$$

If $(d^\nu)^{\mathrm{T}} M d^\nu$ is nonpositive, then with a unit stepsize $\tau_\nu = 1$, we can ensure

$$f(z^\nu) - f(z^{\nu+1}) \geq -\frac{\tau_\nu}{2}(d^\nu)^{\mathrm{T}}(q + Mz^\nu); \tag{3}$$

moreover, the next iterate $z^{\nu+1}$ is equal to $z^{\nu+1/2}$ which is nonnegative. On the other hand, if $(d^\nu)^{\mathrm{T}}Md^\nu$ is positive, then the one-dimensional function

$$g(\tau) = f(z^\nu + \tau d^\nu),\ \tau \in R$$

is strictly convex in τ and its global minimum is attained at the value

$$\bar{\tau}_\nu = -\frac{(d^\nu)^{\mathrm{T}}(q + Mz^\nu)}{(d^\nu)^{\mathrm{T}}Md^\nu} \tag{4}$$

which is clearly positive. If $z^\nu + \bar{\tau}_\nu d^\nu$ is nonnegative, then we must have $\tau_\nu = \bar{\tau}_\nu$; moreover, it is easy to verify that the inequality (3) must hold as an equation in this case. On the other hand, if $z^\nu + \bar{\tau}_\nu d^\nu$ contains at least one negative component, then the stepsize τ_ν as defined in **5.5.2** must satisfy the relation

$$1 \leq \tau_\nu \leq \bar{\tau}_\nu.$$

In particular, τ_ν is bounded away from zero by the constant one. Moreover, it is not diffcult to verify that (3) also holds in this case. (See Exercise **5.11.9** for a summary of these properties which the reader is asked to prove.) We remark that if M is positive definite and if the direction d^ν is nonzero, then $(d^\nu)^{\mathrm{T}}Md^\nu$ must be positive.

The inequality (3) is similar to the corresponding inequality (5.3.2) for Algorithm **5.2.1**. Not only do these inequalities guarantee that the sequence of objective values $\{f(z^\nu)\}$ is nonincreasing (strictly decreasing in case the algorithm has not yet terminated), they ensure a certain positive amount of decrease between two consecutive functional values $f(z^\nu)$ and $f(z^{\nu+1})$. Such decrease is essential for the overall convergence of the respective algorithms.

The following is the main convergence result for Algorithm **5.5.2**.

5.5.3 Theorem. Let M be a symmetric matrix and B be positive definite. Then the sequence $\{z^\nu\}$ produced by **5.5.2** is uniquely defined; moreover, every accumulation point of $\{z^\nu\}$ solves the LCP (q, M). If in addition, the two assumptions (a) and (b) of Lemma **5.3.4** hold, then the sequence $\{z^\nu\}$ is bounded.

Proof. It suffices to prove the convergence and boundedness properties of the sequence $\{z^\nu\}$. Without loss of generality, we may assume that the descent condition $(d^\nu)^{\mathrm{T}}(q + Mz^\nu) < 0$ is satisfied for each ν. Let $\tilde{z}$ be an accumulation point of $\{z^\nu\}$, and let $\{z^\nu : \nu \in \kappa\}$ be a subsequence converging to $\tilde{z}$. Since the sequence $\{f(z^\nu)\}$ is nonincreasing, by the same argument as in the proof of **5.3.3**, we may deduce that the sequence $\{f(z^\nu)\}$ converges. This implies that $f(z^\nu) - f(z^{\nu+1}) \to 0$. Thus, by condition (3), it follows that

$$\lim_{\nu\to\infty} \tau_\nu (d^\nu)^{\mathrm{T}}(q + Mz^\nu) = 0.$$

Suppose that $\liminf\{\tau_\nu : \nu \in \kappa\}$ is positive. Then,

$$\lim_{\nu\in\kappa,\,\nu\to\infty} (d^\nu)^{\mathrm{T}}(q + Mz^\nu) = 0$$

and expression (1) implies $\{d^\nu : \nu \in \kappa\} \to 0$ by the positive definiteness of B. From this, it is easy to deduce that $\tilde{z}$ solves the LCP (q, M).

On the other hand, suppose that $\liminf\{\tau_\nu : \nu \in \kappa\} = 0$. Then, there must exist an infinite subset $\kappa' \subseteq \kappa$ such that for each $\nu \in \kappa'$, $\tau_\nu = \bar{\tau}_\nu$ where $\bar{\tau}_\nu$ is given by (4) and $\lim_{\nu\in\kappa',\nu\to\infty} \tau_\nu = 0$. We claim that the sequence $\{d^\nu : \nu \in \kappa'\}$ is bounded. Assume the contrary. Let $\bar{d}$ be a limit point of the normalized sequence $\{d^\nu/\|d^\nu\| : \nu \in \kappa'\}$ which must exist and be nonzero. Then, the inequality (1) and the positive definiteness of B imply $\bar{d} = 0$ which is a contradiction. Consequently, the boundedness of the sequence $\{d^\nu : \nu \in \kappa'\}$ follows. Without loss of generality, we may assume that $\{d^\nu : \nu \in \kappa'\}$ converges to a vector $\tilde{d}$. Since

$$0 = \lim_{\nu\in\kappa',\,\nu\to\infty} \bar{\tau}_\nu = - \lim_{\nu\in\kappa',\,\nu\to\infty} \frac{(d^\nu)^{\mathrm{T}}(q + Mz^\nu)}{(d^\nu)^{\mathrm{T}} M d^\nu},$$

it follows that

$$(q + M\tilde{z})^{\mathrm{T}}\tilde{d} = 0.$$

Passing to the limit $\{\nu \to \infty, \nu \in \kappa'\}$ in the inequality (1), and using the last equation and the positive definiteness of B, we deduce that $\tilde{d} = 0$. As before, this implies $\tilde{z} \in \mathrm{SOL}(q, M)$.

To establish the last assertion of the theorem, suppose that assumptions (a) and (b) of **5.3.4** hold. Then, the sequence $\{f(z^\nu)\}$ is nonincreasing and bounded below, it therefore converges. Suppose that some subsequence $\{\|z^\nu\| : \nu \in \kappa\} \to \infty$. As we have shown before, we can deduce that for some infinite subset $\kappa' \subseteq \kappa$, $\{d^\nu : \nu \in \kappa'\} \to 0$. Now, by the same proof as that of Lemma **5.3.4**, we can derive a contradiction to the two assumptions (a) and (b). This completes the proof of the theorem. □

Theorem **5.5.3** complements the results of Section 5.3 under the symmetry approach for convergence and is applicable to splitting algorithms in which the splitting (B, C) is not necessarily regular. Some realizations of **5.5.2** include the projected Jacobi method and more generally, all parallel iterative methods generated by a block diagonal matrix B with positive definite diagonal blocks.

It is interesting to compare Theorem **5.5.3** and Proposition **5.3.13**. According to the latter result, given any symmetric positive definite matrix G, it is always possible to scale G and use the scaled matrix to define an iterative scheme for solving the LCP (q, M). This procedure requires the matrix M to be positive definite and differs from **5.5.2** in two major respects. First, the positive definiteness of M is a precondition for the scaling approach to be successful, whereas this property of M is not needed for the convergence theory of **5.5.2**. Second, due to the regularity property of the splitting, the scaling procedure requires no linesearch; on the other hand, the latter step is crucial for Algorithm **5.5.2**.

A symmetric variational inequality approach

Algorithm **5.5.2** and its convergence theory can be extended to more general quadratic programs. In particular, applying the splitting idea to (1.4.2) leads to the family of iterative methods described below. Since the equivalence between the LCP (q, M) and the quadratic program (1.4.2) hinges on the row-sufficiency property of M (see Section 3.5) but does not require the symmetry of M, the approach described below is applicable to an asymmetric LCP of the row sufficient type.

Throughout this subsection, let $f(z)$ denote the objective function of the quadratic program (1.4.2), i.e.,

$$f(z) = z^{\mathrm{T}}(q + Mz).$$

The gradient vector of f is given by

$$\nabla f(z) = q + (M + M^{\mathrm{T}})z.$$

Let N denote the matrix $M + M^{\mathrm{T}}$ which is clearly symmetric. Also, we let $S = \mathrm{FEA}(q, M)$.

According to Theorem **3.5.4**, if M is a row sufficient matrix, the LCP (q, M) is equivalent to the Karush-Kuhn-Tucker conditions of the quadratic program (1.4.2) in the sense made precise in that theorem. In turn, these Karush-Kuhn-Tucker conditions define another LCP which according to the discussion in Section 1.2 is equivalent to the affine variational inequality problem $\mathrm{VI}(S, \nabla f)$. Thus by transitivity, the LCP (q, M) becomes equivalent to the latter variational problem. We summarize the relationship between these two problems in the following lemma which provides the key to the entire approach described in this subsection.

5.5.4 Lemma. Let M be a row sufficient matrix. Then a vector z solves the LCP (q, M) if and only if z solves the problem $\mathrm{VI}(S, \nabla f)$. □

Using the above lemma, we introduce a splitting algorithm for solving the LCP (q, M). The algorithm makes use of an arbitrary splitting (B, C) of the (symmetric) matrix N in which B is symmetric positive definite. Note that with such a matrix B, any variational problem $\mathrm{VI}(S, g)$ where g is the affine mapping

$$g(z) = r + Bz$$

is equivalent to the strictly convex quadratic program

$$\begin{array}{ll} \text{minimize} & r^{\mathrm{T}}z + \frac{1}{2}z^{\mathrm{T}}Bz \\ \text{subject to} & z \in S. \end{array} \tag{5}$$

5.5.5 Algorithm. (A VI-Based Splitting Algorithm)

Step 0. *Initialization.* Let (B, C) be a splitting of the matrix N with B being symmetric positive definite. Let $z^0 \in S$ be arbitrary. Set $\nu = 0$.

Step 1. *General iteration: compute direction.* Given $z^\nu \in S$, solve the quadratic program (5) with

$$r = q + Cz^\nu$$

and let $z^{\nu+1/2}$ denote its unique solution. Set $d^\nu = z^{\nu+1/2} - z^\nu$.

Step 2. *General iteration: compute stepsize.* Define the stepsize τ_ν as follows: if $(d^\nu)^{\mathrm{T}} M d^\nu$ is nonpositive, set $\tau_\nu = 1$; otherwise, let τ_ν be a nonnegative number such that

$$f(z^\nu + \tau_\nu d^\nu) = \min\{f(z^\nu + \tau d^\nu) : z^\nu + \tau d^\nu \in S, \tau \geq 0\}.$$

Step 3. *Test for termination.* Set $z^{\nu+1} = z^\nu + \tau_\nu d^\nu$ and test $z^{\nu+1}$ for termination. If $z^{\nu+1}$ fails the termination test, return to Step 1 with $\nu \leftarrow \nu + 1$.

The main advantage of the above iterative scheme is its applicability to the LCP (q, M) with an arbitrary row sufficient matrix M. Note that the algorithm assumes the feasibility of (q, M). Thus, by **3.5.5**, the existence of a solution to this LCP is not an issue here; instead, it is the convergence of the sequence $\{z^\nu\}$ that is the main concern.

As in the previous splitting algorithms, there are many candidates for the splitting (B, C); of particular importance is the choice of a positive diagonal matrix for B. If B is such a matrix, then each quadratic subprogram (5) has a strictly convex, separable objective function, and the conversion scheme discussed in Section 5.3 offers an effective approach for solving the subproblem (5) by a variety of iterative methods. This overall process is known as *diagonalization*; its underlying idea is to transform (or *diagonalize*) a non-separable problem into a sequence of separable subproblems which presumably can be solved more effectively by some efficient methods. The projected Jacobi method is a simple application of this diagonalization idea.

When each subproblem (5) is in turn solved by an iterative scheme, Algorithm **5.5.5** becomes, overall, a hybrid method which involves two levels of iterations: an inner level and an outer level. The inner iterations refer to the solution of a particular subproblem (5), and the outer iterations pertain to the updates of the subproblems to be solved by the inner scheme. As a solution method for the LCP (q, M), this combined strategy seems a bit cumbersome and artificial; nevertheless, in the absence of additional properties of the matrix M, it provides a promising avenue for solving an LCP of the row sufficient type by a provably convergent iterative method of the kind discussed in this chapter.

The key to the convergence of the sequence $\{z^\nu\}$ produced by **5.5.5** is the following lemma which ensures that the direction d^ν defined in the algorithm satisfies the descent condition $(d^\nu)^{\mathrm{T}} \nabla f(z^\nu) < 0$. The proof of the lemma is exactly the same as that for **5.5.1** and is left to the reader.

5.5.6 Lemma. The direction d^ν generated by **5.5.5** satisfies

$$(d^\nu)^{\mathrm{T}}(q + Nz^\nu) \le -(d^\nu)^{\mathrm{T}} B d^\nu \le 0.$$

Moreover, if $(d^\nu)^{\mathrm{T}}(q + Nz^\nu) = 0$, then z^ν solves the problem VI$(S, \nabla f)$. □

The stepsize analysis for Algorithm **5.5.2** easily carries over to the present context. In particular, one can derive a descent property similar to the inequality (3). Furthermore, by means of a proof analogous to that of **5.5.3**, one can establish the following main convergence result for **5.5.5**.

5.5.7 Theorem. Let M be a row sufficient matrix and B be symmetric positive definite. Suppose that the LCP (q, M) is feasible. Then, the sequence $\{z^\nu\}$ produced by **5.5.5** is uniquely defined; moreover, every accumulation point of $\{z^\nu\}$ solves the LCP (q, M). If, in addition, the level set

$$\{z \in S : f(z) \le f(z^0)\} \tag{6}$$

is bounded, then so is the sequence $\{z^\nu\}$.

According to Proposition **3.9.23**, if $M \in \boldsymbol{R_0}$, then the level set (6) must be bounded. In particular, this holds if M is a $\boldsymbol{P}$-matrix. Since the LCP (q, M) has a unique solution when $M \in \boldsymbol{P}$, it follows from **5.5.7** that the sequence $\{z^\nu\}$ produced by **5.5.5** is bounded and has a unique accumulation point which must be the solution of (q, M). Consequently, we have proved

5.5.8 Corollary. Let $M \in \boldsymbol{P}$ and B be a symmetric positive definite matrix. Then, the uniquely defined sequence $\{z^\nu\}$ generated by Algorithm **5.5.5** converges to the unique solution of the LCP (q, M). □

It is rather evident that the diagonalization idea and, more generally, Algorithm **5.5.5** are not just restricted to the LCP (q, M), but can be applied to a general quadratic program. The matrix N and the set S are related in the context of the LCP (q, M); by taking N to be an arbitrary symmetric matrix and S an arbitrary polyhedral set, the variational inequality problem $\mathrm{VI}(S, \nabla f)$ becomes the stationary point problem of the general quadratic program

$$\begin{array}{ll} \text{minimize} & f(z) = q^{\mathrm{T}}z + \frac{1}{2}z^{\mathrm{T}}Nz \\ \text{subject to} & z \in S. \end{array}$$

Algorithm **5.5.5** applies without any change. In this case, one can show—parallel to **5.5.7**—that if the above quadratic program is feasible, then every accumulation point of the sequence produced by **5.5.5** is a stationary point of the program, and under the additional assumption of bounded level sets, that such a sequence must be bounded. Of course, if N is positive semi-definite, then every stationary point of the quadratic program is a global minimum.

5.6 Regularization Algorithms

Since a positive semi-definite matrix is row sufficient, Algorithm **5.5.5** is applicable to the LCP (q, M) where M is positive semi-definite. For an LCP of this type, alternative iterative methods can be derived by a process which we term *regularization*. The essential idea involved in this process is to transform the given LCP into a sequence of linear complementarity subproblems defined by positive definite matrices. Presumably, such transformation is desirable for two reasons. First, each subproblem is in turn amenable to solution by a broad family of iterative methods. More importantly, as we shall see in Section 5.10 and Chapter 7, an LCP of the positive definite type is globally more "stable" than one of the positive semi-definite type. Incidentally, Algorithm **5.5.5** may also be thought of as providing a way of "regularizing" the LCP (q, M), but in a more general sense. There, the connection between the LCP (q, M) and the quadratic program (1.4.2) is exploited, and **5.5.5** is actually a regularization algorithm applied to (1.4.2). The resulting algorithm yields subproblems each of which is a strictly convex quadratic program instead of a positive definite LCP.

A broad class of regularization algorithms for solving the LCP (q, M) can be derived from the basic splitting algorithm **5.2.1**. To motivate this derivation, we ask the question of how to generate from M a positive definite matrix B. An obvious answer to this question is to simply add an arbitrary positive definite matrix to M. Since M is positive semi-definite and the added matrix $B - M$ is positive definite, the resulting matrix B being the sum of M and $B - M$ is indeed positive definite. When $B - M$ is a positive multiple of the identity matrix, the resulting algorithm becomes the so-called *proximal point algorithm* in the theory of monotone operators. Generalizing this choice for $B - M$, we require in the sequel that $B - M$ be symmetric as well as positive definite.

The following is the main convergence result for the above class of regularization algorithms.

5.6.1 Theorem. Let M be a positive semi-definite matrix. Let (B, C) be a splitting of M such that $B - M$ is symmetric positive definite. Let $q \in K(M)$ and $z^0 \geq 0$ be arbitrary. Then, the uniquely defined sequence of vectors $\{z^\nu\}$ produced by **5.2.1** converges to some solution of the LCP (q, M).

Proof. The uniqueness of each iterate $z^{\nu+1}$ follows from the positive definiteness of B. Since the matrix $B - M$ is symmetric posititive definite, it defines the elliptic norm $\| \cdot \|_{B-M}$. In order to simplify the notation somewhat, we shall omit the subscript $B - M$ from this norm.

We first show that $\{z^\nu\}$ is bounded. Let $\tilde{z}$ be a solution of the LCP (q, M). Since $z^{\nu+1}$ solves (q^ν, B), we derive the inequality

$$0 \geq (z^{\nu+1} - \tilde{z})^{\mathrm{T}}(C(z^\nu - \tilde{z}) + B(z^{\nu+1} - \tilde{z})).$$

Substituting $B = (B - M) + M$, rearranging terms and using the positive semi-definiteness of M, we obtain from the above inequality,

$$-(z^{\nu+1} - \tilde{z})^{\mathrm{T}}C(z^\nu - \tilde{z}) \geq (z^{\nu+1} - \tilde{z})^{\mathrm{T}}(B - M)(z^{\nu+1} - \tilde{z}). \tag{1}$$

Note that $-C = B - M$. Therefore, by the symmetry and positive definiteness of $B - M$ and the Cauchy-Schwartz inequality, it follows that

$$\|z^{\nu+1} - \tilde{z}\| \leq \|z^\nu - \tilde{z}\|$$

which implies that $\{z^\nu\}$ is bounded; moreover, the limit

$$c = \lim_{\nu \to \infty} \|z^\nu - \tilde{z}\| \tag{2}$$

exists and is finite. Next, we show that

$$\lim_{\nu \to \infty} \|z^\nu - z^{\nu+1}\| = 0. \tag{3}$$

We have

$$\|z^{\nu+1} - z^\nu\|^2 = \|z^{\nu+1} - \tilde{z}\|^2 - 2(z^{\nu+1} - \tilde{z})^{\mathrm{T}}(B - M)(z^\nu - \tilde{z}) + \|z^\nu - \tilde{z}\|^2$$

which in view of (1), implies

$$\|z^{\nu+1} - z^\nu\|^2 \leq \|z^\nu - \tilde{z}\|^2 - \|z^{\nu+1} - \tilde{z}\|^2.$$

The limit (3) follows readily from (2). By the definition of each iterate $z^{\nu+1}$ and the limit (3), it is easy to show that every accumulation point of the sequence $\{z^\nu\}$ solves the LCP (q, M).

To establish the convergence of $\{z^\nu\}$, let u^1 and u^2 be any two limit points of $\{z^\nu\}$. Then, with $\tilde{z}$ replaced by u^i (for $i = 1, 2$), it follows that the limits

$$c_i = \lim_{\nu \to \infty} \|z^\nu - u^i\|$$

exist for $i = 1, 2$. Clearly,

$$\|z^\nu - u^1\|^2 = \|z^\nu - u^2\|^2 + \|u^1 - u^2\|^2 + 2(u^2 - u^1)^{\mathrm{T}}(B - M)(z^\nu - u^2).$$

Thus, $\lim_{\nu \to \infty}(u^2 - u^1)^{\mathrm{T}}(B - M)(z^\nu - u^2)$ exists and in fact is equal to zero because u^2 is a limit point of $\{z^\nu\}$. Consequently, we deduce,

$$c_1^2 = c_2^2 + \|u^1 - u^2\|^2.$$

Reversing the role of u^1 and u^2, we also have

$$c_2^2 = c_1^2 + \|u^1 - u^2\|^2.$$

Thus, $u^1 = u^2$. This completes the proof. □

The proof of Theorem **5.6.1** is a kind of a nonexpansive argument; it can be used to analyze splitting methods satisfying some related properties. The essential idea may be summarized as follows. Suppose $\{z^\nu\}$ is a sequence generated by the splitting method of **5.2.1** whose convergence is desired. One first demonstrates that the limit (2) exists for an arbitrary solution $\tilde{z}$ of the LCP (q, M); next, one shows that the sequence of differences of consecutive iterates, namely $\{z^{\nu+1} - z^{\nu}\}$, converges to zero. Finally, from these two conclusions, one establishes the convergence of the entire sequence $\{z^{\nu}\}$ by the same argument as given above. Typically, the first task relies on some special property of the splitting (B, C) . The convergence of $\{z^{\nu+1} - z^{\nu}\}$ to zero is intimately related to the assertion that every accumulation point of $\{z^{\nu}\}$—if it exists—solves the problem (q, M); see Exercise **7.6.3**. By following this line of reasoning, one may generalize Theorem **5.3.12** to an LCP with a special kind of positive semi-definite matrix. This generalization is outlined in Exerecise **5.11.8**.

The regularization idea can be generalized to the LCP (q, M) with $M \in \boldsymbol{P}_0$. Indeed, if M is a $\boldsymbol{P}_0$-matrix, then for each $\varepsilon > 0$, $M + \varepsilon I \in \boldsymbol{P}$ by **3.4.2**. Thus, the LCP $(q, M + \varepsilon I)$ has a unique solution $z(\varepsilon)$ which can be obtained, in turn, by the iterative scheme **5.5.5**. By taking a sequence of positive scalars $\{\varepsilon_\nu\} \to 0$, we generate a sequence $\{z^{\nu}\}$ with $z^{\nu} = z(\varepsilon_\nu)$ for each ν. The following theorem establishes two convergence properties of these iterates. Exercise **5.11.10** provides a third property under a different assumption.

5.6.2 Theorem. Let M be a $\boldsymbol{P}_0$-matrix. Let $\{\varepsilon_\nu\}$ be a decreasing sequence of positive scalars with $\varepsilon_\nu \to 0$. For each ν, let z^{ν} be the unique solution of the LCP $(q, M + \varepsilon_\nu I)$.

(a) If $M \in \boldsymbol{R}_0$, then the sequence $\{z^{\nu}\}$ is bounded; moreover, every accumulation point of $\{z^{\nu}\}$ solves the LCP (q, M);

(b) If M is positive semi-definite and the LCP (q, M) is solvable, then the sequence $\{z^{\nu}\}$ converges to the least l_2-norm solution of (q, M).

We clarify the conclusions of the theorem before proving it. According to **3.9.22**, any matrix $M \in \boldsymbol{P}_0 \cap \boldsymbol{R}_0$ must be a $\boldsymbol{Q}$-matrix. Thus, under the assumption of part (a) in **5.6.2**, it follows that (q, M) is solvable. The main emphasis of this part of the theorem is the convergence of the sequence $\{z^{\nu}\}$ and not the solvability of (q, M). It is also interesting to compare this convergence result which applies to an LCP with an arbitrary $\boldsymbol{P}_0$-matrix and **5.5.7** which applies to an LCP with a row sufficient matrix (the latter is a special kind of $\boldsymbol{P}_0$-matrix). The two iterative schemes in question are quite different in nature, although both may be regarded as derived from the same regularization idea.

The solution set of the LCP (q, M) with M being positive semi-definite is a convex polyhedron by **3.1.7**(c). Thus, the least l_2-norm solution of (q, M)—which is the vector in SOL(q, M) that is closest to the origin in the l_2-norm—exists and is unique. The conclusion of part (b) in **5.6.2** asserts that this special solution is the limit of the sequence $\{z^\nu\}$ produced by the regularization algorithm when M is a positive semi-definite matrix. This part of the theorem can be compared to **5.6.1** which also deals with a positive semi-definite LCP. Theorem **5.6.1** applies to a broader class of iterative methods in that there is basically no restriction on the matrix B as long as $B - M$ is symmetric positive definite. On the other hand, Theorem **5.6.2** concerns a different iterative scheme which involves a special regularization mechanism, namely, that of adding to M a sequence of decreasingly small positive multiples of the identity matrix.

Proof of 5.6.2. To start, we point out that if the sequence $\{z^\nu\}$ is bounded, then it is a simple matter to verify that every accumulation point of $\{z^\nu\}$ solves the LCP (q, M).

Suppose that $M \in \boldsymbol{P_0} \cap \boldsymbol{R_0}$ and that the sequence $\{z^\nu\}$ is unbounded. Without loss of generality, we may assume that $\{\|z^\nu\|\} \to \infty$. The normalized sequence $\{z^\nu/\|z^\nu\|\}$ has at least one accumulation point, say $\tilde{z}$. We may assume, without loss of generality, that $\tilde{z}$ is the limit of the entire sequence $\{z^\nu/\|z^\nu\|\}$. Clearly, $\tilde{z}$ is nonnegative and nonzero. We have, for each ν,

$$\begin{aligned} q + Mz^\nu + \varepsilon_\nu z^\nu &\geq 0 \\ z^\nu &\geq 0 \\ (z^\nu)^{\mathrm{T}}(q + Mz^\nu + \varepsilon_\nu z^\nu) &= 0. \end{aligned}$$

Dividing the first inequality by $\|z^\nu\|$ and the equation by $\|z^\nu\|^2$, and passing to the limit $\nu \to \infty$, we easily deduce $\tilde{z} \in$ SOL$(0, M)$. But this is a contradiction because $M \in \boldsymbol{R_0}$. Consequently, the sequence $\{z^\nu\}$ is bounded. By the remark made at the beginning of the proof, it follows that every accumulation point of $\{z^\nu\} \in$ SOL(q, M).

Suppose now that M is positive semi-definite and the LCP (q, M) has a solution, say $\hat{z}$. In this part of the proof, $\|\cdot\|$ denotes the l_2-norm of vectors. We show that the sequence $\{z^\nu\}$ is bounded. (Note that this does not follow from part (a).) Write

$$w^\nu = q + Mz^\nu + \varepsilon_\nu z^\nu, \quad \text{and} \quad \tilde{w} = q + M\tilde{z}.$$

Then, we have

$$\begin{aligned} 0 &\geq (z^\nu - \tilde{z})^{\mathrm{T}}(w^\nu - \tilde{w}) \\ &= (z^\nu - \tilde{z})^{\mathrm{T}} M (z^\nu - \tilde{z}) + \varepsilon_\nu (z^\nu - \tilde{z})^{\mathrm{T}} z^\nu \\ &\geq \varepsilon_\nu (z^\nu - \tilde{z})^{\mathrm{T}} z^\nu \end{aligned}$$

where the last inequality follows from the positive semi-definiteness of M. Consequently, we deduce

$$\|z^\nu\|^2 \leq \tilde{z}^{\mathrm{T}} z^\nu \leq \|\tilde{z}\| \, \|z^\nu\|$$

which implies that $\|z^\nu\| \leq \|\tilde{z}\|$, and the boundedness of the sequence $\{z^\nu\}$ is thus established. Again, by the remark made at the beginning of the proof, it follows that every accumulation point of $\{z^\nu\} \in \mathrm{SOL}(q, M)$. If x and y are any two such accumulation points, it follows from the derivation above (with $\tilde{z} = x$) that $\|y\| \leq \|x\|$. Reversing the role of x and y, we may conclude that $\|x\| = \|y\|$ and this common value must be less than or equal to $\|\tilde{z}\|$ for any solution $\tilde{z}$ of (q, M). Since there is a unique solution with the least l_2-norm, it follows that this least l_2-norm solution must be the limit of the sequence $\{z^\nu\}$. This completes the proof of the theorem. □

5.6.3 Remark. It is easy to see that part (a) of **5.6.2** remains valid if the assumption $M \in \boldsymbol{R_0}$ is replaced by that of the boundedness of the set

$$\{z \in R^n_+ : z^{\mathrm{T}}(q + Mz) \leq 0\}.$$

The regularization idea for the LCP discussed above can be extended to a convex quadratic program (1.2.1). There are two ways to implement this extension: one is to apply it directly to the program (1.2.1), and the other to the equivalent LCP formulation (1.2.2). In the former approach, one regularizes the problem (1.2.1) by creating a sequence of strictly convex quadratic programs; a prominent algorithm resulting from this is the proximal point algorithm which requires solving subproblems of the form

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x + \tfrac{1}{2}(x^{\mathrm{T}}Qx + \varepsilon_\nu(x - x^\nu)^{\mathrm{T}}(x - x^\nu)) \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array} \tag{4}$$

where $\{\varepsilon_\nu\}$ is an arbitrary sequence of positive scalars. This algorithm is particularly useful in the design of parallel methods for solving the important subclass of linear programs (which have $Q = 0$). There are variations to the subproblem (4) but the essential idea remains the same.

5.7 Generalized Splitting Methods

The basic splitting algorithm given in **5.2.1** can be generalized in a number of ways. We have discussed some of these in Section 5.5 in conjunction with a linesearch procedure. In this section, we discuss a few more generalizations, but do not analyze their convergence in detail. The extended analysis is not difficult to carry out under appropriate assumptions.

An inexact splitting method

In many realizations of the splitting algorithm **5.2.1**, the subproblems are themselves LCPs that are not entirely trivial to solve. Sometimes, it might even be profitable to solve these subproblems by an iterative procedure. In practice, such a procedure produces only an approximate solution. Rigorously speaking, the convergence results established thus far fail to be valid when the subproblems are solved *inexactly*; this is because these results all require that each $z^{\nu+1}$ be an *exact* solution of the subproblem $(q + Cz^\nu, B)$. This consideration leads us to the study of the *inexact splitting methods*. Instead of discussing these inexact methods in their full generality, we focus on a two-stage splitting method which contains the essential idea of a typical inexact splitting method.

As the name suggests, a *two-stage splitting method* solves the LCP (q, M) by a two-stage process. In order to explain this in more detail, let (B, C) be a splitting of the matrix M. The *outer stage* of the method refers to the (iterative) solution of the subproblem $(q + Cz^\nu, B)$ at a specific (outer) iteration ν; whenever the iteration count ν is replaced by $\nu + 1$, we say that a *new outer stage is entered*. Each *inner stage* corresponds to the actual iterations for solving a given $(q + Cz^\nu, B)$; these *inner iterations* are defined by a splitting of the matrix B given by

$$B = E + F.$$

Thus, an inner stage of the method generates a sequence $\{y^{\nu,\iota}\}$ where each iterate $y^{\nu,\iota+1}$ is an exact solution of the LCP $(q + Cz^\nu + Ey^{\nu,\iota}, F)$. For practical purposes, we terminate the inner iterations when an iterate $y^{\nu,\iota+1}$ satisfies a prescribed termination rule; when this happens, the iterate $y^{\nu,\iota+1}$ is deemed satisfactory and taken to be the next outer iterate, i.e., $z^{\nu+1} = y^{\nu,\iota+1}$; a new outer stage is then entered.

Consequently, each iterate $z^{\nu+1}$ is an inexact solution of the problem $(q+Cz^\nu, B)$, the inaccuracy of $z^{\nu+1}$ as a solution of the latter LCP depends on the criterion we use to terminate the inner iterations. One such rule is the following:

$$\| y^{\nu,\iota+1} - y^{\nu,\iota} \|_2 \leq r_\nu / (\| y^{\nu,\iota+1} - z^\nu \|_2 \; \| E \|_2) \tag{1}$$

where r_ν is a prescribed scalar. This rule is practically implementable in the sense that the resulting (inexact) method can actually be carried out in practice. Notice that if $y^{\nu,\iota+1} = y^{\nu,\iota}$, then (1) is clearly satisfied; in this case, $y^{\nu,\iota+1}$ becomes an exact solution of $(q + Cz^\nu, B)$.

We summarize the above discussion and present a full description of the two-stage splitting method for solving the LCP (q, M). We assume that the two splittings

$$M = B + C, \quad B = E + F$$

are given. Also given are prescribed rules for terminating the inner and outer iterations.

5.7.1 Algorithm. (The Two-Stage Splitting Method)

Step 0. *Initialization*. Let z^0 and $y^{0,0}$ be arbitrary nonnegative vectors, set $\nu = 0$.

Step 1. *Inner iterations*. Given z^ν and $y^{\nu,0}$, generate a sequence $\{y^{\nu,\iota+1}\}$ by letting each $y^{\nu,\iota+1} \in \text{SOL}(q + Cz^\nu + Ey^{\nu,\iota}, F)$. Let $y^{\nu,\bar{\iota}+1}$ be the vector obtained when the prescribed termination rule for the inner iterations is satisfied. Set $z^{\nu+1} = y^{\nu,\bar{\iota}+1}$.

Step 2. *Outer iteration and termination test.* If $z^{\nu+1}$ satisfies a prescribed termination rule for the outer iterations, terminate; otherwise, set $y^{\nu+1,0} = z^{\nu+1}$ and return to Step 1 with ν replaced by $\nu + 1$.

5.7.2 Remark. In the above algorithm, each inner iteration is started with the initial iterate $y^{\nu,0} = z^{\nu}$. Other choices are also possible; this one is for the sake of convenience.

Under appropriate choices of the inner termination rule, it is possible to generalize the convergence results established in the previous sections to the present context of **5.7.1**. For instance, with the rule (1), one can show that essentially all the results obtained under the symmetry approach for convergence will continue to hold provided that the sequence of positive scalars $\{r_\nu\}$ satisfies the property

$$\sum_{\nu=0}^{\infty} r_\nu < \infty.$$

This condition therefore provides the needed assumption on the amount of inaccuracy that the inner iterations can sustain in order to preserve the convergence of the overall iterative process in the case of the symmetric LCP.

Variable splittings and underrelaxation

Another generalization of the basic splitting method given in **5.2.1** is obtained with the use of a sequence of *variable splittings*. Specifically, let $\{(B_\nu, C_\nu)\}$ be a sequence of splittings of the matrix M. At iteration ν, the splitting (B_ν, C_ν) is used to define the subproblem $(q + C_\nu z^\nu, B_\nu)$. By imposing a uniformity assumption on the sequence $\{(B_\nu, C_\nu)\}$, the convergence results established under the symmetry and the contraction approaches all remain valid. In order to illustrate such a uniformity assumption, consider the symmetry approach for convergence in the case of solving the symmetric LCP. In this case, the needed assumption is that there exists a constant $\sigma > 0$ such that for all ν and all vectors $x \in R^n$,

$$x^{\mathrm{T}}(B_\nu - C_\nu)x \geq \sigma x^{\mathrm{T}}x;$$

in other words, we require the smallest eigenvalue of the symmetric part of $B_\nu - C_\nu$ be bounded below by the positive scalar σ for all ν, i.e., the sequence $\{B_\nu - C_\nu\}$ is *uniformly positive definite.*

We may further generalize the variable splitting scheme with the use of an *underrelaxation parameter* $\tau \in [0, 1)$. More specifically, if y^ν denotes a solution of the subproblem $(q + C_\nu z^\nu, B_\nu)$, the next iterate $z^{\nu+1}$ is defined to be

$$z^{\nu+1} = \tau z^\nu + (1 - \tau)y^\nu. \tag{2}$$

A complete convergence theory can be developed for this generalization which is parallel to that in the preceding sections.

It is easy to see that the iterate $z^{\nu+1}$ defined in (2) is a solution of the complementarity system below:

$$\begin{aligned} (1-\tau)q + (C_\nu - \tau M)z^\nu + B_\nu z^{\nu+1} &\geq 0 \\ z^{\nu+1} &\geq \tau z^\nu \\ (z^{\nu+1} - \tau z^\nu)^{\mathrm{T}}((1-\tau)q + (C_\nu - \tau M)z^\nu + B_\nu z^{\nu+1}) &= 0, \end{aligned}$$

which we recognize as an instance of the implicit complementarity problem discussed in Section 1.5.

5.8 A Damped-Newton Method

In most of the iterative methods discussed in the previous sections, the notion of a matrix splitting has played a major role. In this section, we discuss the application of the classical damped-Newton method for solving systems of equations to the LCP. This approach relies on the equivalent formulation of the LCP (q, M) as a system of piecewise linear equations. As explained in Section 1.4, there are several such formulations. Our discussion below centers on the system

$$H(z) := \min(z, q + Mz) = 0. \tag{1}$$

The reader can easily apply the same ideas to other related systems (see Exercise **5.11.12**).

5.8.1 Remark. The above function $H(z)$ is the same as the function $H_{q,M}(z)$ introduced in Definition **1.4.3**. Throughout this section, we shall say that a vector z is *nondegenerate* to mean that z is a nondegenerate vector with respect to the function $H_{q,M}$ as defined in Definition **1.4.3**; the reference to the function $H_{q,M}$ is omitted from this terminology.

Closely related to the zero-finding problem (1) is the unconstrained norm-minimization problem

$$\text{minimize} \quad \theta(z) = \tfrac{1}{2}H(z)^{\mathrm{T}}H(z).$$

Obviously, the following three statements are equivalent: (i) z solves (q, M), (ii) $H(z) = 0$ and (iii) z is a global minimum point of the function θ and $\theta(z) = 0$. The central idea of the damped-Newton method for solving (q, M) is to find a vector z satisfying the last condition (iii). Of course, what makes the minimization of θ a non-trivial problem is its non-F(réchet) differentiability. For this reason, it is useful to summarize some important differentiability properties of the functions H and θ. For an arbitrary vector z, define the index sets

$$\begin{aligned} \alpha(z) &= \{i : z_i > (q + Mz)_i\} \\ \beta(z) &= \{i : z_i = (q + Mz)_i\} \\ \gamma(z) &= \{i : z_i < (q + Mz)_i\}. \end{aligned} \tag{2}$$

Note that the vector z is not required to be a solution of (q, M); these index sets generalize those defined in **3.9.15** when $z \in \mathrm{SOL}(q, M)$. Moreover, the notion of a nondegenerate vector z defined in **1.4.3** corresponds to the case where $\beta(z)$ is empty. Elements in $\beta(z)$ are called *degenerate indices*, and $\beta(z)$ is called the *degenerate (index) set*. As the following result shows, the set $\beta(z)$ plays a fundamental role in the differentiability of the functions H and θ.

5.8.2 Proposition. Let $q \in R^n$ and $M \in R^{n\times n}$ be given, and $z \in R^n$ be arbitrary. Then,

(a) the functions H and θ are everywhere directionally differentiable; their directional derivatives are given by

$$(H'(z,d))_i = \begin{cases} (Md)_i & \text{if } i \in \alpha(z) \\ \min(d_i, (Md)_i) & \text{if } i \in \beta(z) \\ d_i & \text{if } i \in \gamma(z) \end{cases}$$

$$\theta'(z,d) = \sum_{i\in\alpha(z)}(q+Mz)_i(Md)_i + \sum_{i\in\beta(z)} z_i \min(d_i,(Md)_i) + \sum_{i\in\gamma(z)} z_i d_i\,;$$

(b) the function H is F-differentiable at z if and only if for each degenerate index i, the i-th row of M is equal to the i-th unit vector;

(c) the function θ is F-differentiable at z if for each degenerate index i, either the i-th row of M is equal to the i-th unit vector, or

$$z_i = (q+Mz)_i = 0. \tag{3}$$

Moreover, if z solves the LCP (q, M), then θ must be F-differentiable at z with $\nabla\theta(z) = 0$, and the following limit holds

$$\lim_{(u,v)\to(z,z)} \frac{\theta(u)-\theta(v)}{\|u-v\|} = 0. \tag{4}$$

Proof. The formulae for the directional derivatives in part (a) can be verified by a straightforward calculation. To prove part (b), suppose that for each degenerate index i, the i-th row of M is equal to the i-th unit vector. It then follows easily that the directional derivative $H'(z,d)$ is linear in the argument d. Moreover, one can verify that the limit condition

$$\lim_{\|d\|\to 0} \frac{H(z+d)-H(z)-H'(z,d)}{\|d\|} = 0$$

holds. This establishes the F-differentiability of H at the point z. Conversely, if H is F-differentiable at z, then the directional derivative $H'(z,d)$ must be a linear function in d. By linearity, it is easy to show that for each index $i \in \beta(z)$, the desired assertion about the i-th row of M must hold. This proves part (b). The proof of part (c) is similar and is left to the reader. For the final assertion, it suffices to show that if z solves (q, M), then the limit property (4) holds. The proof of this expression is easy by noting that we may write

$$\frac{\theta(u)-\theta(v)}{\|u-v\|} = \frac{(H(u)-H(v))^{\mathrm{T}}(H(u)+H(v))}{2\|u-v\|}$$

and observing that the function H is Lipschitzian and both $H(u)$ and $H(v)$ approach 0 because $H(z) = 0$. □

Several implications follow from Proposition **5.8.2**. First, both functions H and θ are F-differentiable at a nondegenerate vector because the assumptions in parts (b) and (c) are then satisfied vacuously. Second, it is possible for the norm function θ to be F-differentiable at a vector z without H being differentiable there. Third, the function θ must be F-differentiable at a solution point of the LCP (q, M); moreover, the F-derivative there is strong, (i.e., the limit (4) holds).

We have mentioned how the LCP (q, M) is related to the minimization of the norm function θ. In this context, we may ask the question of when a stationary point of θ is a solution of (q, M). To provide an answer to this question, we introduce the following concept.

5.8.3 Definition. A vector $z \in R^n$ is said to be *regular* (with respect to the function $H_{q,M}$) if

(a) the principal submatrix $M_{\alpha\alpha}$ is nonsingular

(b) the Schur complement

$$M_{\beta\beta} - M_{\beta\alpha}M_{\alpha\alpha}^{-1}M_{\alpha\beta}$$

is a $\boldsymbol{Q}$-matrix.

Here, α and β denote the index sets $\alpha(z)$ and $\beta(z)$ respectively, see (2). The vector z is said to be *strongly regular* (with respect to the same min function) if conditions (a) and (b′) hold:

(b′) the Schur complement

$$M_{\beta\beta} - M_{\beta\alpha}M_{\alpha\alpha}^{-1}M_{\alpha\beta}$$

is a $\boldsymbol{P}$-matrix.

Note that if $M \in \boldsymbol{P}$, then all vectors are strongly regular. Also, if z is a nondegenerate vector, then the strong regularity property of z is the same as the regularity property, and both reduce to the nonsingularity of the principal submatrix $M_{\alpha\alpha}$. The following result relates the regularity property to the norm function θ.

5.8.4 Proposition. Suppose that z is a stationary point of the norm function θ, i.e., suppose that

$$\theta'(z, d) \geq 0, \quad \text{for all } d \in R^n.$$

If z is regular, then z solves the LCP (q, M).

Proof. By regularity, it follows that for each vector $r \in R^n$, the system

$$\begin{aligned} (Md)_i &= r_i \quad \text{for } i \in \alpha(z) \\ \min(d_i, (Md)_i) &= r_i \quad \text{for } i \in \beta(z) \\ d_i &= r_i \quad \text{for } i \in \gamma(z) \end{aligned} \tag{5}$$

has a solution d; indeed, by substituting the variables $d_i = r_i$ for $i \in \gamma(z)$ and $d'_i = d_i - r_i$ for $i \in \beta(z)$, the system (5) becomes the mixed LCP

$$\begin{aligned} M_{\alpha\alpha}d_\alpha + M_{\alpha\beta}d'_\beta &= r'_\alpha \\ M_{\beta\alpha}d_\alpha + M_{\beta\beta}d'_\beta &\geq r'_\beta \\ d'_\beta &\geq 0 \\ (d'_\beta)^{\mathrm{T}}(M_{\beta\alpha}d_\alpha + M_{\beta\beta}d'_\beta - r'_\beta) &= 0 \end{aligned}$$

where α and β denote the index sets $\alpha(z)$ and $\beta(z)$ respectively, and where for $\delta = \alpha \cup \beta$,

$$r'_\delta = r_\delta - M_{\delta\gamma}r_\gamma - M_{\delta\beta}r_\beta,$$

the existence of a solution to the above mixed LCP follows from **3.12.5**.

Now, suppose that $z \notin \mathrm{SOL}(q, M)$. Then, $\min(z_i, (q + Mz)_i) \neq 0$ for some index i. Let $r = -\min(z_i, (q + Mz)_i)e_i$ and d be a solution to the corresponding system (5). It then follows from the expression for the directional derivative $\theta'(z, d)$ given in part (a) of **5.8.2** that

$$\theta'(z, d) = -(\min(z_i, (q + Mz)_i))^2 < 0$$

which contradicts the assumption that z is a stationary point of θ. □

Description and convergence of the method

We now explain the damped-Newton method for solving the problem (q, M). Suppose that a nondegenerate vector z^ν is given. According to **5.8.2**, the function H is F-differentiable at z^ν. Let α and γ denote the index sets $\alpha(z^\nu)$ and $\gamma(z^\nu)$ respectively (note that $\beta(z^\nu)$ is empty). Write $w^\nu = q + Mz^\nu$ and set up the Newton equation in order to solve for the direction d^ν,

$$H(z^\nu) + \nabla H(z^\nu)d^\nu = 0. \tag{6}$$

In terms of the index sets α and γ, the latter equation simplifies to

$$\begin{bmatrix} w^\nu_\alpha \\ z^\nu_\gamma \end{bmatrix} + \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\gamma} \\ 0 & I \end{bmatrix} \begin{bmatrix} d^\nu_\alpha \\ d^\nu_\gamma \end{bmatrix} = 0$$

which yields

$$d^\nu_\gamma = -z^\nu_\gamma \tag{7}$$

$$d^\nu_\alpha = -z^\nu_\alpha - M^{-1}_{\alpha\alpha} q_\alpha \tag{8}$$

provided that $M_{\alpha\alpha}$ is nonsingular. Since

$$\nabla\theta(z^\nu)^{\mathrm{T}} d^\nu = H(z^\nu)^{\mathrm{T}} \nabla H(z^\nu) d^\nu = -H(z^\nu)^{\mathrm{T}} H(z^\nu),$$

the vector d^ν is therefore a descent direction for the norm function θ at the point z^ν if $H(z^\nu) \neq 0$, i.e., if z^ν does not solve (q, M).

Since z^ν is nondegenerate, there exists a scalar $\delta_\nu \in (0,1)$ such that for all $\tau \in [0, \delta_\nu)$, the vector

$$z^\nu(\tau) = z^\nu + \tau d^\nu$$

is nondegenerate and

$$\alpha(z^\nu(\tau)) = \alpha(z^\nu), \text{ and } \gamma(z^\nu(\tau)) = \gamma(z^\nu).$$

To see how δ_ν is determined, it is convenient to define the pair of vectors u and v :

$$u_\alpha = -M^{-1}_{\alpha\alpha} q_\alpha, \; u_\gamma = 0 \tag{9}$$

and

$$v = q + Mu. \tag{10}$$

Notice that $v_\alpha = 0$; thus the vectors u and v are complementary. Moreover, u solves (q, M) if and only if u_α and v_γ are nonnegative. In general, the pair (u, v) is determined by the index set α which in turn is derived from the iterate z^ν. As the algorithm proceeds, different (u, v) pairs will be generated. However, since there are only finitely many index subsets of $\{1, \ldots, n\}$, there can be a finite number of such (u, v) pairs (which presumably correspond to an infinite sequence of iterates $\{z^\nu\}$).

In terms of the vector u, we may write

$$z^\nu(\tau) = (1-\tau)z^\nu + \tau u; \tag{11}$$

moreover, by letting $w^\nu(\tau) = q + Mz^\nu(\tau)$, we have

$$w^\nu(\tau) = (1-\tau)w^\nu + \tau v. \tag{12}$$

Suppose that $u \notin \text{SOL}(q, M)$. Define the two ratios

$$\rho_1 = \min\{\frac{z_i^\nu - w_i^\nu}{z_i^\nu - w_i^\nu - u_i} : i \in \alpha, u_i < 0\}$$

$$\rho_2 = \min\{\frac{w_i^\nu - z_i^\nu}{w_i^\nu - z_i^\nu - v_i} : i \in \gamma, v_i < 0\}.$$

It is then easy to see that the scalar δ_ν is given by

$$\delta_\nu = \min(\rho_1, \rho_2).$$

Since u is assumed not to be a solution of (q, M), the so defined δ_ν lies in the interval $(0, 1)$. Moreover, it is not difficult to verify that for any $\tau \in [0, \delta_\nu]$, the inequality holds

$$\theta(z^\nu) - \theta(z^\nu(\tau)) \geq \tau\theta(z^\nu).$$

Since $\theta(z^\nu)$ is positive, it follows that for an arbitrary $\sigma \in (0, 1/2)$, we have

$$\theta(z^\nu) - \theta(z^\nu(\tau)) > 2\sigma\tau\theta(z^\nu) \tag{13}$$

for any $\tau \in (0, \delta_\nu]$. By continuity, there exists a scalar $\bar{\delta}_\nu > \delta_\nu$ such that for any $\tau \in (\delta_\nu, \bar{\delta}_\nu]$, the inequality (13) continues to hold. Note that $z^\nu(\delta_\nu)$ is no longer a nondegenerate vector. However, for some $\delta'_\nu > \delta_\nu$, $z^\nu(\tau)$ will be nondegenerate for all $\tau \in (\delta_\nu, \delta'_\nu]$. Consequently, by taking the smaller of $\bar{\delta}_\nu$ and δ'_ν, we conclude that there is an interval beyond δ_ν such that the inequality (13) holds and the vector $z^\nu(\tau)$ is nondegenerate for all τ within this interval.

Summarizing the above analysis, we state the following algorithm **5.8.5** for solving the LCP (q, M). The algorithm starts with an arbitrary nondegenerate vector z^0 and generates a sequence of nondegenerate iterates $\{z^\nu\}$ which must be well-defined provided that M is a nondegenerate matrix. In general, the starting (nondegenerate) iterate z^0 is not always readily available, but often it is not difficult to obtain; for example, if the vector q contains no zero component, then the zero vector is nondegenerate and hence can be used to start the algorithm.

5.8.5 Algorithm. (The Damped-Newton Algorithm)

Step 0. *Initialization.* Let $\sigma \in (0, 1/2)$ and $\rho \in (0, 1)$ be given scalars, and z^0 be a nondegenerate vector. Set $\nu = 0$.

Step 1. *Compute direction.* Given the nondegenerate vector z^ν, let $\alpha = \alpha(z^\nu)$ and $\gamma = \gamma(z^\nu)$. Compute the vectors u and v by (9) and (10) respectively.

Step 2. *Test for termination: I.* Terminate if both u and v are nonnegative. The vector u solves (q, M). Otherwise, continue.

Step 3. *Compute stepsize.* Compute the scalar δ_ν. Let m_ν be the smallest nonnegative integer such that with

$$\tau_\nu = \delta_\nu + (1 - \delta_\nu)\rho^{m_\nu},$$

the vector

$$z^{\nu+1} = (1 - \tau_\nu)z^\nu + \tau_\nu u$$

is nondegenerate and the inequality (13) holds with $\tau = \tau_\nu$.

Step 4. *Test for termination: II.* Test $z^{\nu+1}$ for termination. Return to Step 1 with $\nu \leftarrow \nu + 1$ if $z^{\nu+1}$ fails the prescribed termination rule.

According to the preceding discussion, the integer m_ν can be determined in a finite number of trials by starting with $m = 0$ and successively testing the values $m = 1, 2, \ldots$. The scalar ρ is the *backtracking factor*. What Step 3 does is that it first tests if the inequality (13) holds with τ equal to unity (i.e., with $z^{\nu+1} = u$); if this fails, then the stepsize is scaled back by the factor ρ and the next vector (corresponding to $m = 1$) is tested. Such a test continues until the desired integer m_ν is obtained. There are two reasons for computing the stepsize τ_ν in this manner (in particular, why we want $\tau_\nu > \delta_\nu$): one reason is that the vector $z^\nu(\tau)$ cannot be a solution of (q, M) for $\tau \in [0, \delta_\nu]$, so there is no need to search in this interval; the second reason is that by enforcing $\tau_\nu > \delta_\nu$, we are assured of a change in the index sets α and γ, thereby avoiding the possibility of *jamming* (or stalling) at the same pair of index sets.

When Algorithm **5.8.5** terminates in a finite number of iterations, then either an exact solution (given by u) or an approximate solution $z^{\nu+1}$ of the LCP (q, M) is obtained. In the following analysis, we assume that the algorithm generates an infinite sequence of iterates $\{z^\nu\}$. By the infinite nature of the sequence, we must have $\theta(z^\nu) > 0$ for each ν. We study the convergence property of the sequence $\{z^\nu\}$.

By the inequality (13), we have

$$\theta(z^\nu) - \theta(z^{\nu+1}) > 2\sigma\tau_\nu\theta(z^\nu) > 0. \tag{14}$$

As we have pointed out, the nondegeneracy of the matrix M provides a sufficient condition for the sequence $\{z^\nu\}$ to be well defined; the same nondegeneracy property of M also ensures that this sequence $\{z^\nu\}$ must be bounded. The proof of this statement is an immediate consequence of the inequality (14) and the following lemma which provides a sufficient condition for the norm function θ to have bounded level sets.

5.8.6 Lemma. Let M be a nondegenerate matrix. Then the norm function $\theta(z)$ has bounded level sets, i.e., for all scalars c, the set

$$L(c) = \{z \in R^n : \|\min(z, q + Mz)\|_2 \leq c\}$$

is bounded.

Proof. The proof is by contradiction. Suppose that $\{z^\nu\}$ is an unbounded sequence in the set $L(c)$. For each ν, there is a subset α (depending on ν), such that after a suitable permutation of the components if necessary, we may write

$$\min(z^\nu, q + Mz^\nu) = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ 0 & I \end{bmatrix} \begin{bmatrix} z^\nu_\alpha \\ z^\nu_{\bar{\alpha}} \end{bmatrix} + \begin{bmatrix} q_\alpha \\ 0 \end{bmatrix}. \tag{15}$$

Since there are only finitely many subsets α, there must exist a certain α and a subsequence $\{z^\nu : \nu \in \kappa\}$ such that (15) holds for all $\nu \in \kappa$. The nondegeneracy of M implies that $\{z^\nu : \nu \in \kappa\}$ must be bounded. This contradiction establishes the lemma. □

In general, despite the fact that each iterate z^ν produced by **5.8.5** is a nondegenerate vector, we can not establish that a limit point of $\{z^\nu\}$ must be nondegenerate. As a matter of fact, a consequence of the main convergence result below is that if $\tilde{z}$ is a nondegenerate accumulation point of $\{z^\nu\}$, then $\tilde{z}$ must be a solution of the LCP (q, M). The more difficult argument concerns those accumulation points (if there are any) which are not nondegenerate vectors.

The theorem below is the main convergence result for the damped-Newton algorithm. It establishes several necessary and sufficient conditions for an arbitrary accumulation point of an infinite sequence $\{z^\nu\}$ produced by the method to be a solution of the LCP (q, M). Note that the result does not assert that every limit point of $\{z^\nu\}$ must solve (q, M).

5.8.7 Theorem. Suppose that $\tilde{z}$ is an accumulation point of an infinite sequence $\{z^\nu\}$ produced by **5.8.5**. The following statements are equivalent:

(a) $\tilde{z}$ solves the LCP (q, M)

(b) for each $i \in \beta(\tilde{z})$,

$$\tilde{z}_i = (q + M\tilde{z})_i = 0,$$

(c) for each $i \in \beta(\tilde{z})$,

$$\tilde{z}_i = (q + M\tilde{z})_i \geq 0,$$

(d) the norm function θ has a strong F-derivative at $\tilde{z}$.

In particular, (a) holds if $\tilde{z}$ is a nondegenerate vector.

Proof. Clearly, (a) $\Rightarrow$ (b) $\Rightarrow$ (c); moreover, by **5.8.2**, (a) $\Rightarrow$ (d). Hence it suffices to establish [(c) $\Rightarrow$ (a)] and [(d) $\Rightarrow$ (a)]. We prove the former implication first. Suppose that (c) holds. Let $\{z^\nu : \nu \in \kappa\}$ be a subsequence converging to $\tilde{z}$. Write $\tilde{w} = q + M\tilde{z}$.

The sequence $\{\theta(z^\nu)\}$ is nonincreasing and nonnegative, so it converges; hence $\theta(z^\nu) - \theta(z^{\nu+1}) \to 0$. The inequality (14) implies

$$\lim_{\nu \to \infty} \tau_\nu \theta(z^\nu) = 0.$$

If $\liminf_{\nu \to \infty, \nu \in \kappa} \tau_\nu > 0$, then $\theta(\tilde{z}) = 0$ and $\tilde{z}$ solves (q, M). Suppose that

$$\liminf_{\nu \to \infty, \nu \in \kappa} \tau_\nu = 0.$$

By restricting attention to a sub-subsequence if necessary, we may assume that

$$\lim_{\nu\to\infty,\nu\in\kappa} \tau_\nu = 0, \quad \text{which implies} \quad \lim_{\nu\to\infty,\nu\in\kappa} m_\nu = \infty.$$

We claim for every $\nu \in \kappa$ sufficiently large such that $m_\nu > n+1$, an integer $k_\nu \in \{1, \cdots, n, n+1\}$ exists such that with $\tau'_\nu = \delta_\nu + (1-\delta_\nu)\rho^{m_\nu - k_\nu}$, the vector $x^\nu = z^\nu(\tau'_\nu)$ is nondegenerate and

$$\theta(z^\nu) - \theta(x^\nu) \leq 2\sigma\tau'_\nu\theta(z^\nu). \tag{16}$$

The existence of the integer k_ν is due to two facts: (i) there are at most n values of $\tau \in [0,1]$ for which $z^\nu(\tau)$ is degenerate—this in turns is a consequence of the fact that for $i = 1, \cdots, n$, $z_i^\nu(\tau)$ is a linear function of $\tau \in [0,1]$; and (ii) m_ν is the smallest nonnegative integer for which the first inequality in (14) holds with such a τ_ν. It is easy to see that

$$\lim_{\nu\to\infty,\nu\in\kappa} \tau'_\nu = 0.$$

Hence, the sequence $\{x^\nu : \nu \in \kappa\}$ also converges to $\tilde{z}$. Then, for all $\nu \in \kappa$ sufficiently large, we have

$$\alpha(\tilde{z}) \subseteq \alpha(z^\nu) \cap \alpha(x^\nu), \text{ and } \gamma(\tilde{z}) \subseteq \gamma(z^\nu) \cap \gamma(x^\nu).$$

In view of these set inclusions, we may write

$$\theta(z^\nu) - \theta(x^\nu) = \tfrac{1}{2}(T_1 + T_2 + T_3)$$

where

$$T_1 = \sum_{i\in\alpha(\tilde{z})} ((w_i^\nu)^2 - (y_i^\nu)^2)$$

$$T_2 = \sum_{i\in\beta(\tilde{z})} (\min(z_i^\nu, w_i^\nu)^2 - \min(x_i^\nu, y_i^\nu)^2)$$

$$T_3 = \sum_{i\in\gamma(\tilde{z})} ((z_i^\nu)^2 - (x_i^\nu)^2)$$

and where $y^\nu = w^\nu(\tau'_\nu)$. Consider an index $i \in \alpha(\tilde{z})$. Then $i \in \alpha(z^\nu)$ for all $\nu \in \kappa$ sufficiently large. Hence, by (12) and the fact that $v_i = 0$, we obtain

$$(w_i^\nu)^2 - (y_i^\nu)^2 = (w_i^\nu)^2 - (1-\tau'_\nu)^2(w_i^\nu)^2 = 2\tau'_\nu(w_i^\nu)^2(1 - \tfrac{1}{2}\tau'_\nu). \tag{17}$$

Consequently, we deduce

$$\lim_{\nu\to\infty,\,\nu\in\kappa} \frac{T_1}{\tau'_\nu} = 2 \sum_{i\in\alpha(\tilde{z})} \tilde{w}_i^2.$$

By a similar argument, we derive

$$\lim_{\nu\to\infty,\,\nu\in\kappa} \frac{T_3}{\tau'_\nu} = 2 \sum_{i\in\gamma(\tilde{z})} \tilde{z}_i^2.$$

Consider the term T_2. By assumption, we may write

$$T_2 = T_{2,+} + T_{2,0}$$

where

$$\begin{aligned} T_{2,+} &= \textstyle\sum_{i\in\beta_+(\tilde{z})}(\min(z_i^\nu, w_i^\nu)^2 - \min(x_i^\nu, y_i^\nu)^2) \\ T_{2,0} &= \textstyle\sum_{i\in\beta_0(\tilde{z})}(\min(z_i^\nu, w_i^\nu)^2 - \min(x_i^\nu, y_i^\nu)^2) \end{aligned}$$

with

$$\beta_+(\tilde{z}) = \{i \in \beta(\tilde{z}) : \tilde{z}_i > 0\}, \quad \beta_0(\tilde{z}) = \{i \in \beta(\tilde{z}) : \tilde{z}_i = 0\}.$$

Notice that there is no negative component in $(\tilde{z}_i : i \in \beta(\tilde{z}))$ by assumption (c). Consider an index $i \in \beta_+(\tilde{z})$. It follows that for all $\nu \in \kappa$ large enough, we have all four quantities $z_i^\nu, w_i^\nu, x_i^\nu$ and y_i^ν positive. The index i must belong to either $\alpha(z^\nu)$ or $\gamma(z^\nu)$. Suppose that $i \in \alpha(z^\nu)$. Then, $\min(z_i^\nu, w_i^\nu) = w_i^\nu$. It follows that for $\nu \in \kappa$ large enough,

$$(\min(z_i^\nu, w_i^\nu))^2 - (\min(x_i^\nu, y_i^\nu))^2 \geq (w_i^\nu)^2 - (y_i^\nu)^2$$

because x_i^ν and y_i^ν are positive. Hence, by (17), we obtain

$$(\min(z_i^\nu, w_i^\nu))^2 - (\min(x_i^\nu, y_i^\nu))^2 \geq 2\tau'_\nu(\min(z_i^\nu, w_i^\nu))^2(1 - \tfrac{1}{2}\tau'_\nu).$$

By a similar argument, we may derive this same inequality if $i \in \gamma(z^\nu)$. Consequently, it follows that

$$\lim_{\nu\to\infty,\,\nu\in\kappa} \frac{T_{2,+}}{\tau'_\nu} \geq 2 \sum_{i\in\beta_+(\tilde{z})} (\min(\tilde{w}_i, \tilde{z}_i))^2.$$

Finally, by using an argument similar to the proof of the expression (4), we may derive

$$\lim_{\nu\to\infty,\,\nu\in\kappa} \frac{T_{2,0}}{\tau'_\nu} = 0 = 2 \sum_{i\in\beta_0(\tilde{z})} (\min(\tilde{w}_i, \tilde{z}_i))^2.$$

Now, summarizing the above derivation, we conclude that

$$\lim_{\nu\to\infty,\nu\in\kappa} \frac{\theta(z^\nu)-\theta(x^\nu)}{\tau'_\nu} \geq 2\theta(\tilde{z}).$$

Therefore, dividing the inequality (16) by τ'_ν, and passing to the limit $\nu \to \infty$, $\nu \in \kappa$, we obtain

$$2\theta(\tilde{z}) \leq 2\sigma\theta(\tilde{z})$$

which implies $\theta(\tilde{z}) = 0$ because $2\sigma < 1$. This establishes the implication [(c) $\Rightarrow$ (a)].

It remains to verify the implication (d) $\Rightarrow$ (a). Suppose that θ has a strong F-derivative at $\tilde{z}$. Then the following limit holds

$$\lim_{\nu\to\infty,\nu\in\kappa} \frac{\theta(x^\nu)-\theta(z^\nu)-\nabla\theta(z^\nu)^{\mathrm{T}}(x^\nu-z^\nu)}{\|x^\nu-z^\nu\|} = 0.$$

By the definition of x^ν, we have

$$x^\nu - z^\nu = \tau'_\nu d^\nu$$

where d^ν is defined in (7) and (8). Clearly, $\{d^\nu : \nu \in \kappa\}$ is bounded. Hence

$$\lim_{\nu\to\infty,\nu\in\kappa} \frac{\theta(x^\nu)-\theta(z^\nu)-\nabla\theta(z^\nu)^{\mathrm{T}}(x^\nu-z^\nu)}{\tau'_\nu} = 0.$$

In view of the equation (6), it is easy to see

$$\nabla\theta(z^\nu)^{\mathrm{T}}(x^\nu - z^\nu) = -2\tau'_\nu\theta(z^\nu).$$

Consequently, we deduce

$$\lim_{\nu\to\infty,\nu\in\kappa} \frac{\theta(z^\nu)-\theta(x^\nu)}{\tau'_\nu} = 2\theta(\tilde{z})$$

which in view of (16) yields $\theta(\tilde{z}) = 0$ as before. This completes the proof of the theorem. □

5.8.8 Remark. In parts (b) and (c) of Theorem **5.8.7**, if $i \in \beta(\tilde{z})$, then $\tilde{z}_i$ and $(q + M\tilde{z})_i$ are already equal by the definition of the set $\beta(\tilde{z})$; what is required in these parts is that the common values of $\tilde{z}_i$ and $(q + M\tilde{z})_i$ not be negative.

The major operation in each iteration of the damped-Newton algorithm **5.8.5** is the computation of the pair of vectors (u, v) in (9) and (10) which in turn requires the solution of the system of linear equations

$$q_\alpha + M_{\alpha\alpha} u_\alpha = 0.$$

In this respect, the algorithm resembles a (block) pivoting method because solving such linear equations is essentially a pivot step. Besides its infinite nature, a special feature of **5.8.5** which distinguishes this algorithm from all the pivoting methods in Chapter 4 is the presence of a linesearch procedure on the merit function $\theta(z)$; of course, this latter step is the key to the determination of the index set α which, in turn, dictates what the next "pivot" is, i.e., which system of equations to solve.

5.9 Interior-Point Methods

In Section 5.6, we have described an iterative method for solving the LCP (q, M) when M is a $\boldsymbol{P}_0$-matrix. This method is based on the regularization idea and requires solving a sequence of subproblems which themselves are LCPs. In this section, we discuss two alternate methods for solving an LCP of same type which rely on the existence of a strictly feasible point of the problem (recall that this is a vector $z > 0$ satisfying $q + Mz > 0$). Starting at such a point, both methods generate a sequence of interior points in FEA(q, M); it is for this reason that the methods are called *interior-point methods.*

The interior-point methods for the linear complementarity problem have their roots in solving linear programs and possess some interesting computational complexity properties when specialized to a positive semi-definite LCP. In the following subsections, we describe the methods in their "infinite" version and discuss some of their convergence properties; complexity results are omitted but can be found in the references cited in **5.12.22**.

Throughout this section, the matrix M is assumed to be in the class $\boldsymbol{P}_0$. We further assume that the LCP (q, M) has a strictly feasible vector.

A merit function approach

The first interior-point method to be discussed is one of the descent type. It starts at a given interior point of FEA(q, M), computes a descent direction along which a certain merit function will be decreased, and obtains the next iterate that remains a strictly feasible point. In order to introduce this method, fix a scalar $\zeta > n$ and consider the real-valued function $\phi : R^n_{++} \times R^n_{++} \rightarrow R$ defined by

$$\phi(z, w) = \zeta \log(z^{\mathrm{T}} w) - \sum_{i=1}^{n} \log(z_i w_i). \tag{1}$$

This is the merit function for this class of interior-point methods; it is well defined whenever z and w are positive vectors. The following result lists some useful properties of the function $\phi(z, w)$.

5.9.1 Lemma. Let z and w be two positive n-vectors and $\zeta > n$. Then

$$\phi(z, w) \geq (\zeta - n) \log(z^{\mathrm{T}} w), \tag{2}$$

$$(\nabla_z \phi(z, w))_i = w_i \Big(\frac{\zeta}{z^{\mathrm{T}} w} - \frac{1}{z_i w_i} \Big), \quad \text{for all } \; i, \tag{3}$$

$$(\nabla_w \phi(z, w))_i = z_i \Big(\frac{\zeta}{z^{\mathrm{T}} w} - \frac{1}{z_i w_i} \Big), \quad \text{for all } \; i, \tag{4}$$

$$(\nabla_z \phi(z, w))^{\mathrm{T}} \nabla_w \phi(z, w) > 0. \tag{5}$$

Proof. It is fairly straightforward to verify the first three expressions. We now prove (5). Suppose that $(\nabla_z \phi(z, w))^{\mathrm{T}} \nabla_w \phi(z, w) = 0$. Since each product $(\nabla_z \phi(z, w))_i (\nabla_w \phi(z, w))_i$ is nonnegative, it follows that for all i,

$$\frac{\zeta}{z^{\mathrm{T}} w} - \frac{1}{z_i w_i} = 0.$$

This implies

$$\zeta z^{\mathrm{T}} w = n z^{\mathrm{T}} w$$

which is a contradiction because $\zeta > n$. □

The next result is an immediate consequence of the expressions (3), (4) and (5) in the above lemma, and provides an important justification for the descent step of the interior-point method to be described.

5.9.2 Corollary. If $M \in \boldsymbol{P_0}$, then for $z, w > 0$,

$$\nabla_z \phi(z, w) + M^{\mathrm{T}} \nabla_w \phi(z, w) \neq 0.$$

Proof. Suppose the contrary. By (3) and (4), it follows that for all i,

$$(\nabla_w \phi(z,w))_i (M^{\mathrm{T}} \nabla_w \phi(z,w))_i = -(\nabla_w \phi(z,w))_i (\nabla_z \phi(z,w))_i \leq 0. \quad (6)$$

The expression (5) implies that $\nabla_w \phi(z,w) \neq 0$. Since M (and hence M^{T}) is in the class $\boldsymbol{P}_0$, it follows that there must exist an index i for which $(\nabla_w \phi(z,w))_i \neq 0$ and $(\nabla_w \phi(z,w))_i (M^{\mathrm{T}} \nabla_w \phi(z,w))_i \geq 0$. In view of (6), we deduce that for such an i,

$$(\nabla_w \phi(z,w))_i (\nabla_z \phi(z,w))_i = 0$$

which implies that $(\nabla_w \phi(z,w))_i = 0$ by (3) and (4). This leads to a contradiction. □

We now describe the merit reduction interior-point method for solving the LCP (q, M) when $M \in \boldsymbol{P}_0$. In the algorithm, the scalar $\beta \in (0,1)$ controls the stepsize in each descent iteration and ensures the strict feasibility of the iterates obtained; $\rho \in (0,1)$ is the usual backtracking factor required in the linesearch step; and $\sigma \in (0, \frac{1}{2})$ determines the amount of sufficient decrease in the linesearch.

5.9.3 Algorithm. (The Merit Reduction Interior-Point Method)

Step 0. *Initialization.* Let $\beta, \rho \in (0,1)$ and $\sigma \in (0, \frac{1}{2})$ be given. Let z^0 be a strictly feasible point of (q, M) and let $w^0 = q + Mz^0$. Set $\nu = 0$.

Step 1. *Compute direction and stepsize.* Given the pair $(z^\nu, w^\nu) > 0$, let

$$\nabla_z \phi_\nu = \nabla_z \phi(z^\nu, w^\nu), \qquad \nabla_w \phi_\nu = \nabla_w \phi(z^\nu, w^\nu),$$

and

$$Z^\nu = \operatorname{diag}(z^\nu), \qquad W^\nu = \operatorname{diag}(w^\nu).$$

Solve the following minimization problem to obtain the search direction (d_z^ν, d_w^ν) :

$$\begin{array}{ll} \text{minimize} & (\nabla_z \phi_\nu)^{\mathrm{T}} d_z + (\nabla_w \phi_\nu)^{\mathrm{T}} d_w \\ \text{subject to} & d_w = M d_z \\ & \|(Z^\nu)^{-1} d_z\|_2^2 + \|(W^\nu)^{-1} d_w\|_2^2 \leq \beta^2. \end{array}$$

Set $\Delta_\nu = (\nabla_z\phi_\nu)^{\mathrm{T}}d_z^\nu + (\nabla_w\phi_\nu)^{\mathrm{T}}d_w^\nu$. Let m_ν be the smallest nonnegative integer m such that

$$\phi(z^\nu + \rho^m d_z^\nu, w^\nu + \rho^m d_w^\nu) - \phi(z^\nu, w^\nu) \leq \sigma\rho^m\Delta_\nu.$$

Step 2. *Termination test.* Set

$$(z^{\nu+1}, w^{\nu+1}) = (z^\nu, w^\nu) + \rho^{m_\nu}(d_z^\nu, d_w^\nu).$$

If $(z^{\nu+1}, w^{\nu+1})$ satisfies a prescribed termination rule, stop; otherwise, return to Step 1 with ν replaced by $\nu + 1$.

The search direction (d_z^ν, d_w^ν) can be computed from an explicit expression. Indeed, let

$$p^\nu = \nabla_z\phi_\nu + M^{\mathrm{T}}\nabla_w\phi_\nu,$$

$$M^\nu = (Z^\nu)^{-2} + M^{\mathrm{T}}(W^\nu)^{-2}M;$$

the vector p^ν is nonzero by **5.9.2**, and the matrix M^ν is clearly symmetric and positive definite; hence, the scalar

$$\lambda_\nu = \frac{\sqrt{(p^\nu)^{\mathrm{T}}(M^\nu)^{-1}p^\nu}}{\beta}$$

is positive. It is easy to show that

$$d_z^\nu = -\frac{1}{\lambda_\nu}(M^\nu)^{-1}p^\nu, \qquad d_w^\nu = Md_z^\nu.$$

Since

$$(\nabla_z\phi_\nu)^{\mathrm{T}}d_z^\nu + (\nabla_w\phi_\nu)^{\mathrm{T}}d_w^\nu = -\lambda_\nu\beta^2 < 0, \tag{7}$$

it follows that (d_z^ν, d_w^ν) indeed provides a descent direction for the function $\phi(z, w)$ at the iterate (z^ν, w^ν). Moreover, it is obvious that for any scalar $\tau \in [0, 1)$, the vector pair

$$(z^\nu(\tau), w^\nu(\tau)) = (z^\nu, w^\nu) + \tau(d_z^\nu, d_w^\nu)$$

remains positive; in particular, so is the pair $(z^{\nu+1}, w^{\nu+1})$ defined in the algorithm.

In view of (7), the sequence $\{z^\nu\}$ satisfies the inequality

$$\phi(z^{\nu+1}, w^{\nu+1}) - \phi(z^\nu, w^\nu) \leq -\sigma\beta^2\rho^{m_\nu}\lambda_\nu < 0 \tag{8}$$

which implies that the sequence $\{\phi(z^\nu, w^\nu)\}$ is decreasing. Thus, by (2), the sequence

$$\{(z^\nu)^{\mathrm{T}} w^\nu\}$$

is bounded. Since z^ν and w^ν are both positive, it follows that for each i, the componentwise sequence $\{z_i^\nu w_i^\nu\}$ is also bounded. The result below shows that if $M \in \boldsymbol{R_0}$, then the sequence $\{z^\nu\}$ must be bounded.

5.9.4 Proposition. Let $M \in \boldsymbol{P_0} \cap \boldsymbol{R_0}$. Then the sequence $\{z^\nu\}$ generated by **5.9.3** is bounded.

Proof. Suppose the contrary. Let $\{z^\nu : \nu \in \kappa\}$ be a subsequence such that $\{\|z^\nu\| : \nu \in \kappa\} \to \infty$. The normalized sequence $\{z^\nu/\|z^\nu\| : \nu \in \kappa\}$ has at least one accumulation point, say $\tilde{z}$. Clearly, $\tilde{z}$ is nonzero and nonnegative; moreover, the vector $\tilde{w} = M\tilde{z}$, being the limit of the sequence $\{w^\nu/\|z^\nu\| : \nu \in \kappa\}$, must be nonnegative.

As pointed out before, for each i, the sequence $\{z_i^\nu w_i^\nu\}$ is bounded; thus for some constant c_i, we have

$$z_i^\nu w_i^\nu \leq c_i$$

for all ν. Dividing both sides in the last inequality by $\|z^\nu\|^2$ and passing to the limit $\{\nu \in \kappa, \nu \to \infty\}$, we deduce

$$\tilde{z}_i \tilde{w}_i \leq 0, \qquad \text{for all } i.$$

Since $\tilde{z}$ and $\tilde{w}$ are nonnegative vectors, equalities must hold for all i. This shows that $\tilde{z}$ is a nonzero solution of the homogeneous LCP $(0, M)$ which contradicts the assumption that $M \in \boldsymbol{R_0}$. Therefore, the lemma is established. □

5.9.5 Remark. The assumption $M \in \boldsymbol{R_0}$ in Proposition **5.9.4** is rather restrictive if one considers the LCP (q, M) for an individual vector q. There are several ways to relax this condition. For example, if the matrix M is copositive (in addition to being in the class $\boldsymbol{P_0}$), then it is enough to assume that the implication (5.3.4) holds. Alternatively, the conclusion of **5.9.4** remains valid if $M \in \boldsymbol{P}_0$ and if the level set

$$L_\tau = \{z \in \text{FEA}(q, M) : z^{\mathrm{T}}(q + Mz) \leq \tau\}$$

is bounded for every $\tau \geq 0$. By the proof of **3.9.23**, it is not hard to show that $M \in \boldsymbol{R}_0$ if and only if the set L_τ is bounded for every vector q and every scalar $\tau \geq 0$.

5.9.6 Remark. We recall from Theorem **3.9.22** that if $M \in \boldsymbol{P}_0 \cap \boldsymbol{R}_0$, then $M \in \boldsymbol{Q} \subseteq \boldsymbol{S}$. In this case, the LCP (q, M) has a strictly feasible vector z for each vector q. Consequently, the assumption of the existence of such a vector z becomes redundant under the hypothesis of Proposition **5.9.4**.

Having established sufficient conditions for the sequence $\{z^\nu\}$ to be bounded, we now turn to its convergence. The following is the main result of this kind.

5.9.7 Theorem. Let M be a $\boldsymbol{P}_0$-matrix. Suppose that the LCP (q, M) has a strictly feasible solution. Then every accumulation point of the sequence $\{z^\nu\}$ produced by **5.9.3** solves (q, M).

Proof. Let $\tilde{z}$ be the limit of a subsequence $\{z^\nu : \nu \in \kappa\}$ and let $\tilde{w} = q + M\tilde{z}$. Clearly, the pair $(\tilde{z}, \tilde{w})$ is nonnegative. Since $\phi(\tilde{z}, \tilde{w}) < \infty$, it follows that either $\tilde{z}^{\mathrm{T}}\tilde{w} = 0$ or $(\tilde{z}, \tilde{w}) > 0$. Without loss of generality, we may assume that the latter holds. Let $\tilde{p}$ and $\tilde{M}$ denote the limits of the sequences $\{p^\nu : \nu \in \kappa\}$ and $\{M^\nu : \nu \in \kappa\}$ respectively. The matrix $\tilde{M}$ remains positive definite; moreover, the sequence of scalars $\{\lambda_\nu : \nu \in \kappa\}$ converges to

$$\tilde{\lambda} = \frac{\sqrt{\tilde{p}^{\mathrm{T}}\tilde{M}^{-1}\tilde{p}}}{\beta}$$

which is positive, and the sequence of directions $\{(d_z^\nu, d_w^\nu) : \nu \in \kappa\}$ converges to $(\tilde{d}_z, \tilde{d}_w)$ where

$$\tilde{d}_z = -\tilde{M}^{-1}\tilde{p}/\tilde{\lambda}, \quad \tilde{d}_w = M\tilde{d}_z.$$

Since the sequence $\{\phi(z^{\nu+1}, w^{\nu+1}) - \phi(z^\nu, w^\nu)\}$ converges to zero, the inequality (8) implies

$$\lim_{\nu \to \infty, \nu \in \kappa} \rho^{m_\nu} = 0$$

or equivalently,

$$\lim_{nu \to \infty, \nu \in \kappa} m_\nu = \infty.$$

Hence, both sequences

$$\{(z^{\nu+1}, w^{\nu+1}) : \nu \in \kappa\} \quad \text{and} \quad \{(z^\nu + \tau_\nu d_z^\nu, w^\nu + \tau_\nu d_w^\nu) : \nu \in \kappa\}$$

where $\tau_\nu = \rho^{m_\nu - 1}$, converge to $(\tilde{z}, \tilde{w})$. By the definition m_ν, we have

$$\frac{\phi(z^\nu + \tau_\nu d_z^\nu, w^\nu + \tau_\nu d_w^\nu) - \phi(z^\nu, w^\nu)}{\tau_\nu} > -\sigma \beta^2 \lambda_\nu.$$

On the other hand, (8) implies

$$\frac{\phi(z^{\nu+1}, w^{\nu+1}) - \phi(z^\nu, w^\nu)}{\rho^{m_\nu}} \leq -\sigma \beta^2 \lambda_\nu.$$

Passing to the limit $\{\nu \to \infty, \nu \in \kappa\}$ and noting that ϕ is F-differentiable at $(\tilde{z}, \tilde{w})$, we deduce

$$(\nabla_z \phi(\tilde{z}, \tilde{w}))^{\mathrm{T}} \tilde{d}_z + (\nabla_w \phi(\tilde{z}, \tilde{w}))^{\mathrm{T}} \tilde{d}_w = -\sigma \tilde{\lambda} \beta^2.$$

Similarly, passing to the same limit in (7), we obtain

$$(\nabla_z \phi(\tilde{z}, \tilde{w}))^{\mathrm{T}} \tilde{d}_z + (\nabla_w \phi(\tilde{z}, \tilde{w}))^{\mathrm{T}} \tilde{d}_w = -\tilde{\lambda} \beta^2$$

which is a contradiction. This establishes the theorem. □

It goes without saying that the applicability of the interior-point method depends crucially on the existence of a strictly feasible vector of the LCP (q, M). In general, such a vector is not always available. However, the augmented LCP (3.7.10) introduced in Section 3.7 must possess one which is trivial to obtain. Recall that this augmented LCP (q', M') is defined by

$$q' = \begin{bmatrix} q \\ a \end{bmatrix}, \quad M' = \begin{bmatrix} M & I \\ -I & 0 \end{bmatrix};$$

if $a > 0$, then any pair $(z, y) > 0$ with z sufficiently small and y sufficiently large is strictly feasible to (q', M'). Moreover, if $M \in \boldsymbol{P_0}$, then so is M'. Consequently, one can apply **5.9.3** to this augmented LCP. Note that M' is not an $\boldsymbol{R_0}$-matrix; nevertheless, one can show that the sequence $\{(z^\nu, y^\nu)\}$ produced must be bounded (see Exercise **5.11.14**). According to Theorem **5.9.7**, every limit point $(\tilde{z}, \tilde{y})$ of $\{(z^\nu, y^\nu)\}$ must be a solution of the augmented LCP (q', M'). The remaining question is how we can recover a solution of the original LCP (q, M) if it exists. When M is column sufficient, one can invoke Theorem **3.7.17** to complete this task.

A continuation approach

Another interior-point method for solving the problem (q, M) is derived from applying the idea of *numerical continuation* to a certain system of (nonlinear) equations that is a one-parameter perturbation of the LCP system. In order to explain this technique, we recall the notion of the Hadamard product of two vectors introduced in **4.1.6**.

In terms of the Hadamard product, Algorithm **5.9.3** can be described as generating a sequence of positive vector pairs (w^ν, z^ν) each of which satisfies the system

$$w - Mz = q \tag{9}$$

$$w * z = c \tag{10}$$

for some positive vector $c \in R^n$. The goal of this algorithm is to drive the sequence $\{c^\nu\}$, where $c^\nu = w^\nu * z^\nu$, to zero. The system (9) motivates the definition of the mapping $\Phi : R^{2n}_+ \to R^n \times R^n_+$ given by

$$\Phi(w, z) = \begin{bmatrix} w - Mz \\ w * z \end{bmatrix}. \tag{11}$$

This mapping plays a central role in the continuation interior-point method for solving the LCP (q, M) corresponding to arbitrary vectors q. In order to prepare for the description of this method, we first derive several properties of Φ.

Clearly, Φ is continuously differentiable on its domain and its Jacobian matrix is easily computed to be

$$\nabla\Phi(w, z) = \begin{bmatrix} I & -M \\ Z & W \end{bmatrix}$$

where $W = \text{diag}(w)$ and $Z = \text{diag}(z)$. The following result provides a simple characterization for the nonsingularity of this Jacobian matrix for $(w, z) > 0$.

5.9.8 Lemma. A necessary and sufficient condition for the Jacobian matrix $\nabla\Phi(w,z)$ to be nonsingular for all $(w,z) > 0$ is that $M \in \boldsymbol{P}_0$.

Proof. Let $(w,z) > 0$ be given. It is easy to show that a pair of vectors u and v satisfies

$$\nabla\Phi(w,z)\begin{bmatrix} u \\ v \end{bmatrix} = 0$$

if and only if for all i,

$$v_i = -\frac{z_i}{w_i}(Mv)_i.$$

This observation, together with the sign reversing property of a $\boldsymbol{P}_0$-matrix (see part (b) in Theorem **3.4.2**), readily yields the desired conclusion of the lemma. □

An immediate consequence of Lemma **5.9.8** is that if M is a $\boldsymbol{P}_0$-matrix, then the mapping Φ is a local homeomorphism from R^{2n}_{++} into $R^n \times R^n_{++}$ (by Theorem **2.1.22**). Moreover, since R^{2n}_{++} is an open set in R^{2n}, Corollary **2.1.23** implies that the set $\Phi(R^{2n}_{++}) \subseteq R^n \times R^n_{++}$ is open in R^{2n}. Thus, we have established the following result.

5.9.9 Corollary. Let $M \in \boldsymbol{P}_0 \cap R^{n\times n}$. Then $\Phi(R^{2n}_{++}) \subseteq R^n \times R^n_{++}$ is an open subset of R^{2n}. □

The next result shows that if $M \in \boldsymbol{P}_0$, then Φ is an injective mapping on R^{2n}_{++}.

5.9.10 Lemma. Suppose that $M \in \boldsymbol{P}_0 \cap R^{n\times n}$. Then $\Phi : R^{2n}_{+} \to R^n \times R^n_{+}$ is an injection when restricted to R^{2n}_{++}. This restriction can be removed if $M \in \boldsymbol{P}$. □

The proof of this lemma follows easily from the inequality

$$(a-b)(c-d) \leq |ac - bd| \tag{12}$$

which must hold for any nonnegative scalars a, b, c and d. The reader is asked to supply the omitted proof of this inequality and the lemma in Exercise **5.11.16**.

With the above preliminary results in place, we are now ready to establish a global surjectivity property of the mapping Φ.

5.9.11 Theorem. Let $M \in R^{n\times n} \cap \boldsymbol{P_0} \cap \boldsymbol{R_0}$. Then $\Phi(R^{2n}_+) = R^n \times R^n_+$. Thus, Φ is a global homeomorphism from R^{2n}_+ onto $R^n \times R^n_+$ if $M \in \boldsymbol{P}$.

Proof. It suffices to establish the inclusion

$$R^n \times R^n_+ \subseteq \Phi(R^{2n}_+).$$

Suppose that this does not hold. Let $(q, c) \in (R^n \times R^n_+) \setminus \Phi(R^{2n}_+)$. We claim that there exists a vector $(\bar{q}, \bar{c}) \in R^{2n}$ for which

$$(\bar{q}, \bar{c}) \in (R^n \times R^n_+) \setminus \Phi(R^{2n}_+), \quad \text{and} \quad (\bar{q}, \bar{c}) \in \operatorname{cl} \Phi(R^{2n}_+) \tag{13}$$

where $\operatorname{cl} A$ denotes the closure of a set A in R^{2n}. This claim is easily seen to be true if $\Phi(R^{2n}_{++}) = R^n \times R^n_{++}$; in this case, the pair (q, c) itself would do the job. On the other hand, if $\Phi(R^{2n}_{++})$ is a proper subset of $R^n \times R^n_{++}$, then since $\Phi(R^{2n}_{++})$ is open by Corollary **5.9.9**, it follows that there exists $(\bar{q}, \bar{c}) \in R^n \times R^n_{++} \setminus \Phi(R^{2n}_{++})$ and $(\bar{q}, \bar{c}) \in \operatorname{cl} \Phi(R^{2n}_{++})$. Clearly, this pair $(\bar{q}, \bar{c})$ must satisfy the conditions in (13).

With the existence of $(\bar{q}, \bar{c})$ established, let $\{(q^k, c^k)\}$ be a sequence in $\Phi(R^{2n}_+)$ converging to $(\bar{q}, \bar{c})$. For each k, let $(w^k, z^k) \in R^{2n}_+$ satisfy

$$w^k - Mz^k = q^k$$

$$w^k * z^k = c^k.$$

If the sequence $\{(w^k, z^k)\}$ is bounded, then any one of its subsequential limits can be used to produce a contradiction to the condition that the limit $(\bar{q}, \bar{c}) \notin \Phi(R^{2n}_+)$. Hence, the sequence $\{(w^k, z^k)\}$ must be unbounded. In this case, a subsequential limit of the sequence $\{z^k/\|(w^k, z^k)\|\}$ can easily be shown to be a nonzero solution of the homogeneous LCP $(0, M)$. But this contradicts the $\boldsymbol{R_0}$-property of M. This contradiction establishes the surjectivity of Φ. If $M \in \boldsymbol{P}$, then Lemma **5.9.10** implies that Φ is injective on its domain R^{2n}_+. Hence, the last conclusion of the theorem follows. □

5.9.12 Remark. The proof of Theorem **5.9.11** provides an alternative demonstration of the matrix class inclusion $\boldsymbol{P_0} \cap \boldsymbol{R_0} \subseteq \boldsymbol{Q}$ (cf. **3.9.22**).

When one considers the solution of an LCP (q, M) corresponding to a given vector q, the global surjectivity of the mapping Φ is an excessively strong property; in this instance, one is merely interested in the solvability of the system (9), (10) with nonnegative w and z, for arbitrary nonnegative vectors c (and for q fixed). As a result of this consideration, one would like to be able to weaken the assumption in Theorem **5.9.11** in order to better accomodate the case of an individual vector q. The following result is the analog of **5.9.11** along this line. (See also Remark **5.9.5**.)

5.9.13 Theorem. Let $M \in R^{n \times n} \cap \boldsymbol{P_0}$ and $q \in R^n$. Suppose that (q, M) has a strictly feasible solution, and that the level set

$$L_\tau = \{z \in \text{FEA}(q, M) : z^{\text{T}}(q + Mz) \leq \tau\}$$

is bounded for every $\tau \geq 0$. Then for every $c \in R^n_+$, the system (9), (10) has a solution $(w, z) \geq 0$. Moreover, the solution is unique if $c > 0$.

Proof. Let $c \in R^n_+$ be given, and $(w^0, z^0) > 0$ be such that $w^0 - Mz^0 = q$. Set $c^0 = w^0 * z^0$. Then $(q, c^0) \in \Phi(R^{2n}_{++})$. Consider the system

$$w - Mz = q \tag{14}$$

$$w * z = tc + (1 - t)c^0 \tag{15}$$

for $t \in [0, 1]$. This system has a positive solution for $t = 0$; our goal is to show that it has a nonnegative solution for $t = 1$. Define

$$t^* = \sup\{t \in [0, 1] : (q, t'c + (1 - t')c^0) \in \Phi(R^{2n}_+) \text{ for all } t' \in [0, t)\}.$$

Since $\Phi(R^{2n}_{++})$ is an open set and contains (q, c^0), it follows that there must exist a $\delta > 0$ such that $(q, t'c+(1-t')c^0) \in \Phi(R^{2n}_{++})$ for all $t' \in [0, \delta)$. Hence, t^* is well defined and positive. We claim that $(q, t^*c+(1-t^*)c^0) \in \Phi(R^{2n}_+)$. Indeed, if $\{t_k\}$ is a sequence of scalars in the interval $[0, t^*)$ converging to t^*, then for each k, there exists $(w^k, z^k) \in R^{2n}_+$ satisfying the system (14) and (15) for $t = t_k$. Clearly, the sequence $\{z^k\} \subseteq L_\tau$ for some $\tau > 0$ sufficiently large. By the boundedness assumption of the set L_τ, it follows that the sequence $\{(w^k, z^k)\}$ has a limit point which must be a solution of the system (14), (15) for $t = t^*$. Again, using the fact that the image $\Phi(R^{2n}_{++})$ is an open set, we can establish that $t^* = 1$, completing the proof that the system (9), (10) has a nonnegative solution. The uniqueness assertion of the solution for $c > 0$ follows from Lemma **5.9.10**. $\square$

5.9.14 Remark. When Theorem **5.9.13** is specialized to the case where $c = 0$, it yields a sufficient condition for the existence of a solution of the LCP (q, M) for $M \in \boldsymbol{P_0}$. Moreover, the boundedness assumption of the level sets L_τ implies that SOL$(q, M)(= L_0)$ is a compact set. This specialized existence result can be compared to Theorem **3.8.6** which also provides a sufficient condition for the nonemptiness of SOL(q, M), but for the case of a copositive LCP. It would be interesting to explore the commonality of these two results and derive a unification. (One difference between them is the boundedness of SOL(q, M) which must hold in **5.9.13** but is not ensured in **3.8.6**.)

5.9.15 Remark. The proof of Theorem **5.9.13** is based on the classical *homotopy principle* and the related *continuation property* of mappings. These are well known concepts in the theory of solving systems of nonlinear equations. The key idea involved is that in order to demonstrate the existence of a solution to a certain system of equations—in this case, (9), (10)—we deform the system to one which possesses a known solution—in this case, (14), (15) with $c = 0$. The latter solution is then taken as the starting vector for a curve that is comprised of solutions of the deformed systems. The main assertion of the continuation property is that this curve eventually ends at a solution of the original system. The computational process of tracing this solution curve is known as *numerical continuation.*

Consider now an LCP (q, M) which satisfies the assumptions in Theorem **5.9.13**. Let $(w^0, z^0) > 0$ be such that $q = w^0 - Mz^0$, and let $c^0 = w^0 * z^0$. According to this theorem, for each scalar $t \in [0, 1)$, there exists a unique vector pair $(w(t), z(t)) > 0$ satisfying

$$w - Mz = q \tag{16}$$

$$w * z = (1 - t)c^0; \tag{17}$$

moreover, as a function in t, the solution curve $(w(t), z(t))$ is continuous (by the local homeomorphism of Φ on R^{2n}_{++}). As we have briefly outlined in **5.9.15**, the principal idea underlying the continuation interior-point method is to numerically trace this solution curve, starting at the value $t = 0$ and eventually reaching $t = 1$ (or some positive value which is sufficiently close to 1), at which point, an exact solution (or an approximate solution) of the LCP (q, M) is obtained.

A practical way to implement the above conceptual scheme is to carry out, at each iteration, one step of Newton's method on the system (16),

(17) for an appropriately chosen sequence of scalars $\{t_\nu\} \subset [0,1)$ converging to 1; this generates a sequence of vectors $\{(w^\nu, z^\nu)\}$ that remain strictly feasible to (q, M), and which will, under some suitable conditions, converge to a desired solution of the given LCP. More precisely, the generation of the sequence $\{(w^\nu, z^\nu)\}$ is as follows. At the beginning of iteration $\nu+1$, we are given the scalar $t_{\nu+1}$ and the pair $(w^\nu, z^\nu) > 0$ that satisfies $w^\nu - Mz^\nu = q$. (Note: in the practical implementation scheme, it is generally not true that $w^\nu * z^\nu = (1 - t_\nu)c^0$ for $\nu > 0$.) A direction pair (d_w^ν, d_z^ν) is generated by solving the following system of linear equations:

$$\begin{bmatrix} I & -M \\ Z^\nu & W^\nu \end{bmatrix} \begin{bmatrix} d_w^\nu \\ d_z^\nu \end{bmatrix} = \begin{bmatrix} 0 \\ (1 - t_{\nu+1})c^0 - w^\nu * z^\nu \end{bmatrix} \tag{18}$$

where $W^\nu = \operatorname{diag}(w^\nu)$ and $Z^\nu = \operatorname{diag}(z^\nu)$. Notice that the lower part of the above equation corresponds to the linearization of the (nonlinear) equation $w * z = (1 - t_{\nu+1})c^0$ at the pair (w^ν, z^ν). (We recall that this linearization is precisely the main idea in Newton's method for solving nonlinear equations.) By observing that the matrix defining the system (18) is just the Jacobian $\nabla\Phi(w^\nu, z^\nu)$, it follows from Lemma **5.9.8** that (18) has a unique solution.

Having obtained the pair (d_w^ν, d_z^ν), we set the next iterate to be

$$(w^{\nu+1}, z^{\nu+1}) = (w^\nu, z^\nu) + \tau_\nu (d_w^\nu, d_z^\nu) \tag{19}$$

where $\tau_\nu \in (0, 1]$ is the steplength determined according to a certain criterion that ensures, among other things, the positivity of the new pair $(w^{\nu+1}, z^{\nu+1})$. The iterations continue until a prescribed stopping rule is satisfied.

Summarizing the above discussion, we describe below a general framework for the implementation of a continuation interior-point method for solving an LCP (q, M) that satisfies the assumptions of **5.9.13**.

5.9.16 Algorithm. (A Continuation Interior-Point Method)

Step 0. *Initialization.* Let $(w^0, z^0) > 0$ satisfy $q = w^0 - Mz^0$. Set $c^0 = w^0 * z^0$ and $\nu = 0$.

Step 1. *Compute direction.* Choose a positive scalar $t_{\nu+1} \in (0, 1)$ and solve the system of linear equations (18) to get the search direction pair (d_w^ν, d_z^ν).

Step 2. *Determine stepsize.* Choose a stepsize $\tau_\nu > 0$ so that the new iterate $(w^{\nu+1}, z^{\nu+1})$ defined by (19) remains positive.

Step 3. *Termination test.* If $(w^{\nu+1}, z^{\nu+1})$ satisfies a prescribed termination rule, stop; otherwise, return to Step 1 with ν replaced by $\nu + 1$.

The above algorithm involves two different parameters $t_{\nu+1}$ and τ_ν. We have left open the question of how they ought to be chosen in order to ensure the global convergence of the algorithm and to enhance its computational efficiency. In theory, one can show that it is possible to choose these parameters so that if the assumptions of Theorem **5.9.13** hold, then given any prescribed tolerance, $\varepsilon > 0$, the overall algorithm will terminate in a finite number of steps with an iterate (w^ν, z^ν) that satisfies the stopping rule:

$$(w^\nu)^{\mathrm{T}} z^\nu \leq \varepsilon;$$

furthermore, if ε is small enough, then an exact solution of the LCP (q, M) can be recovered by solving a system of linear equations of order n. See Exercise **5.11.15** for more details on the recovery procedure. In practice, one may use the merit function (1) as a guide for the choice of the parameters $t_{\nu+1}$ and τ_ν in order to ensure the computational effectiveness of the algorithm.

A compact form of the continuation method

In effect, the interior-point method of the last subsection (Algorithm **5.9.16**) is derived from the numerical continuation technique applied to the $2n \times 2n$ system of equations (16), (17). It turns out that if the matrix $M \in \boldsymbol{P_0}$ has a positive diagonal, then one may simplify this system of equations by eliminating the w-variables, thereby obtaining a reduced system of order n. In what follows, we explain how this reduction can be accomplished and introduce the underlying mapping, called Ψ in the sequel, that plays a similar role as Φ (see (11)).

Consider the system (9), (10) where c is a positive vector. By substituting $w = q + Mz$ into the equation involving the Hadamard product, this system is recognized as equivalent to

$$z_i\Big(q_i + m_{ii}z_i + \sum_{j\neq i} m_{ij}z_j\Big) = c_i, \quad i = 1, \dots, n.$$

Note that the i-th of these equations can be considered as a quadratic equation in the variable z_i. Solving for z_i and taking the positive root, we obtain after a rearrangement of terms,

$$2m_{ii}z_i + \sum_{j\neq i} m_{ij}z_j + q_i - \sqrt{\Big(\sum_{j\neq i} m_{ij}z_j + q_i\Big)^2 + 4m_{ii}c_i} = 0 \qquad (20)$$

for $i = 1, \dots, n$. Let $\Psi_i(z)$ denote the expression on the left-hand side of the above equation, and $\Psi : R^n \to R^n$ be the resulting mapping. Note that as c changes, so does this mapping.

Clearly, solving for a *positive* solution (w, z) of the system (9), (10) is equivalent to finding a zero of the mapping Ψ. The noteworthy property of this mapping is that any one of its zeroes is necessarily a positive vector. (The fact that c is a positive vector is essential to the truth of this statement.) Unlike the case of (9), (10), it does not seem possible to derive a "global" version of Ψ that applies universally to all vectors q (cf. the mapping Φ in (11) which is independent of q); in other words, the dependence of the mapping Ψ on the vector q seems to be an intrinsic feature of this approach and cannot be removed.

The main idea of this continuation method is to numerically trace a solution curve of the system (20) for a family of vectors $c^\nu = t_\nu c^0$ where $\{t_\nu\}$ is a sequence of positive scalars converging to zero.

5.10 Residues and Error Bounds

In the previous sections, we have discussed and analyzed the convergence of many iterative procedures for solving the LCP. Since in general, these methods do not terminate finitely, it is essential that there be rules which can be employed to stop their execution (cf. the discussion of the inexact splitting method in Section 5.7). Ideally, these rules should have the property that the iterate obtained at termination is a "satisfactory (approximate) solution" to the problem. This consideration immediately raises the question: when is a vector qualified as a satisfactory approximate solution of the LCP? In general, there are two ways to answer this question: one is that if an iterate satisfies all the defining conditions of the LCP (i.e., feasibility and complementarity) within a prescribed tolerance, then it is deemed acceptable; the other answer is that if the iterate is within a prescribed distance to the solution set of the problem, then it is considered satisfactory. In order to employ these ideas to devise termination rules for an iterative method, it is important to be able to *quantify* the notion of "satisfying the defining conditions of the LCP approximately" and to have a measure of the distance between a vector and the solution set of the problem. In theory, the latter is readily available. Indeed, the quantity

$$d(x, S) = \inf\{\,\|x - z\| : z \in S\}$$

where $S = \mathrm{SOL}(q, M)$ and $\|\cdot\|$ denotes a given vector norm, measures the distance from an arbitrary vector x to S. If this quantity $d(x, S)$ were available, then a condition such as

$$d(x, S) \leq \varepsilon \tag{1}$$

where ε is a given positive scalar (called the *tolerance*) could be used as a termination criterion. Unfortunately, the practicality of a stopping rule based directly on this distance measure is slight because the solution set S is generally unknown. (If it were known, there would be no need to solve the problem in the first place.) Consequently, a different and more practical approach is called for.

Quantification of the idea of approximately satisfying the constraints of the LCP leads to the notion of a residue. In general, a *residue* for the LCP (q, M) may be defined as a function $r(\cdot, q, M) : R^n \to R_+$ that does not depend on the solution(s) of (q, M) and possesses the property that $r(x, q, M) = 0$ if and only if $x \in \mathrm{SOL}(q, M)$. The quantity $r(x, q, M)$ is called the residue of the vector x. The following expressions give two examples of a residue

$$r(x, q, M) = \|((q + Mx)_-, x_-, x^{\mathrm{T}}(q + Mx))\| \tag{2}$$

$$r(x, q, M) = \|\min(x, q + Mx)\|. \tag{3}$$

Since $r(x, q, M)$ depends only on the given vector x and the data of the LCP (q, M), and not on the solution(s) of (q, M), a condition such as

$$r(x, q, M) \leq \varepsilon \tag{4}$$

can be used as a termination rule for an iterative algorithm; any vector x satisfying this rule may be considered an acceptable (approximate) solution of (q, M). An obvious difference between the two rules (1) and (4) is that the former is only a conceptual rule and can not be implemented during the execution of an iterative scheme, whereas the latter can certainly be used to test if an iterate z^ν is a satisfactory (approximate) solution or not.

Given the two measures $d(\cdot, S)$ and $r(\cdot, q, M)$, it is natural to ask if there is any quantitative relationship between them. Ideally, one would like to obtain positive constants $c_1(q, M)$ and $c_2(q, M)$ such that

$$c_1(q, M)r(x, q, M) \leq d(x, S) \leq c_2(q, M)r(x, q, M) \tag{5}$$

for all vectors x in a certain domain $\mathcal{D}$ that is of interest. Among their implications, these inequalities would ensure that by an appropriate choice of the tolerance ε and by enforcing the rule (4), one can bound the distance from the vector x to the solution set S to any prescribed accuracy. Another consequence of (5) is that if one can derive (upper or lower) bounds for either quantity $d(x, S)$ or $r(x, q, M)$, then an implied bound on the other quantity can be obtained immediately from (5). The remainder of this section is devoted to the investigation of the expression (5) under some specific form of the residue function $r(\cdot, q, M)$.

To begin, we illustrate the fact that the constants c_1 and c_2 need not exist if the domain $\mathcal{D}$ is the whole space R^n.

5.10.1 Example. Consider the vector q and matrix M given by

$$q = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

The unique solution of this LCP (q, M) is $z = (1, 1)$. Let $r(x, q, M)$ be given in (3) using the l_2-norm. With $x(t) = (t, 1)$, it is easy to verify that

$$\frac{\|x(t) - z\|_2}{r(x(t), q, M)} = t - 1, \quad \text{for } t \geq 2,$$

which tends to ∞ as $t \to \infty$. Thus, the constant c_2 can not exist for this particular residue if $\mathcal{D}$ is the entire space R^2.

On the other hand, if $r(x, q, M)$ is given by (2) with the l_2-norm, and if $x(t) = tz$, then as $t \to \infty$

$$\frac{\|x(t) - z\|_2}{r(x(t), q, M)} \leq \frac{2}{t} \to 0$$

which shows that the constant c_1 can not exist for this residue if $\mathcal{D}$ is R^2.

Note that the two residues $r(x, q, M)$ given by (2) and (3) are continuous functions of x. In the following result, we demonstrate that under a Lipschitzian property on the residue function $r(\cdot, q, M)$, the existence of the constant $c_1(q, M)$ is always guaranteed. This Lipschitzian property is satisfied by the residue function (3) on an arbitrary domain $\mathcal{D}$ and also by (2) if the domain $\mathcal{D}$ is compact.

5.10.2 Proposition. Suppose $S = \text{SOL}(q, M) \neq \emptyset$ and that the residue function $r(\cdot, q, M) : R^n \to R_+$ is Lipschitz continuous in the domain $\mathcal{D} \supseteq S$ with modulus $\tilde{c}(q, M) > 0$. Then, the left-hand inequality in (5) holds with $c_1(q, M) = \tilde{c}(q, M)^{-1}$ in $\mathcal{D}$.

Proof. Let $z \in S$ be arbitrary. Then, $r(z, q, M) = 0$ and the Lipschitzian property of $r(\cdot, q, M)$ implies

$$r(x, q, M) = r(x, q, M) - r(z, q, M) \leq \tilde{c}(q, M)\|x - z\|$$

from which the desired inequality follows easily. □

The above result also suggests that the constant $c_1(q, M)$ may be taken as the inverse of the Lipschitz modulus of the residue function $r(\cdot, q, M)$ if $r(x, q, M)$ is Lipschitz continuous in x. Unfortunately, the other constant $c_2(q, M)$ is generally not as easy to obtain without restricting the matrix M.

A fundamental constant of a $\boldsymbol{P}$-matrix

The fundamental role played by the class of $\boldsymbol{P}$-matrices in LCP theory is well documented. In this subsection, we introduce a key constant associated with a $\boldsymbol{P}$-matrix which is central to the derivation of error bounds of approximation solutions to the corresponding LCP.

For an arbitrary matrix $M \in R^{n \times n}$, the quantity

$$c(M) = \min_{\|z\|_\infty = 1} \{ \max_{1 \leq i \leq n} z_i(Mz)_i \} \tag{6}$$

is well defined. Moreover, according to the characterization **3.3.4**(b) of a $\boldsymbol{P}$-matrix, $c(M) > 0$ if and only if $M \in \boldsymbol{P}$. Clearly, the inequality

$$\max_{1 \leq i \leq n} z_i(Mz)_i \geq c(M)\|z\|_\infty^2 \tag{7}$$

holds for an arbitrary vector z. If the matrix M is nonsingular, we may replace M with its inverse M^{-1} in (7), make the substitution $y = M^{-1}z$, and obtain the inequality

$$\max_{1 \leq i \leq n} y_i(My)_i \geq c(M^{-1})\|My\|_\infty^2. \tag{8}$$

In principle, we could use an arbitrary vector norm $\|\cdot\|$ instead of the l_∞-norm to define $c(M)$; the latter is chosen to conform with the quantity

$$\pi_M(z) = (\max_{1 \leq i \leq n} z_i(Mz)_i)^{1/2}$$

which is motivated by the characterizing property **3.3.4**(b) when M is a $\boldsymbol{P}$-matrix. If M is the identity matrix, $\pi_M(z)$ reduces to $\|z\|_\infty$; thus, $\pi_M(z)$ may be thought of as a generalization of the l_∞-norm of vectors to an arbitrary $\boldsymbol{P}$-matrix M (just like the generalization of the l_2-norm to an elliptic norm $\|\cdot\|_A$ in the case of a symmetric positive definite matrix A). Note that in general, $\pi_M(z)$ does not define a vector norm; as a matter of fact, it is not difficult to show that if $M \in \boldsymbol{P}$, then $\pi_M(z)$ defines a norm on R^n if and only if M is a positive diagonal matrix (see Exercise **5.11.17**).

The constant $c(M)$ is in general difficult to compute. However, it is easy to derive upper bounds for $c(M)$ when $M \in \boldsymbol{P}$. For this purpose, we introduce a related quantity for a $\boldsymbol{P}$-matrix M:

$$\delta(M) = \min\{\sigma(M_{\alpha\alpha}) : \alpha \subseteq \{1, \ldots, n\}\} \tag{9}$$

where $\sigma(M_{\alpha\alpha})$ denotes the smallest of the real eigenvalues (if any exists) of $M_{\alpha\alpha}$. The above minimum ranges over those principal submatrices of M having real eigenvalues. Included in this range are the singletons; these correspond to the diagonal entries of M which must be positive. In general, according to the characterization **3.3.4**(c) of a $\boldsymbol{P}$-matrix, each of the real eigenvalues of a principal submatrix of a $\boldsymbol{P}$-matrix must be positive. Consequently, $\delta(M)$ is a well-defined, finite and positive quantity; moreover, we have

$$\delta(M) \leq \min_{1 \leq i \leq n} m_{ii}.$$

If M is a symmetric $\boldsymbol{P}$-matrix, then the constant $\delta(M)$ reduces to the smallest eigenvalue of M by the interlacing property of eigenvalues; see Exercise **5.11.18**. The following result shows that $c(M)$ is always bounded above by $\delta(M)$ if M is a $\boldsymbol{P}$-matrix.

5.10.3 Proposition. Let M be an $n \times n$ $\boldsymbol{P}$-matrix. Then

$$c(M) \leq \delta(M) \leq \min_{1 \leq i \leq n} m_{ii}.$$

Proof. It suffices to prove the first inequality. Write $\delta = \delta(M)$. By the definition of δ, the matrix $M - \delta I \notin \boldsymbol{P}$. Thus, there exists a vector z with $\|z\|_\infty = 1$ such that

$$\max_{1 \leq i \leq n} z_i((M - \delta I)z)_i \leq 0.$$

This implies

$$c(M) \leq \max_{1 \leq i \leq n} z_i(Mz)_i \leq \delta$$

as desired. □

This proposition shows that the quantity $c(M)$ admits an upper bound which is in terms of the smallest of the real eigenvalues of the principal submatrices of M. It seems natural to ask whether a lower bound for $c(M)$ can be obtained in terms of such eigenvalues only. More specifically, one may ask the question: Does there exist a constant $\lambda > 0$ and a function f both depending only on n such that $c(M) \geq \lambda f(\delta(M))$ for an arbitrary $\boldsymbol{P}$-matrix M? The following simple 2×2 matrix answers this question in the negative. The same example also demonstrates that lower bounds for $c(M)$ must involve quantities other than these eigenvalues, and that the off-diagonal entries of M possibly play some role in these bounds.

5.10.4 Example. Consider the matrix

$$M = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

where t is any nonzero number. Clearly, M and its principal submatrices all have eigenvalues equal to 1. It is easy to show, however, that $c(M) \leq 1/t^2$ which implies that $c(M)$ tends to 0 as $t \to \infty$.

We postpone the derivation of lower bounds for the constant $c(M)$ until a later subsection.

Absolute and relative error bounds

In the sequel, we demonstrate how the quantity $c(M)$ can be used to derive the desired constant $c_2(q, M)$ in (5) when M is a $\boldsymbol{P}$-matrix and the residue $r(x, q, M)$ is given by (3). We also show how the same constant $c(M)$ is instrumental in the derivation of relative error bounds for the LCP with a $\boldsymbol{P}$-matrix.

5.10.5 Proposition. Let M be an $n \times n$ $\boldsymbol{P}$-matrix. Let z denote the unique solution of the LCP (q, M) and let x be an arbitrary n-vector. Then

$$\|z - x\|_\infty \leq \frac{1 + \|M\|_\infty}{c(M)} \| \min(x, q + Mx)\|_\infty. \tag{10}$$

Proof. Let $v = \min(x, q+Mx)$ and $w = q+Mz$. Then the vector $y = x - v$ satisfies the complementarity system

$$y \geq 0, \quad u = q + (M - I)v + My \geq 0, \quad y^{\mathrm{T}}u = 0.$$

Thus, we have for each $i = 1, \ldots, n$,

$$\begin{aligned} 0 \geq (y - z)_i(u - w)_i &= (x - v - z)_i(-v + M(x - z))_i \\ &\geq -(x - z)_i v_i - v_i(M(x - z))_i + (x - z)_i(M(x - z))_i. \end{aligned}$$

In particular, for i such that

$$(x - z)_i(M(x - z))_i = \max_j (x - z)_j(M(x - z))_j,$$

we derive from the above inequality and the condition (7),

$$\begin{aligned} c(M)\|x - z\|_\infty^2 &\leq (x - z)_i v_i + v_i(M(x - z))_i \\ &\leq (1 + \|M\|_\infty)\|v\|_\infty\|x - z\|_\infty \end{aligned}$$

from which the desired inequality (10) follows readily. □

Combining Propositions **5.10.2** and **5.10.5**, we derive the following absolute error bounds for the LCP with a $\boldsymbol{P}$-matrix.

5.10.6 Theorem. Let M be an $n \times n$ $\boldsymbol{P}$-matrix. Let z denote the unique solution of the LCP (q, M), and x be an arbitrary n-vector. Let $r(x, q, M) = \| \min(x, q + Mx)\|_\infty$. Then

$$\frac{1}{\max(1, \|M\|_\infty)} r(x, q, M) \leq \|z - x\|_\infty \leq \frac{1 + \|M\|_\infty}{c(M)} r(x, q, M). \tag{11}$$

Proof. It suffices to verify the Lipschitzian property

$$\| \min(u, q + Mu) - \min(v, q + Mv)\|_\infty \leq \max(1, \|M\|_\infty)\|u - v\|_\infty. \quad (12)$$

Let $x = \min(u, q + Mu)$ and $y = \min(v, q + Mv)$. Consider an arbitrary component i, and suppose that

$$|x_i - y_i| = x_i - y_i.$$

Suppose also that $\min(v_i, (q + Mv)_i) = v_i$. Then

$$|x_i - y_i| \leq u_i - v_i \leq \max(1, \|M\|_\infty)\|u - v\|_\infty.$$

By a similar argument, we may establish the validity of the inequality

$$|x_i - y_i| \leq \max(1, \|M\|_\infty)\|u - v\|_\infty$$

in all other cases. This proves the desired Lipshitzian property (12) of the "min" function. □

The inequality (11) gives upper and lower bounds of the error $\|z - x\|_\infty$ in terms of the residue measure $\| \min(x, q + Mx)\|_\infty$. Often, it is useful to bound the relative error $\|z - x\|_\infty / \|z\|_\infty$ in terms of a "relative residue". To derive the latter bounds, we establish the following bounds for the exact solution of the LCP (q, M).

5.10.7 Proposition. Let M be an $n \times n$ $\boldsymbol{P}$-matrix and let z be the unique solution of the LCP (q, M). Then,

$$c(M^{-1})\|(-q)_+\|_\infty \leq \|z\|_\infty \leq c(M)^{-1}\|(-q)_+\|_\infty. \quad (13)$$

Proof. Without loss of generality, we may assume that q is not nonnegative, or equivalently, that z is nonzero. Since $z \in \text{SOL}(q, M)$, we have by (7),

$$\begin{aligned} c(M)\|z\|_\infty^2 &\leq \max_{1 \leq i \leq n} z_i(Mz)_i = \max_{1 \leq i \leq n} z_i(-q)_i \\ &= \max_{1 \leq i \leq n} z_i((-q)_+)_i \leq \|z\|_\infty \|(-q)_+\|_\infty \end{aligned}$$

from which we obtain the right-hand inequality in (13).

To prove the other inequality, we note that $Mz \geq -q$. Thus,

$$|Mz| \geq (Mz)_+ \geq (-q)_+,$$

and

$$\|Mz\|_\infty \geq \|(-q)_+\|_\infty.$$

By (8) and the fact that $z_i(q + Mz)_i = 0$ for each i, we deduce

$$\begin{aligned} \|(-q)_+\|_\infty^2 &\leq c(M^{-1})^{-1} \max\nolimits_{1\leq i\leq n} z_i(Mz)_i \\ &\leq c(M^{-1})^{-1} \max\nolimits_{1\leq i\leq n} z_i(-q)_i \\ &\leq c(M^{-1})^{-1} \|z\|_\infty \|(-q)_+\|_\infty \end{aligned}$$

from which the desired left-hand inequality in (13) follows. □

Notice that it is the vector $(-q)_+$, and not $-q$, that serves to define the magnitude of the bounding term in the inequality (13). This is consistent with the fact that zero is the unique solution of the LCP (q, M) if and only if q is a nonnegative vector (assuming that $M \in \boldsymbol{P}$).

Combining **5.10.6** and **5.10.7**, we immediately obtain the following relative error bounds for the LCP with a $\boldsymbol{P}$-matrix.

5.10.8 Theorem. Let M be an $n \times n$ $\boldsymbol{P}$-matrix. Let z denote the unique solution of the LCP (q, M), and x be an arbitrary n-vector. Let $r(x, q, M) = \|\min(x, q + Mx)\|_\infty$. Assume that $(-q)_+ \neq 0$. Then

$$\frac{c(M)}{\max(1, \|M\|_\infty)} \frac{r(x, q, M)}{\|(-q)_+\|_\infty} \leq \frac{\|z - x\|_\infty}{\|z\|_\infty} \leq \frac{(1 + \|M\|_\infty)}{c(M)c(M^{-1})} \frac{r(x, q, M)}{\|(-q)_+\|_\infty}. \tag{14}$$

□

5.10.9 Remark. The two multipliers of the relative residue term in expression (14) are, respectively, less than or equal to and greater than 1. Indeed, the lower-bound multiplier is less than or equal to 1 because of **5.10.3**; the fact that the upper-bound multiplier is greater than 1 can be seen by setting $y = x$ in (8) and multiplying the resulting inequality by (7) in order to yield $c(M)c(M^{-1}) \leq 1$.

The error bounds obtained in **5.10.6** and **5.10.8** pertain to arbitrary vectors $x \in R^n$. For such "global" bounds to hold, the matrix M is assumed to be in the class $\boldsymbol{P}$. This $\boldsymbol{P}$-property can be somewhat relaxed if one restricts attention to vectors x which are close to the solution z of (q, M). The derivation of such local error-bound results is considerably more complicated and depends crucially on the notion of a *stable solution* of the LCP. We postpone this development until Chapter 7 where we discuss the sensitivity and stability issues (see Section 7.3 and Corollary **7.3.14**).

An important feature of the error bounds derived herein is that the bounding constants are all independent of the vector q; in other words, the same constants apply uniformly to all vectors q (for a fixed matrix M). This turns out to be very crucial when these error bound results are employed in an analysis of the convergence rate of the splitting method **5.2.1** for solving the LCP (see Theorem **7.3.15**).

Lower bounds for $c(M)$

The derivation of lower bounds for the constant $c(M)$ is not a trivial task. Exercise **5.11.19** discusses more about this matter for a general $\boldsymbol{P}$-matrix. In the sequel, we consider the classes of diagonally stable matrices and $\boldsymbol{H}$-matrices with positive diagonals, and show how lower bounds for $c(M)$ can be derived for these matrices. We note from the various error bound inequalities (11) and (14) that any lower bound for $c(M)$ immediately yields an implied bound for the error of an approximate solution of the LCP.

Define the symmetric rank-two matrix

$$A_i = \tfrac{1}{2}(e_i e_i^T M + M^T e_i e_i^T)$$

for $i = 1, \ldots, n$. This matrix A_i is obtained by symmetrizing the i-th row of M and substituting zeroes in all other entries of M. It is easy to see

$$\pi_M(x)^2 = \max_{1 \leq i \leq n} x^{\mathrm{T}} A_i x.$$

The following result gives a lower bound for the quantity $c(M)$ when M is diagonally stable.

5.10.10 Proposition. Let M be an arbitrary $n \times n$ diagonally stable matrix. Then,

$$c(M) \geq \max\Big\{\mu_1\Big(\sum_{i=1}^{n} \lambda_i A_i\Big) : \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1\Big\} > 0 \tag{15}$$

where $\mu_1(N)$ denotes the smallest eigenvalue of a symmetric matrix N. In particular, if M is itself positive definite, then

$$c(M) \geq \mu_1(\tilde{M})/n$$

where $\tilde{M}$ denotes the symmetric part of M.

Proof. Let

$$L(M) = \max\Big\{\mu_1\Big(\sum_{i=1}^{n} \lambda_i A_i\Big) : \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1\Big\}.$$

Since each A_i is a symmetric matrix, and since the smallest eigenvalue is a continuous function of the entries of the matrix, it follows that the above maximum is actually attained. Since M is diagonally stable, there exists a positive diagonal matrix D such that DM is positive definite. Without loss of generality, we may assume that the diagonal entries of D sum to unity. Clearly, $\sum_{i=1}^{n} d_{ii} A_i$ is just the symmetric part of DM whose smallest eigenvalue is positive. Consequently, $L(M) > 0$.

Let $x \in R^n$ be an arbitrary vector $\|x\|_\infty = 1$. Suppose that the maximum in $L(M)$ is achieved by the vector $\bar{\lambda}$. Then, we have,

$$\begin{aligned} \pi_M(x)^2 &= \max_{1 \leq i \leq n} x^{\mathrm{T}} A_i x \\ &\geq \sum_{i=1}^{n} \bar{\lambda}_i x^{\mathrm{T}} A_i x \geq \mu_1\Big(\sum_{i=1}^{n} \bar{\lambda}_i A_i\Big) \end{aligned}$$

from which the desired inequality (15) follows. Finally, if M is itself positive definite, then $\tilde{M}/n$ is equal to the convex combination of the A_i's with each λ_i equal to $1/n$. Consequently, the desired bound on $c(M)$ follows. □

The computation of the lower bound $L(M)$ is not completely trivial, but can be somewhat simplified by noting that

$$L(M) = \max\Big\{\mu_1\Big(\sum_{i=1}^{n} \lambda_i A_i\Big) : \sum_{i=1}^{n} \lambda_i = 1\Big\} \tag{16}$$

in which the multipliers λ_i are not required to be nonnegative. The justification for the simplified expression of $L(M)$ is due to the fact that the least eigenvalue of a symmetric matrix is always bounded above by the smallest diagonal element of the matrix; thus the maximum in (16) can not occur at a vector λ with a negative component. In turn, by eliminating one of the λ_i variables using the summation equation, the computation of (16) can be turned into an unconstrained optimization problem involving the maximization of the minimum eigenvalue of a symmetric matrix. This latter eigenvalue-maximization problem has been the subject of intensive research and some highly efficient solution methods have been developed; see **5.12.25** for a reference.

When M is an $\boldsymbol{H}$-matrix with positive diagonals, a particularly simple lower bound for $c(M)$ can be derived.

5.10.11 Proposition. Let M be an $\boldsymbol{H}$-matrix with positive diagonals and $\bar{M}$ denote its comparison matrix. Then, for any vector $p > 0$, the vector $d = \bar{M}^{-1}p > 0$ and

$$c(M) \geq \frac{(\min_i p_i)(\min_i d_i)}{(\max_j d_j)^2}. \tag{17}$$

Proof. By Theorem **3.11.10**, the matrix $\bar{M}$ has a nonnegative inverse; thus the vector $d = \bar{M}^{-1}p$ is positive for any positive p.

To prove the inequality (17), consider first the case where d is the vector of all ones. (This is equivalent to the case where the matrix M is strictly row diagonally dominant.) Let $x \in R^n$ be an arbitrary vector with $\|x\|_\infty = 1$ and i be an index where the maximum in $\|x\|_\infty$ is achieved. Without loss of generality, we may assume that $x_i > 0$. Then $x_i = 1$ and it is easy to verify that

$$\pi_M(x)^2 \geq \Big(m_{ii} - \sum_{j \neq i} |m_{ij}|\Big) x_i^2 = p_i.$$

Thus, the inequality (17) holds in this case.

In general, let M be an $\boldsymbol{H}$-matrix with positive diagonals, and let p and d be as given. Let $D = \text{diag}(d_i)$. Then, the matrix $N = MD$ is strictly row diagonally dominant and the above derivation implies that $c(N) \geq \min_i p_i$. Let $x \in R^n$ with $\|x\|_\infty = 1$ be a vector achieving the minimum in the quantity $c(M)$. With the change of variables $y = D^{-1}x$, we obtain

$$c(M) = \max_i d_i y_i (Ny)_i \geq (\min_i d_i)(\max_i y_i (Ny)_i)$$

which, by (7), implies

$$c(M) \geq (\min_i d_i) c(N) \|y\|_\infty^2.$$

Since $x = Dy$, we derive $\|y\|_\infty \geq (\max_j d_j)^{-1}$ in view of the fact that $\|x\|_\infty = 1$. Consequently, using the established inequality for $c(N)$, we obtain the desired inequality (17). □

The positive semi-definite case

Theorems **5.10.6** and **5.10.8** have shown that when M is a $\boldsymbol{P}$-matrix, the quantity

$$\| \min(x, q + Mx) \|_\infty \tag{18}$$

can be used to bound the distance between an arbitrary vector x and the exact solution of the LCP (q, M). In this subsection, we consider the case where M is positive semi-definite.

To begin the discussion, we make two preliminary observations. First, as example **5.10.1** shows, the quantity (18) is no longer a valid residue. Second, as M is not assumed to be a $\boldsymbol{P}$-matrix, the LCP (q, M) may have multiple solutions (the case where $\text{SOL}(q, M) = \emptyset$ is of no interest in the present context). Hence, unlike the previous theorems which bound the distance between x and the unique solution z of (q, M), generalized results which provide bounds for $d(x, S)$ where $S = \text{SOL}(q, M)$ should be derived. More specifically, the error bound results obtained in the sequel all assert that for a given vector x, there exists a solution $z \in S$ (dependent on x) such that $\|x - z\|_\infty$ is bounded above by an appropriate residue multiplied by a constant which is dependent on q and M only.

The analysis of error bounds for the positive semi-definite LCP is based on two known results: one concerns a special property of the solutions of an LCP of this type (Theorem **3.1.7**), and the other gives a bound on the distance between an arbitrary vector and a polyhedron (Exericse **2.10.22**). For ease of reference, we restate these results in the lemmas below.

5.10.12 Lemma. Let $M \in R^{n \times n}$ be positive semi-definite, and $q \in R^n$ be arbitrary. Suppose that $\text{SOL}(q, M) \neq \emptyset$. Then, there exist a vector $d \in R^n$ and a scalar σ such that for any $\bar{z} \in \text{SOL}(q, M)$,

$$(M + M^{\mathrm{T}})\bar{z} = d \quad \text{and} \quad q^{\mathrm{T}}\bar{z} = \sigma;$$

moreover,

$$\mathrm{SOL}(q, M) = \{z \in R^n_+ : q + Mz \geq 0, z^{\mathrm{T}}(q + d) + \sigma \leq 0, (M + M^{\mathrm{T}})z = d\}.$$

□

The proof of the above lemma follows rather easily from Theorem **3.1.7**, and is left to the reader.

5.10.13 Lemma. Let $P = \{x \in R^n : Ax = b,\ Cx \geq d\}$ be a nonempty polyhedron. Then, there exists a constant $\lambda > 0$, dependent on A, C, b and d, such that for any vector $a \in R^n$,

$$\|a - \Pi_P(a)\|_\infty \leq \lambda(\|(b - Aa, (Ca - d)^-)\|_\infty)$$

where $\Pi_P(a)$ denotes any point in P that is nearest to a under the l_∞-norm. □

Combining these two lemmas, we immediately obtain the following error bound result for a positive semi-definite LCP.

5.10.14 Theorem. Let $M \in R^{n \times n}$ be positive semi-definite, and $q \in R^n$ be arbitrary. Suppose that $\mathrm{SOL}(q, M) \neq \emptyset$. Let d and σ be as given in **5.10.12**. Then, there exists a constant $c > 0$, dependent on q and M only, such that for any vector $z \in R^n$ (with $w = q + Mz$),

$$\|z - \Pi_S(z)\|_\infty \leq c\|(z^-, w^-, (z^{\mathrm{T}}(q + d) + \sigma)^+, (M + M^{\mathrm{T}})z - d)\|_\infty \quad (19)$$

where $S = \mathrm{SOL}(q, M)$. □

The above theorem shows that for a (solvable) LCP of the positive semi-definite type, the quantity on the right-hand side of (19) less the multiplier c can be used as a residue function for the problem. An undesirable feature in this quantity is the presence of the pair (d, σ), which, albeit a constant of the problem (q, M), is generally not available without obtaining at least one solution. In Exercise **5.11.21**, the reader is asked to derive an alternate residue function which depends on another constant of the problem (q, M). In what follows, we consider two special cases in which a residue function can be prescribed that depends only on the test vector z in question.

5.10.15 Corollary. Let $M \in R^{n \times n}$ be positive semi-definite, and $q \in R^n$ be arbitrary. Suppose that $\mathrm{SOL}(q, M) \neq \emptyset$. Then, there exists a constant $c' > 0$, dependent on q and M only, such that for any vector $z \in \mathrm{FEA}(q, M)$,

$$\| z - \Pi_S(z) \|_\infty \leq c'(z^{\mathrm{T}} w + \sqrt{z^{\mathrm{T}} w}). \tag{20}$$

Proof. Let $\bar{z}$ be an arbitrary solution of (q, M), and $\bar{w} = q + M\bar{z}$. Since $z \in \mathrm{FEA}(q, M)$, we have $z^- = w^- = 0$, and

$$0 \leq (z - \bar{z})^{\mathrm{T}} M (z - \bar{z}) = z^{\mathrm{T}} w - (z^{\mathrm{T}} \bar{w} + \bar{z}^{\mathrm{T}} w) \leq z^{\mathrm{T}} w.$$

Furthermore, it is easy to show that

$$z^{\mathrm{T}}(q + d) + \sigma = z^{\mathrm{T}} \bar{w} + \bar{z}^{\mathrm{T}} w \leq z^{\mathrm{T}} w. \tag{21}$$

Since $M + M^{\mathrm{T}}$ is symmetric positive semi-definite, it follows from Exercise **2.10.11** that there exists a constant $\tau > 0$ such that

$$\begin{aligned}
\| (M + M^{\mathrm{T}}) z - d \|_\infty^2 &\leq \| (M + M^{\mathrm{T}}) z - d \|_2^2 \\
&= \| (M + M^{\mathrm{T}})(z - \bar{z}) \|_2^2 \\
&\leq \tau (z - \bar{z})^{\mathrm{T}} (M + M^{\mathrm{T}})(z - \bar{z}) \\
&\leq 2\tau z^{\mathrm{T}} w.
\end{aligned}$$

The desired inequality (20) now follows easily by combining the above inequalities. □

The following example shows that the square root term $\sqrt{z^{\mathrm{T}} w}$ in (20) is essential and cannot be dropped.

5.10.16 Example. Consider the data

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We can see that $\mathrm{SOL}(q, M) = \{0\}$ and $\mathrm{FEA}(q, M) = \{x \in R^2_+ : x_2 \leq x_1\}$. Let $z(\varepsilon) = (\varepsilon, \varepsilon^2)$ for $\varepsilon \in [0, 1]$. Then,

$$\frac{\| z(\varepsilon) \|_\infty}{z(\varepsilon)^{\mathrm{T}} w(\varepsilon)} = \frac{\varepsilon}{2\varepsilon^2 + \varepsilon^4} \longrightarrow \infty$$

as $\varepsilon \to 0$. On the other hand, with the constant $c' = 1$, it is easy to show that the inequality (20) holds for $z(\varepsilon)$ with $\varepsilon \in [0,1]$.

The matrix M given above is positive definite; hence, the error bound results of Theorems **5.10.6** and **5.10.8** are applicable to this LCP (q, M). The significance of this observation is that for this problem, while the quantity $x^{\mathrm{T}}(q + Mx)$ is not a valid residue for an arbitrary feasible solution x, the function $\|\min(x, q+Mx)\|$ is a valid residue for all vectors x. Incidentally, it is possible to have a positive semi-definite LCP for which the latter "min" function fails to be a legitimate residue and the square root term is needed in (20); see Exercise **5.11.22**.

The other special case we consider is when the LCP (q, M) possesses a nondegenerate solution. We first show that under this additional assumption, the representation of the solution set of (q, M) can be simplified. Indeed, the following result gives a necessary and sufficient condition for the existence of a nondegenerate solution to a positive semi-definite LCP in terms of such a simplified representation.

5.10.17 Theorem. Let $M \in R^{n\times n}$ be positive semi-definite, and $q \in R^n$ be arbitrary. Suppose that $\mathrm{SOL}(q, M) \neq \emptyset$. Then, (q, M) has a nondegenerate solution if and only if

$$\mathrm{SOL}(q, M) = \{z \in R^n_+ : q + Mz \geq 0, z^{\mathrm{T}}(q + d) + \sigma \leq 0\}, \qquad (22)$$

where d and σ are as given in **5.10.12**.

Proof. Let $\bar{S}$ denote the set on the right-hand side of (22). Clearly, the inclusion $\mathrm{SOL}(q, M) \subseteq \bar{S}$ always holds.

To prove the necessity part, it suffices to show the reverse inclusion $\bar{S} \subseteq \mathrm{SOL}(q, M)$. Let $\bar{z}$ be a nondegenerate solution of (q, M). Then, as derived in (21),

$$z^{\mathrm{T}}(q + d) + \sigma = z^{\mathrm{T}}\bar{w} + \bar{z}^{\mathrm{T}}w.$$

If $z \in \bar{S}$, it follows that $z^{\mathrm{T}}\bar{w} = \bar{z}^{\mathrm{T}}w = 0$. By the nondegeneracy of the solution $\bar{z}$, one can easily show that z and w are complementary. This establishes the equality of the two sets in (22).

Conversely, suppose that (22) holds. To show that a nondegenerate solution of (q, M) exists, consider the following linear program:

$$\begin{aligned} &\text{minimize} && -\tau \\ &\text{subject to} && z \in \bar{S} \\ & && z + (q + Mz) \geq \tau e. \end{aligned}$$

If the LCP (q, M) has no nondegenerate solution, then the optimum objective value of this program must be zero; hence, so is that of the dual program:

$$\begin{aligned} &\text{maximize} && -q^{\mathrm{T}}(u+v) + \sigma\psi \\ &\text{subject to} && u + M^{\mathrm{T}}(u+v) - \psi(q+d) \leq 0 \\ & && e^{\mathrm{T}}u = 1, \quad u, v, \psi \geq 0. \end{aligned}$$

Let (u, v, ψ) be an optimal solution of the dual program. Then setting the optimal dual objective equal to zero and premultiplying the first dual constraint by $(u + v)$ give:

$$\begin{aligned} 0 &\geq (u+v)^{\mathrm{T}}u + (u+v)^{\mathrm{T}}M(u+v) - \psi(u+v)^{\mathrm{T}}(q+d) \\ &= (u+v)^{\mathrm{T}}u + (u+v)^{\mathrm{T}}M(u+v) - \psi d^{\mathrm{T}}(u+v) - \sigma\psi^2 \\ &= (u+v)^{\mathrm{T}}u + (u+v-\psi\bar{z})^{\mathrm{T}}M(u+v-\psi\bar{z}) \end{aligned}$$

where the last equality follows from the definition of σ and d and the identity $\sigma = -\bar{z}^{\mathrm{T}}M\bar{z}$. This is a contradiction because of the constraints $u \geq 0$, $e^{\mathrm{T}}u = 1$ and the positive semi-definiteness of M. □

By combining the above theorem with Lemma **5.10.13**, we immediately obtain the following simplified error bound for a positive semi-definite LCP with a nondegenerate solution.

5.10.18 Corollary. Let $M \in R^{n\times n}$ be positive semi-definite, and $q \in R^n$ be arbitrary. Suppose that SOL(q, M) has a nondegenerate solution. Then, there exists a constant $c'' > 0$, dependent on q and M only, such that for any vector $z \in R^n$,

$$\|z - \Pi_S(z)\|_\infty \leq c''\|(z^-, w^-, (z^{\mathrm{T}}w)^+)\|_\infty. \tag{23}$$

Proof. It suffices to note that the expression (21) implies

$$(z^{\mathrm{T}}(q+d) + \sigma)^+ \leq (z^{\mathrm{T}}w)^+. \quad \square$$

5.11 Exercises

5.11.1 Let M be a positive definite matrix and let z^* be the unique solution of the LCP (q, M). Consider the iteration (5.3.14) for solving the LCP (q, M).

(a) Show that with $\lambda = \|M\|_2^2/\mu$ where μ is the least eigenvalue of the symmetric part of M, the sequence $\{z^\nu\}$ satisfies

$$\|z^{\nu+1} - z^*\|_2 \leq \beta\|z^\nu - z^*\|_2$$

where $\beta = \sqrt{1 - 1/\gamma^2}$ and $\gamma = \|M\|_2/\mu$. Give an argument to explain why this choice of λ is reasonable.

(b) Let the sequence $\{z^\nu\}$ be generated as described in part (a). Show that if $q \not\geq 0$, then there exists an integer $\tilde{\nu} > 0$ which depends on q, M and z^0 only, such that if the index j satisfies

$$z_j^{\tilde{\nu}+1} = \max_i z_i^{\tilde{\nu}+1},$$

then $z_j^* > 0$.

(c) Use the result from part (b) to derive a modification of the iterative method defined by (5.3.14) which will solve the LCP (q, M) in a finite number of iterations.

5.11.2 Let M be an $\boldsymbol{H}$-matrix with positive diagonals. Let $\{z^\nu\}$ be a sequence of vectors generated by the PSOR method as described in Corollary **5.3.16**. Show that part (b) of Exercise **5.11.1** remains valid for this PSOR sequence.

5.11.3 Let M be an arbitrary square matrix, and let $\{z^\nu\}$ be a sequence of vectors generated by Algorithm **5.2.1**. Suppose that $\{z^\nu\}$ converges to a nondegenerate solution z^* of the LCP (q, M). Show that there exists an integer $\tilde{\nu} > 0$ such that for all $\nu \geq \tilde{\nu}$,

$$\{i : z_i^* = 0\} = \{i : z_i^\nu = 0\}$$

$$\{i : w_i^* = 0\} = \{i : w_i^\nu = 0\}$$

where $w^* = q + Mz^*$ and $w^\nu = q + Cz^{\nu-1} + Bz^\nu$.

5.11.4 Consider the LCP (q, M) where q and M are given by (5.3.11). Suppose that Q is a positive diagonal matrix. Describe an efficient implementation of the PSOR method that avoids the explicit formation of the matrix M.

5.11.5 Write a (short) computer program to implement the PSOR method for the LCP (q, M') arising from the convex curve fitting problem as discussed in Exercise **1.6.3**. You may take $x_{i+1} - x_i = 1$ for all $i = 0, \ldots, n$. Try different values of the relaxation parameter $\omega \in (0, 2)$ and report your results.

5.11.6 This exercise is intended to provide an outline of the proof for Proposition **5.4.4**. The notation set forth in the proposition is used.

(a) Show that there exists a vector λ^* such that

$$y^* = y + H^{\mathrm{T}}\lambda^*, \quad \text{and} \quad HH^{\mathrm{T}}\lambda^* + (b + Hy) = 0.$$

(b) Use a spectral decomposition of the matrix HH^{T} to complete the proof of **5.4.4**.

5.11.7 Let $A \in R^{m \times n}$ and $b \in R^m$ be given. Suppose that $A_{i\bullet}(A_{i\bullet})^{\mathrm{T}} = 1$ for $i = 1, \ldots, m$. Let $\omega \in (0, 2)$ and $a \in R^n$ be arbitrary. Suppose that the system of linear inequalities

$$Ax \geq b \tag{1}$$

is consistent. Consider the following iterative method for finding a solution of this system closest to the vector a in the Euclidean norm. Let $z^0 = 0$ and $x^0 = a$. Generate two sequences $\{z^\nu\}$ and $\{x^\nu\}$ in the following way. Given z^ν and x^ν, define for $i = 1, \ldots, m$,

$$z_i^{\nu+1} = \max(0, z_i^\nu - \omega(A_i x^{\nu+(i-1)/m} - b)) \tag{2}$$

$$x^{\nu+i/m} = x^{\nu+(i-1)/m} + (A_{i\bullet})^{\mathrm{T}}(z_i^{\nu+1} - z_i^\nu). \tag{3}$$

(a) Show that the sequence $\{z^\nu\}$ is the specialization of the PSOR method to the LCP (q, M) where $q = -b + Aa$ and $M = AA^{\mathrm{T}}$, and that for each $\nu, x^\nu = a + A^{\mathrm{T}}z^\nu$.

(b) Show that the sequence $\{x^\nu\}$ converges to the unique solution of the system (1) that is closest to the vector a under the Euclidean norm.

(c) If x^* is the limit of $\{x^\nu\}$, show that there exists a constant $c \in (0,1)$ such that for all ν sufficiently large, $\|x^{\nu+1} - x^*\|_2 \leq c\|x^\nu - x^*\|_2$.

5.11.8 This exercise uses the notation and assumption in **5.3.12**.

(a) Show that $\rho(\tilde{B}, C) \leq 1$ if and only if the matrix $B - C^{\mathrm{T}}\hat{B}^{-1}C$, where $\hat{B}$ is the symmetric part of B, is positive semi-definite. (Compare this result with part (a) of Proposition **5.3.13** in which the symmetry of B is also assumed.) Show also that if $\rho(\tilde{B}, C) \leq 1$, then M must be positive semi-definite.

(b) Suppose that M is a positive semi-definite matrix satisfying the implication (2.10.1), i.e.,

$$x^{\mathrm{T}}Mx = 0 \quad \Rightarrow \quad Mx = 0.$$

Show that there exists a constant $\bar{\lambda} > 0$ such that for all $\lambda > \bar{\lambda}$, the quantity $\rho(\tilde{B}, C) \leq 1$ for $B = \lambda I$. Moreover, with the latter matrix B, the following implication holds:

$$x^{\mathrm{T}}(B - C^{\mathrm{T}}\hat{B}^{-1}C)x = 0 \quad \Rightarrow \quad (\hat{B} + C)x = 0. \tag{4}$$

(c) A splitting (B, C) of M is called a *T-splitting* if (i) B is positive definite, (ii) $\rho(\tilde{B}, C) \leq 1$, and (iii) the implication (4) holds. Suppose that (B, C) is a T-splitting of M. Show that for any $z^* \in \mathrm{SOL}(q, M)$ and any $v \in \mathrm{SOL}(q + Cu, B)$,

$$\|v - z^*\|_{\hat{B}} \leq \|u - z^*\|_{\hat{B}}$$

with equality holding only if $v \in \mathrm{SOL}(q, M)$.

(d) Suppose that (B, C) is a T-splitting of M. Show that if $\mathrm{SOL}(q, M)$ is nonempty, then for any $z^0 \geq 0$, the uniquely defined sequence $\{z^\nu\}$ generated by Algorithm **5.2.1** converges to some solution of LCP (q, M).

5.11.9 Let $f(z) = q^{\mathrm{T}}z + \frac{1}{2}z^{\mathrm{T}}Mz$ where $q \in R^n$ and $M \in R^{n\times n}$ is symmetric. Fix a vector $z \in R^n$ and a direction $d \in R^n$ that satisfy $d^{\mathrm{T}}(q + Mz) < 0$ and $d^{\mathrm{T}}Md > 0$. Consider the univariate function

$$g(\tau) = f(z + \tau d), \quad \tau \in R.$$

(a) Show that $g(\tau)$ is strictly convex in $\tau \in R$, and that its (unconstrained) global minimum is attained at the value

$$\bar{\tau} = \frac{-d^{\mathrm{T}}(q + Mz)}{d^{\mathrm{T}}Md}.$$

Show also that

$$g(\bar{\tau}) = f(z) + \frac{\bar{\tau}}{2} d^{\mathrm{T}}(q + Mz).$$

(b) Consider the constrained 1-dimensional minimization problem

$$\begin{array}{ll} \text{minimize} & g(\tau) \\ \text{subject to} & \tau \geq 0,\ z + \tau d \geq 0. \end{array}$$

If τ^* denotes the minimum point of this problem, show that

$$g(\tau^*) \leq f(z) + \frac{\tau^*}{2} d^{\mathrm{T}}(q + Mz).$$

5.11.10 Let $M \in \boldsymbol{P_0} \cap R^{n \times n}$ and $q \in R^n$. Suppose that (q, M) has a solution z^* satisfying (i) $M_{\alpha\alpha}$ is nonsingular (where $\alpha = \operatorname{supp} z^*$), and (ii) z^* is nondegenerate (i.e., $z^* + q + Mz^* > 0$).

(a) For each $\varepsilon > 0$, let $z(\varepsilon)$ be the unique solution of the LCP $(q, M + \varepsilon I)$. Show that $z(\varepsilon)$ converges to z^* as $\varepsilon \to 0$.

(b) Deduce from (a) that z^* is the only solution of (q, M) possessing the two given properties.

5.11.11 Supply the missing details in the proof of Proposition **5.8.2**.

5.11.12 Develop a damped-Newton method for solving the LCP (q, M) that is based on the equation

$$q + Mz^+ - z^- = 0.$$

5.11.13 Consider the damped-Newton method **5.8.5** for solving the LCP (q, M) where M is of order n. Suppose that $M \in \boldsymbol{Z}$ and that q contains no zero component. Prove that if the initial vector z^0 is chosen to be zero, then we may conclude that the problem is infeasible if, during any interation, the algorithm produces a u_α which is not strictly positive. From this, show that if **5.8.5** does not terminate with a solution to the problem within n steps, then we may conclude the problem is infeasible.

5.11.14 Consider the interior-point method **5.9.3** applied to the augmented LCP (q', M') given by (3.7.10) where $a > 0$. Show that any sequence produced by the method must be bounded.

5.11.15 Let $M \in R^{n\times n}$ and $q \in R^n$. Suppose the vector $\bar{z} \in \text{FEA}(q, M)$ satisfies the condition (with $\bar{w} = q + M\bar{z}$) :

$$\bar{z}^{\mathrm{T}}\bar{w} \leq \varepsilon$$

where ε is a certain positive scalar. Show that there exists a value $\bar{\varepsilon} > 0$ such that if $0 < \varepsilon < \bar{\varepsilon}$, then the LCP (q, M) has a solution z^* with $w^* = q + Mz^*$ and

$$z_i^* = 0, \text{ for all } i \in I, \quad \text{and} \quad w_i^* = 0, \text{ for all } i \in J,$$

where

$$I = \{i : \bar{z}_i \leq \sqrt{\varepsilon}\} \quad \text{and} \quad J = \{i : \bar{w}_i \leq \sqrt{\varepsilon}\}.$$

The proof of this is based on the following considerations: (i) the set FEA(q, M) has a finite number of extreme points, (ii) there must exist a positive scalar δ such that if z is any one of these extreme points with $w = q + Mz$, then

$$[z_i > 0 \;\Rightarrow\; z_i \geq \delta] \quad \text{and} \quad [w_i > 0 \;\Rightarrow\; w_i \geq \delta],$$

and (iii) any feasible vector of (q, M) is the sum of a convex combination of extreme points and a nonnegative combination of extreme rays of the feasible region.

5.11.16 Let a, b, c and d be nonnegative scalars. Show that the inequality (5.9.12) holds. Use this inequality to prove Lemma **5.9.10**.

5.11.17 Let M be an $n \times n$ $\boldsymbol{P}$-matrix. Show that the quantity

$$\pi_M(z) = (\max_{1\leq i\leq n} z_i(Mz)_i)^{1/2}$$

defines a norm on vectors in R^n if and only if M is diagonal.

5.11.18 An important property of eigenvalues of a symmetric matrix is that of *interlacing*. Specifically, let $A \in R^{n\times n}$ be symmetric, and A_r be any principal submatrix of A of order r. Suppose that the eigenvalues of A and A_r are arranged in nondecreasing order:

$$\lambda_1(A) \leq \ldots \leq \lambda_n(A), \quad \text{and} \quad \lambda_1(A_r) \leq \ldots \leq \lambda_r(A_r).$$

Then for each integer k such $1 \leq k \leq r$, we have

$$\lambda_k(A) \leq \lambda_k(A_r) \leq \lambda_{k+n-r}(A).$$

Use this fact to show that if M is a symmetric $\boldsymbol{P}$-matrix, then the constant $\delta(M)$ defined in (5.10.9) is equal to $\lambda_1(M)$.

5.11.19 Let M be an $n \times n$ $\boldsymbol{P}$-matrix and let $\delta(M)$ be defined by (5.10.9). Let

$$\delta = \delta(M), \quad \zeta = \max_{i \neq j} |m_{ij}|.$$

Define the matrix $M(\delta, \zeta, n)$ by

$$(M(\delta,\zeta,n))_{ij} = \begin{cases} 0 & \text{if } i > j \\ \delta & \text{if } i = j \\ -\zeta & \text{if } i < j. \end{cases}$$

(a) Show that $M(\delta, \zeta, n) \in \boldsymbol{K}$ and $c(M) \geq c(M(\delta, \zeta, n))$.

(b) Show that

$$(1+\delta/\zeta)^2 \frac{\delta}{(1+\zeta/\delta)^{2(n-1)}} \geq c(M(\delta,\zeta,n)) \geq \frac{\delta}{(1+\zeta/\delta)^{2(n-1)}}.$$

5.11.20 Let M be a positive definite matrix. Let z denote the unique solution of the LCP (q, M) and x be an arbitrary vector. Show that

$$\|z - x\|_2 \leq \frac{1 + \|M\|_2}{\lambda_1(\tilde{M})} \| \min(x, q + Mx)\|_2$$

where $\tilde{M}$ is the symmetric part of M and $\lambda_1(\tilde{M})$ denotes the least eigenvalue of $\tilde{M}$.

5.11.21 Refine the argument in Corollary **5.10.15** to show that under the assumptions of Theorem **5.10.14**, there exist positive constants c_1 and

c_2, both dependent on q and M only, such that for any vector $z \in R^n$ (with $w = q + Mz$),

$$\| z - \Pi_S(z) \|_\infty \le c_1 [\, \| (z^-, w^-, (z^T w)^+) \|_\infty + (z^T w + c_2 \| (z^-, w^-) \|_\infty)^{1/2}].$$

As a matter of fact, we have $c_2 = \min\{ \| (\bar{w}, \bar{z}) \|_1 : \bar{z} \in S \}$, where as usual, $S = \mathrm{SOL}(q, M)$. Exercise **7.6.7** shows how bounds on c_2 can be derived under a strict feasibility assumption.

5.11.22 Consider the pair

$$q = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Show that for this LCP, the "min" function given by (5.10.3) fails to be a valid residue for arbitrary (feasible) vectors in R^2, and that Corollary **5.10.15** fails to hold without the square root term in (5.10.20).

5.12 Notes and References

5.12.1 The early study of iterative methods for solving the LCP was mainly concerned with the symmetric problem and its application to a non-negatively constrained convex quadratic program. Hildreth (1954, 1957) developed a projected Gauss-Seidel relaxation method for solving a strictly convex quadratic program with only inequality constraints; his method actually solved the dual problem which was equivalent to an LCP (although it was not recognized as such at that time). In turn, Hildreth's procedure was closely related to some relaxation methods for solving a system of linear inequalities that were proposed a few years earlier in Agmon (1954), and Motzkin and Schoenberg (1954). These papers were among the earliest published articles on this subject. A more contemporary treatment of these relaxation methods and analysis of their convergence rates can be found in Goffin (1980), Mandel (1984b) and Iusem and De Pierro (1990). One important application of these iterative methods for solving linear inequalities is image reconstruction from projections, see the monograph of Herman (1980) for more details.

5.12.2 Except for some scattered papers, the study of iterative methods for quadratic programs and/or the LCP was not very intense in the nineteen sixties. Then, a paper by Cryer (1971b) was published which developed the first SOR method for quadratic programming. Cryer's work was extended by Cottle, Golub and Sacher (1978), Mangasarian (1977) and Cottle and Pang (1980). More will be said about these papers in subsequent notes.

5.12.3 The finite-dimensional contact problem was sketchily formulated as an LCP by Fridman and Chernina (1967) who proposed an iterative scheme (the projected Gauss-Seidel method) for solving it. Our treatment of contact problems is based mainly on Conry and Seireg (1971). For other relevant work, see Fischer (1974), Chand, Haug and Rim (1976), and Eckhardt (1978). The paper by Maier et al. (1979) features an interesting application of the LCP to a contact problem concerned with the design of underwater pipelines. For comprehensive studies on the mechanics of contact problems see Kalker (1975, 1977) and Panagiotopoulos (1985).

5.12.4 The free-boundary problem for journal bearings came to the attention of the optimization community largely through the publications of Cryer (1971a, 1971b). Cryer's work was, in turn, inspired by that of Christoperson (1941). For the most part, the material presented here is based on the Ph.D. thesis of Sacher (1974) and the paper by Cottle, Golub and Sacher (1978). The monograph by Crank (1984) contains a nice account of the problem and the LCP formulation.

5.12.5 The network equilibrium problem is a special instance of the market equilibrium problem discussed in Section 1.2 in which the supply side linear program is a minimum cost network flow problem. In addition to the classic treatise by Takayama and Judge (1971), there is a vast literature on the network equilibrium problem. The monograph edited by Harker (1985) contains a sample of research articles on this problem. Our treatment in Section 5.1 follows the discussion in Glassey (1978) and Pang and Lee (1981). Incidentally, these two papers describe some specialized pivoting methods for solving the LCP arising from the single commodity affine case of the network equilibrium problem.

5.12.6 A number of papers have reported computational results with the use of SOR methods for solving LCPs arising from the application problems discussed in Section 5.1. Cottle, Golub and Sacher (1978), Cottle and Goheen (1978) pertain to certain free-boundary problems; Pang (1982), Güder

(1989) and Güder and Morris (1989) concern the network equilibrium problem. Cottle (1984) applied an SOR method to solve a constrained matrix problem (formulated as an LCP with network structure) and reported computational experience with the method.

5.12.7 Mangasarian (1977) proposed a very general iterative scheme for solving the symmetric LCP and established its convergence under some fairly broad assumptions. This paper can be credited as being the first one in which a systematic study of the convergence of iterative methods for the LCP was carried out; it has provided the impetus for much subsequent research on this subject, among which is the work of Aganagić (1978a) and Ahn (1981) who investigated the convergence of iterative methods for solving the asymmetric LCP.

5.12.8 Inspired by the aforementioned papers of Mangasarian, Aganagić, and Ahn, Pang (1982) introduced a matrix-splitting algorithm as a unification of many of the iterative methods for solving the LCP. Incidentally, Mangasarian's scheme is general enough to include the splitting algorithm as a special case. Nevertheless, the way Mangasarian (1977) specified his algorithm restricted it to be one of the relaxation type; he made no mention of casting it in the form of a splitting method. The advantage of the splitting framework is its simplicity and ease of analysis. Pang's splitting algorithm is the main topic discussed in Sections 5.2 and 5.3; the results presented therein appear in the papers by Pang (1982, 1984, 1986a).

5.12.9 Aganagić (1978a) initiated the use of the norm contraction approach in the convergence analysis of the simple iteration (5.3.14). The latter is the well known *projection method* for variational inequality problems specialized to the LCP. For references on the general projection method, see Glowinski, Lions and Trémolières (1981), and Pang and Chan (1982). Aganagić (1978a) is believed to be the first one to employ it for solving the asymmetric LCP.

The notion of a T-splitting of a matrix defined in Exercise **5.11.8** was introduced in Iusem (1990b). As shown in this exercise, a special case of this T-splitting idea leads to a projection method for solving an LCP with a certain kind of positive semi-definite matrix. The paper of Bertsekas and Gafni (1982) establishes the convergence of this method in the context of a variational inequality problem of a particular type.

Ahn (1981) was the first one to employ a vector contraction approach to analyze the convergence of Mangasarian's 1977 scheme for an asymmetric LCP. Ahn (1983) used the monotone approach to establish the convergence of this scheme for solving an LCP with a $\boldsymbol{Z}$-matrix and with upper bounds on the (primary) variables.

5.12.10 The family of SOR methods is among the most effective for solving large, sparse LCPs. The efficiency of these methods is crucially dependent on the choice of the relaxation parameter ω, see (5.2.1). There is little theory concerning the choice of an *optimal* parameter value. Generally speaking, if one can identify the positive variables of the limit solution in a finite number of iterations, then the iterative method essentially reduces to that for solving a system of linear equations. (Exercises **5.11.2** and **5.11.3** are relevant to this consideration.) In this case, it becomes possible to borrow from the theory of linear equations to help identify a good value for ω. See the text by Hageman and Young (1981) for more discussion on the latter subject.

5.12.11 The SOR methods, in conjunction with a type of proximal-point scheme (see **5.12.16**), have been used extensively for solving large-scale linear programs. Discussions of these applications can be found in the work of Mangasarian (1981a, 1983, 1984a), and Cheng (1982). Implementation of the resulting algorithms in a parallel computation environment is discussed in Mangasarian and De Leone (1987, 1988b), De Leone and Mangasarian (1988a, 1988b), De Leone, Mangasarian and Shiau (1990), and Pang and Yang (1988). The last paper is the source for the two-stage splitting method discussed in Section 5.7.

5.12.12 The topic treated in Section 5.4 was an open research question for a long time. Inspired by Mangasarian's 1977 paper, many of the early theoretical studies of the iterative methods for solving the symmetric LCP were mainly concerned with the notion of subsequential convergence of the iterates. There was generally a lack of sequential convergence results except in the positive definite case—Cryer (1971b)—and a few special instances—Pang (1986a). Then, a breakthrough occurred with a paper by Luo and Tseng (1991) in which they proved Theorem **5.4.6**, thus settling an outstanding question. This result was independently established by De Pierro and Iusem (1993) whose proof we follow, as it is somewhat easier to comprehend than that of Luo and Tseng.

5.12.13 The family of matrix splitting methods provides one of several approaches for solving large sparse, strictly convex quadratic programs. Related approaches include: the row action methods of Lent and Censor (1980) and Censor (1981), a dual differentiable exact penalty function method proposed in Han and Mangasarian (1983a); a Lagrangean relaxation scheme used by Ohuchi and Kaji (1981, 1984), and Cottle, Duvall and Zikan (1986); a closely related dual active set algorithm of Hager (1987), Hager and Hearn (1993), and the dual conjugate gradient method suggested by Lin and Pang (1987). The last of these papers is a survey article that provides more detailed discussion of these other approaches, and contains some computational results comparing several of the corresponding methods.

5.12.14 The diagonalization process introduced in Section 5.5 is an effective tool for solving non-separable strictly convex quadratic programs; some computational experience with this approach is reported in Lin and Pang (1987). Application of the diagonalization idea for solving linearly constrained convex (nonlinear) programs is discussed in Feijou and Meyer (1984). This technique has been used quite successfully for solving some practical market equilibrium problems; see Ahn (1979) and Ahn and Hogan (1982). Its generalization to the context of the variational inequality problem is discussed in Pang and Chan (1982).

5.12.15 The development of the symmetric variational inequality approach in Section 5.5 follows that in the paper by Pang (1991a). Among the specific algorithms resulting from this approach is the gradient projection method for solving the LCP with a $\boldsymbol{P}$-matrix discussed in Cheng (1984).

5.12.16 The proximal point algorithm is a well-known iterative scheme for finding a zero of a maximal monotone operator. This algorithm is based on a fundamental result of Minty (1962) concerning a proximal map. The paper by Rockafellar (1976a) gives an excellent exposition of the algorithm in this general context, and Rockafellar (1976b) discusses its applications to convex programming. The former paper also contains a brief historical account of this important algorithm. Further studies of the proximal point algorithm can be found in Spingarn (1983) and Ha (1990). Part (b) of Theorem **5.6.2** is the specialization of a general result that holds in monotone operator theory; see Brézis (1973). Subramanian (1988b) discusses this result in the context of the nonlinear complementarity problem.

5.12.17 Gana (1982) and Venkateswaran (1993) discuss the regularization idea applied to an LCP with a $\boldsymbol{P}_0$-matrix as outlined in the paragraph preceding Theorem **5.6.2**. Kostreva (1989) employs the same idea to the LCP arising from a convex quadratic program. Some convergence results similar to this theorem are established, including one that requires a nondegeneracy assumption on the vector q; see Exercise **5.11.10**. In Gana (1982) and Kostreva (1989), the authors claim the convergence of the entire sequence of iterates; however, their "proofs" are based on an invalid argument. The paper of Venkateswaran (1993) also describes an algebraic scheme to implement this regularization approach. The resulting algorithm becomes a generalized Bard-type pivoting method that performs computations with rational functions in the parameter ε.

5.12.18 Mangasarian's 1977 scheme is a type of variable splitting method that also incorporates the underrelaxation step. More discussion of this and other generalized splitting methods can be found in the paper by Luo and Tseng (1991). Mangasarian (1991) and Li (1993) consider an inexact splitting method and investigates its convergence in the case of a symmetric positive semi-definite LCP.

5.12.19 The damped-Newton method (Algorithm **5.8.5**) is a specialization of some Newton-type methods for solving certain B-differentiable systems of nonsmooth equations proposed by Pang (1990a). Our presentation of this algorithm is based on Harker and Pang (1990b) in which some computational results are reported. A refinement of the algorithm that involves solving LCPs of smaller sizes can be derived from the method described in Pang (1991b). The theoretical advantage of the refined algorithm is that a quadratic rate of convergence can be established under the assumption of sequential convergence. Harker and Xiao (1990) and Xiao (1990) discuss extensively the application of the nonsmooth Newton methods for solving the nonlinear complementarity problem. The computational results they report provide evidence of the practical efficiency of these methods for solving realistic applied problems.

5.12.20 The notion of a strongly regular vector in Definition **5.8.3** originates from that of a strongly regular solution of a *generalized equation*; the latter concept was introduced by Robinson (1980a). More discussion about this will be given in the Notes and References section of Chapter 7.

5.12.21 Kostreva (1976) discussed the application of Newton's method for solving the system of piecewise linear equations (5.8.1). The *all-change algorithm*, as the resulting method was called in the reference, differs from Algorithm **5.8.5** in that it contains no linesearch routine. In the absence of this important step, the algorithm becomes a kind of heuristic procedure for solving the LCP. Indeed, Kostreva offered no theoretical justification for his all-change algorithm. Aganagić (1984) also used the system (5.8.1) in developing a Newton method for the LCP. His development restricts M to be a hidden $\boldsymbol{Z}$-matrix.

5.12.22 The interior-point method (Algorithm **5.9.3**) originates from an algorithm introduced by Karmarkar (1984) for solving linear programs. Due to its remarkable practical efficiency and dramatic departure from the traditional simplex method, there is an abundance of research on the latter algorithm. The volumes edited by Megiddo (1989b) and Gay, Kojima and Tapia (1991) contain excellent collections of papers in this area.

The extension of Karmarkar's algorithm to the LCP has been the subject of many papers. Of particular relevance to our discussion in Section 5.9 are the articles by Kojima, Megiddo and Ye (1988), Ye (1988b), Ye and Pardalos (1991), Todd and Ye (1990) and Kojima, Megiddo, Noma and Yoshise (1991). The last of these is an extensive survey paper which presents a unified approach to the entire subject and contains a long bibliography. The function ϕ in (5.9.1) was introduced by Todd and Ye (1990) for solving linear programs.

A major aspect of the interior-point methods not covered in our presentation is their polynomial computational complexity when they are applied to a positive semi-definite problem. The reader is referred to the excellent lecture notes of Kojima, Megiddo, Noma and Yoshise (1991) which documents many results of this nature.

The compact form of the continuation interior-point method is drawn from the Ph.D. dissertation of Chen (1990). In this work, Chen developed the method for solving a monotone variational inequality and the nonlinear complementarity problem with a ***P**-function*; the latter kind of function is a nonlinear generalization of the class of **P**-matrices. Two related pieces of work that precede Chen's thesis are Kojima, Mizuno and Noma (1989, 1990). Interestingly, this kind of continuation method for solving complementarity problems is intimately connected to a conceptual algorithm analyzed in a paper of McLinden (1980) published almost a decade earlier. The reader is referred to the book by Allgower and Georg (1990) for a general introduction to the subject of numerical continuation methods for solving systems of nonlinear equations.

5.12.23 In Definition **2.9.10**, we introduced the idea of a homotopy. This concept dates back to at least the nineteenth century. Homotopies are a powerful tool both analytically and computationally. They can be used to prove theorems and, indeed, much topological theory is based on them. They can also be used to find numerical solutions to differential equations, integral equations, and nonlinear systems of equations. In Section 5.9, homotopies are used analytically in the proof of Theorem **5.9.13**, and they are used computationally as the basis of Algorithm **5.9.16**. The reader may wish to consult Remark **5.9.15**. Besides this, there are other connections between the LCP and homotopies. One may view Lemke's method as being akin to the homotopy concept (see the beginning of Section 6.3). In addition, homotopy algorithms exist for the general complementarity problem, fixed point problems, and other LCP related problems. The reader is referred to **2.11.1** for a brief historical account and some references concerning these homotopy (fixed-point) methods.

5.12.24 As noted in **5.12.1**, some of the earliest iterative methods for solving the LCP appeared in the nineteen fifties. Nevertheless, in the field of mathematical programming, there are rather few formal studies of residues and error bounds. In the context of the LCP, the papers by Mangasarian and Shiau (1986), Mathias and Pang (1990), Mangasarian (1990a, 1992), and Luo and Tseng (1992b, 1992c) seem to be the only available references. The two related papers, Pang (1986b, 1987), discuss some error-bound results for the nonlinear complementarity problem and the variational inequality problem.

Much of the development in Section 5.10 concerning an LCP with a **P**-matrix is drawn from Mathias and Pang (1990). The basic error-bound result for a positive semi-definite LCP (Theorem **5.10.14**) and its consequence (Corollary **5.10.15**) appear in the article by Mangasarian and Shiau who also present the example in **5.10.1** to illustrate the deficiency of the quantity $\| \min(x, q + Mx) \|$ as a residue function. In turn, the work of these authors is based on the earlier results of Mangasarian (1981b) which introduced the notion of a *condition number* for systems of linear inequalities (see Lemma **5.10.13**). Hoffman (1952) is believed to be the first person to have derived such error-bound results for linear inequalities. Unlike Mangasarian (1981b), Hoffman did not give explicit expressions for the multiplier of the residue (i.e., the constant λ in **5.10.13**), only estimates were given in special cases. The LCP in Exercise **5.11.22** is due to Jun Ren (communicated to us by O.L. Mangasarian).

5.12.25 Defined in equation (5.10.6), the fundamental quantity $c(M)$ associated with a $\boldsymbol{P}$-matrix M appears as an important constant in the complexity analysis of the interior point methods for solving the LCP (q, M); see Kojima, Megiddo and Noma (1989) and Ye (1988b). See also Mathias and Pang (1990) and Note **5.12.24**. The computation of the lower bound $L(M)$ in expression (5.10.16) involves solving an eigenvalue optimization problem; the reader is referred to Overton (1992) for an excellent exposition of the latter subject.

5.12.26 Theorem **5.10.17** was obtained by Ferris and Mangasarian (1991) as a by-product of their study of the "sufficiency" of the minimum principle. The consequence of this theorem, Corollary **5.10.18**, was proved earlier in Mangasarian (1990a). In an interesting paper by Luo and Tseng (1990b), the authors obtain a complete characterization of when the quantity $\| \min(x, q + Mx) \|$ provides a bound on the distance from any feasible point to the solution set of the LCP (q, M) with M positive semi-definite; their study actually concerns the monotone affine variational inequality problem of which the positive semi-definite LCP is a special case. Mangasarian (1992) derives some global error-bound results for the latter problem that are based on Theorem **5.10.14** and Corollary **5.10.18**.

5.12.27 The subject of local error bounds is closely related to that of sensitivity theory. The reader is referred to Section 7.7 for notes and references on the latter subject.

Chapter 6

GEOMETRY AND DEGREE THEORY

We have so far studied the linear complementarity problem algebraically and algorithmically. In this chapter we will consider the geometric side of the LCP. The foundation for this chapter has already been laid in Definition **1.3.2** and in Proposition **1.4.4**.

In Section 1.3 we defined the complementary cones, $\operatorname{pos} C(\alpha)$, of a matrix M. It was also mentioned that the LCP (q, M) has a solution if and only if q is contained in one of the complementary cones. In fact, it will turn out that for "most" q, the number of solutions to the LCP (q, M) equals the number of complementary cones containing q. Thus, we may study the question of the existence and number of solutions to linear complementarity problems by examining the properties of the complementary cones, their facets, and their union. This will lead to new insights concerning matrix classes we already know. In addition, it will suggest the study of some new matrix classes.

In Section 1.4 we discussed piecewise linear functions and considered, in particular, the piecewise linear function $f(x)$ defined in (1.4.8). Proposition

1.4.4 shows that there is a bijection between solutions to $f(x) = q$ and solutions to the LCP (q, M). Since, as we will show, $f(x)$ is just the representation of the complementary cones as a piecewise linear function, we may learn more about the LCP (q, M) and its geometry by examining $f(x)$.

A tool that is quite useful in studying $f(x)$ is degree theory. Initially, one might wonder why degree theory should be needed. The map $f(x)$ associated with an LCP (q, M) is a piecewise linear function with very special structure. It would seem that degree theory, which deals with the global properties of uniformly continuous maps, would add very little to what we might discover by exploiting the special structure of $f(x)$. In essence, it would seem that $f(x)$ is too special a function, and degree theory too general a tool, for there to be much gain in applying one to the other.

However, one finds that $f(x)$ is not too special a function, especially in high dimensions. As mentioned in Section 1.4, and shown in Eaves and Lemke (1981), any piecewise linear equation can be represented by an LCP. While they may, at first, appear to be simple extensions of linear functions, piecewise linear functions can approximate any uniformly continuous function arbitrarily closely. Therefore, while $f(x)$ may seem quite simple in very low dimensions, this simplicity quickly evaporates as the dimension increases. Unfortunately, it is in very low dimensions that we can most easily visualize the complementary cones, and general geometry, of the LCP. Thus, a trap exists for anyone studying the geometry of the LCP in that many patterns and properties which exist (or seem to exist) in R^2 and R^3, often break down in R^4 or R^5. We will encounter some examples of this throughout this chapter.

We have tried to point out that $f(x)$ is not too special a function but, fortunately, it is special enough that degree theory turns out not to be too general a tool. As the next section will show, calculating the degree of $f(x)$ is fairly simple. Also, degree theory is a good framework with which to study the local behavior of $f(x)$. Furthermore, many parametric algorithms for the LCP can be viewed as being based on homotopies, hence are amenable to study using ideas from degree theory. In fact, we will revisit Lemke's algorithm from this viewpoint in Section 6.3.

6.1 Global Degree and Degenerate Cones

In Section 2.9 we discussed degree theory for continuous nondegenerate homogeneous functions in general. In this section we will examine the specific case in which the function represents an LCP. To this end, we bring to the forefront the piecewise linear map given in (1.4.8). The following notation will be used throughout this chapter.

6.1.1 Notation. For any $M \in R^{n\times n}$ we let $f_M : R^n \to R^n$ denote the piecewise linear function given by

$$f_M(x) \;=\; x^+ - Mx^-$$

where, as usual, $x_i^+ = \max(0, x_i)$ and $x_i^- = \max(0, -x_i)$ for all $i = 1, \ldots, n$. We will simply write $f(x)$ when it is clear which M is meant.

The following proposition gives several elementary properties of $f_M(x)$. Before stating this proposition, we wish to point out that the complementary cones of the $n \times n$ identity matrix I are, in fact, the orthants of R^n. More precisely,

$$\operatorname{pos} C_I(\alpha) \;=\; \{x \in R^n : x_i \leq 0 \text{ for } i \in \alpha \text{ and } x_i \geq 0 \text{ for } i \notin \alpha\}$$

for all $\alpha \subseteq \{1, \ldots, n\}$.

6.1.2 Proposition. For any $M \in R^{n\times n}$ the function $f_M : R^n \to R^n$ is continuous, piecewise linear, and positive homogeneous of degree 1. In addition, the pieces of f_M are the orthants of R^n and, given any index set α in $\{1, \ldots, n\}$, we have

$$f_M(x) \;=\; C_{-M}(\alpha)x \qquad \text{for all } x \in \operatorname{pos} C_I(\alpha)\,.$$

Thus, the image of $\operatorname{pos} C_I(\alpha)$ under f_M is precisely the complementary cone $\operatorname{pos} C_M(\alpha)$.

Proof. This is Exercise **6.10.2**. □

We can now use f_M to apply the concepts of index and degree to the LCP. Let $x \in R^n$ be in the interior of some orthant, say $x \in \operatorname{int}(\operatorname{pos} C_I(\alpha))$. From **6.1.2** it follows that f_M is differentiable at x and $\nabla f_M(x) = C_{-M}(\alpha)$.

As $\det C_{-M}(\alpha) = \det M_{\alpha\alpha}$, we conclude from **2.9.3** that the index of f_M at x, if it exists, is $\text{sgn}(\det M_{\alpha\alpha})$. We now restate **2.9.3** and **2.9.4** in the framework of the LCP.

6.1.3 Definition. Let $M \in R^{n\times n}$ be given. Suppose $x \in R^n$ is contained in the interior of $\text{pos}\, C_I(\alpha)$, for some $\alpha \subseteq \{1, \ldots, n\}$, and that $M_{\alpha\alpha}$ is nonsingular. We then define the *index* of M at x, denoted by $\text{ind}_M(x)$, to be equal to the index of f_M at x. Thus, $\text{ind}_M(x) = \text{sgn}(\det M_{\alpha\alpha})$, where $\det M_{\emptyset\emptyset} = 1$. We will denote $\text{ind}_M(x)$ simply by $\text{ind}(x)$ when it is clear which M is meant.

Notice, from the above, that if one point in the interior of $\text{pos}\, C_I(\alpha)$ has a well-defined index, then they all do, and the index is the same for each point. Thus, we may define the *index* of the orthant $\text{pos}\, C_I(\alpha)$ to be the common index of the points in its interior. Of course, if the points in the interior do not have a well-defined index, then neither does $\text{pos}\, C_I(\alpha)$. By the *index* of the complementary cone $\text{pos}\, C_M(\alpha)$ we mean the index of the orthant $\text{pos}\, C_I(\alpha)$. This usage somewhat abuses the definition of index, as a complementary cone is in the range (not the domain) of f_M.

From **1.4.4** we observe that $x \in f_M^{-1}(q)$ if and only if (x^+, x^-) solves the LCP (q, M). Thus, if (w, z) solves the LCP (q, M) for some $q \in R^n$, then we define the *index* of (w, z) to be $\text{ind}_M(w-z)$ and denote it as $\text{ind}_M(w, z)$. Of course, if $\text{ind}_M(w - z)$ is not well-defined, then neither is $\text{ind}_M(w, z)$.

6.1.4 Definition. Let $M \in R^{n\times n}$ be given. Suppose, for some $q \in R^n$, that $f_M^{-1}(q)$ consists of finitely many points and, further, that $\text{ind}_M(x)$ is well-defined (using **6.1.3**) for all $x \in f_M^{-1}(q)$. We then define the *local degree* of M at q, denoted by $\deg_M(q)$, to be equal to the local degree of f_M at q. We will denote this simply by $\deg(q)$ when it is clear which M is meant. Notice,

$$\deg_M(q) = \sum_{x \in f_M^{-1}(q)} \text{ind}_M(x)\,.$$

Thus, the local degree of M at q, if it exists, equals the sum of the indexes of all (w, z) which solve (q, M).

Now that we have a notion of local degree, we should examine R^n to see which points have a well-defined local degree and which do not. To this end, we introduce the following.

6.1.5 Notation. Given any matrix $M \in R^{n\times n}$, let $\mathcal{K}(M)$ denote the union of the facets of all the complementary cones of M. It follows from **6.1.2**, that $\mathcal{K}(M)$ equals the image of the coordinate hyperplanes under f_M.

6.1.6 Definition. Let $M \in R^{n\times n}$ be given. The complementary cone $\operatorname{pos} C_M(\alpha)$ is said to be *full* or *nondegenerate* if $\det M_{\alpha\alpha} \neq 0$, and *degenerate* otherwise.

6.1.7 Remark. Note that Definitions **1.3.2** and **6.1.6** agree on the meaning of the word *full.* Also, looking back at Definition **3.6.1**, we see that a matrix is nondegenerate if and only if all of its complementary cones are nondegenerate.

According to **6.1.3** and **6.1.4**, there are several circumstances under which $\deg(q)$ may not be well-defined. One possibility is that there is some point $x \in f^{-1}(q)$ which is contained in a coordinate hyperplane of R^n. Since x would then not be contained in the interior of one of the pieces of f, i.e., in the interior of an orthant, we cannot be certain that f is differentiable around x. Thus, we may not be able to define an index for x and, indeed, **6.1.3** does not give x an index. In turn, **6.1.4** does not give q a local degree. The image of the coordinate hyperplanes, $\mathcal{K}(M)$, is thus a set of points that do not have well-defined local degrees.

Another circumstance under which $\deg(q)$ fails to be defined is the case where, for some point $x \in f^{-1}(q)$, we find $x \in \operatorname{int}(\operatorname{pos} C_I(\alpha))$ and $\det M_{\alpha\alpha} = 0$. This implies that $q = f(x)$ is contained in a degenerate complementary cone. Thus, the degenerate complementary cones, along with $\mathcal{K}(M)$, are a set of points which are not given a well-defined local degree by **6.1.3** and **6.1.4**. It turns out that we need only consider $\mathcal{K}(M)$.

6.1.8 Theorem. Let $M \in R^{n\times n}$ be given. The local degree of q relative to M is well-defined if and only if $q \notin \mathcal{K}(M)$.

Proof. From **6.1.5**, if $q \in \mathcal{K}(M)$, then there is some $x \in f^{-1}(q)$ which lies on the boundary of an orthant. Thus, $\operatorname{ind}(x)$ is not defined by **6.1.3**, so $\deg(q)$ is not well-defined.

Let $q \in R^n \backslash \mathcal{K}(M)$ be given. We must show that $f^{-1}(q)$ has only finitely many points and, further, that if $x \in f^{-1}(q)$, then $x \in \operatorname{int}(\operatorname{pos} C_I(\alpha))$ and $\det M_{\alpha\alpha} \neq 0$ for some $\alpha \subseteq \{1,\ldots,n\}$.

Suppose $x \in f^{-1}(q)$ is given. If x is not in the interior of some orthant, then x is contained in a coordinate hyperplane, i.e., some component of x is zero. This would imply that q, which equals $f(x)$, is in $\mathcal{K}(M)$. Thus, there is some index set α such that $x \in \operatorname{int}(\operatorname{pos} C_I(\alpha))$.

Now, suppose $\det M_{\alpha\alpha} = 0$. Since $\det C_{-M}(\alpha) = \det M_{\alpha\alpha} = 0$, there exists a $y \in R^n$ with $y \neq 0$ and $C_{-M}(\alpha)y = 0$. One easily sees that a $\lambda \in R$ can be found such that $x + \lambda y$ is contained in the boundary of $\operatorname{pos} C_I(\alpha)$. From **6.1.2**, we have $q = f(x + \lambda y)$. Thus, as above, we have the contradiction that $q \in \mathcal{K}(M)$.

The last thing we must ensure is that $|f^{-1}(q)| < \infty$. If $x \in f^{-1}(q)$, we have shown that $x \in \operatorname{int}(\operatorname{pos} C_I(\alpha))$ and $\det M_{\alpha\alpha} \neq 0$ for some index set α. From **6.1.2**, it follows that $x = (C_{-M}(\alpha))^{-1}q$. Thus, x is the only element of $f^{-1}(q)$ contained in $\operatorname{pos} C_I(\alpha)$. Since there are only finitely many complementary cones, there can be only finitely many elements in $f^{-1}(q)$. □

In the above proof of **6.1.8** we proved some side results which are interesting on their own. We state them now as corollaries.

6.1.9 Corollary. Any degenerate complementary cone is the union of its facets. □

6.1.10 Corollary. Let $M \in R^{n\times n}$ and $q \in R^n$ be given. If $\operatorname{pos} C_M(\alpha)$ is a full complementary cone, then there is at most one element of $f_M^{-1}(q)$ in the orthant $\operatorname{pos} C_I(\alpha)$. □

6.1.11 Corollary. Let $M \in R^{n\times n}$ and $q \in R^n$ be given. If $q \notin \mathcal{K}(M)$, then the number of solutions to the LCP (q, M) equals the number of complementary cones containing q. □

While it is possible to give a well-defined index to some of the points on the coordinate planes, it is not necessary for our efforts in this book. Thus, **6.1.8** answers the question which was raised just before **6.1.5**, to wit, what is the set of points in R^n which have, by **6.1.4**, a well-defined local degree?

We now know that $\deg_M(q)$ is well-defined as long as $q \notin \mathcal{K}(M)$. Therefore, we might wonder how extensive is the set $R^n \setminus \mathcal{K}(M)$. The following theorem answers this question.

6.1.12 Theorem. If $M \in R^{n\times n}$, then $\mathcal{K}(M) \subseteq R^n$ is a closed cone, $\dim \mathcal{K}(M) = n-1$, and $R^n \setminus \mathcal{K}(M)$ is dense in R^n.

Proof. $\mathcal{K}(M)$ is the union over all index sets α and over all $i \in \{1,\ldots,n\}$ of the closed convex cones $\text{pos}\, C(\alpha)_{\bullet\bar{i}}$. Thus, $\mathcal{K}(M)$ is a closed cone.

One can show, see **6.10.1**, that $\text{pos}\, C(\alpha)_{\bullet\bar{i}}$ has a nonempty relative interior in its affine hull, which is the subspace spanned by the columns of $C(\alpha)_{\bullet\bar{i}}$. By **2.9.14** and **2.9.15(a)**, the dimension of $\text{pos}\, C(\alpha)_{\bullet\bar{i}}$ is no greater than $n-1$. Thus, **2.9.16** implies that $\dim \mathcal{K}(M) \leq n-1$.

The rank of $C(\emptyset)_{\bullet\bar{1}}$ is $n-1$ as $C(\emptyset)$ is the identity matrix. Thus, the above argument shows that $\text{pos}\, C(\emptyset)_{\bullet\bar{1}}$ has dimension exactly equal to $n-1$. Thus, $\dim \mathcal{K}(M) \geq n-1$ and, so, $\dim \mathcal{K}(M) = n-1$. The theorem's final conclusion follows from **2.9.17** and from what we have already shown. □

Throughout Section 2.9 we required that the function f be a continuous nondegenerate homogeneous function in order for index, local degree, and degree to have meaning. While f_M is continuous and homogeneous, we have not required f_M to be nondegenerate in the definitions of index and local degree given here. The reason for this omission can be explained as follows.

By Definition **2.9.1**, f_M is degenerate if there is some $x \neq 0$ such that $f_M(x) = 0$. We could, of course, make f_M nondegenerate by restricting the domain of f_M to consist only of those orthants $\text{pos}\, C_I(\alpha)$ for which $0 \neq x \in \text{pos}\, C_I(\alpha)$ implies $f_M(x) \neq 0$. The key thing to observe is that this does not affect the points with a well-defined index and, hence, it does not affect the points with a well-defined local degree. For if $\text{pos}\, C_I(\alpha)$ is an orthant excluded from the domain of f_M, then for some $x \neq 0$ in $\text{pos}\, C_I(\alpha)$ we have $0 = f_M(x) = C_{-M}(\alpha)x$. Thus, $0 = \det C_{-M}(\alpha) = \det M_{\alpha\alpha}$. Therefore, if $x \in \text{pos}\, C_I(\alpha)$, then Definition **6.1.3** does not, as it stands, give a meaning to $\text{ind}_M(x)$. We do not need to add the requirement that f_M is nondegenerate to obtain a valid definition of index and local degree. However, to extend things to encompass degree in the global sense, we must require nondegeneracy of f_M. This calls for making a further distinction among degenerate cones.

6.1.13 Definition. Let $M \in R^{n\times n}$ be given. The complementary cone $\text{pos}\, C_M(\alpha)$ is said to be *strongly degenerate* if there exists a nonzero and nonnegative vector x such that $C_M(\alpha)x = 0$. Notice, by Definition **6.1.6**, a strongly degenerate cone must be degenerate. If a cone is degenerate, but not strongly degenerate, it is said to be *weakly degenerate*.

From the comments before the definition, it is clear that f_M is nondegenerate over all of R^n if and only if M has no strongly degenerate complementary cones. With this is mind, we can state the following theorem which follows directly from Corollary **2.9.7**.

6.1.14 Theorem. Let $M \in R^{n\times n}$ be given. If no complementary cone of M is strongly degenerate, then the value of $\deg_M(q)$ is the same for all $q \notin \mathcal{K}(M)$. □

We have already encountered the set of matrices which have no strongly degenerate complementary cones. It is the class $\boldsymbol{R_0}$ defined in **3.8.7**, as the reader is asked to prove in Exercise **6.10.7**. It seems the $\boldsymbol{R_0}$-matrices are particularly amenable to degree-theoretic analysis. Theorem **6.1.14** leads us to the following terminology.

6.1.15 Definition. Let $M \in R^{n\times n} \cap \boldsymbol{R_0}$ be given. We define the *degree* of M, denoted by $\deg M$, to be the common value of $\deg_M(q)$ for all $q \notin \mathcal{K}(M)$.

6.1.16 Examples. Consider the matrices

$$M_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

$M_1 \in \boldsymbol{P} \subset \boldsymbol{R_0}$. The complementary cones of M_1 are depicted in Figure 1.2. The set $\mathcal{K}(M_1) \subseteq R^2$ consists of the rays along the vectors $(1,0)$, $(0,1)$, $(-2,1)$, and $(1,-2)$. For any point $q \notin \mathcal{K}(M_1)$, the set $f_{M_1}^{-1}(q)$ consists of exactly one point with index $+1$. Since M_1 has no degenerate complementary cones, we see that $\deg M_1$ is well-defined and equal to $+1$.

The complementary cones of M_2 are depicted in Figure 1.4 The set $\mathcal{K}(M_2) \subseteq R^2$ consists of the rays along the vectors $(1,0)$, $(0,1)$, $(-1,1)$, and $(1,-1)$. If $q \notin \mathcal{K}(M_2)$ and $e^{\mathrm{T}}q > 0$, then $f_{M_2}^{-1}(q)$ consists of exactly one point with index $+1$. If $e^{\mathrm{T}}q < 0$, then $q \notin \mathcal{K}(M_2)$ and $f_{M_2}^{-1}(q)$ is empty, so $\deg_{M_2}(q) = 0$. Thus, M_2 does not have a well-defined degree. Notice, $\operatorname{pos} C_{M_2}(\{1,2\})$ is strongly degenerate.

In the above example, M_2 does not have a well-defined degree. However, a fair amount can still be said. If we restrict f_{M_2} to be defined only for those $x \in R^2$ not contained in $\text{int}(\text{pos}\, C_I(\{1,2\}))$, then f_{M_2} is now nondegenerate. By **2.9.6** and **6.1.9**, $\deg_{M_2}(q)$ is invariant within any connected component of $R^2 \setminus f_{M_2}(\text{pos}\, C_I(\{1,2\}))$. On examination, we find that there are two connected components consisting of the two halfspaces on either side of the hyperplane $\{q \in R^2 : e^{\mathrm{T}}q = 0\}$. This is all consistent with the given example. It also suggests the following result.

6.1.17 Theorem. Let $M \in R^{n\times n}$ be given. Let $\mathcal{C}$ be the union of the strongly degenerate complementary cones of M. We then have $\deg_M(q) = \deg_M(q')$ for any $q, q' \in R^n \setminus \mathcal{K}(M)$ which are in the same connected component of $R^n \setminus \mathcal{C}$.

Proof. Let $D \subseteq R^n$ be the union of all the orthants of R^n which are not mapped by f_M into strongly degenerate complementary cones. Thus, $f_M : D \to R^n$ is nondegenerate. It is clear that $f_M(\text{bd}\, D)$ is contained in the union of the strongly degenerate complementary cones. The theorem now follows from **2.9.6**. □

6.1.18 Remark. Notice that Corollary **6.1.9** implies $\mathcal{C} \subseteq \mathcal{K}(\mathcal{M})$.

Clearly, it seems that a closer look at the strongly degenerate complementary cones is in order. Since we are attempting to view things geometrically, we would like a geometric characterization of the strongly degenerate cones. It turns out that, except for a trivial case, the strongly degenerate cones are the complementary cones which are not pointed in the sense of Definition **2.6.25**.

6.1.19 Theorem. Let $M \in R^{n\times n}$ be given. For any index set α, the complementary cone $\text{pos}\, C_M(\alpha)$ is strongly degenerate if and only if $\text{pos}\, C_M(\alpha)$ is not pointed or $C_M(\alpha)$ contains a zero column.

Proof. If $C(\alpha)_{\cdot i} = 0$, then $C(\alpha)e_i = 0$. This shows that $\text{pos}\, C(\alpha)$ is strongly degenerate. We now assume that $C(\alpha)$ has no zero columns.

Suppose $\text{pos}\, C(\alpha)$ is not pointed. It then contains some $x \neq 0$ in its lineality space. This means there exists $y, z \geq 0$ such that $C(\alpha)y = x$ and $C(\alpha)z = -x$. Thus, $C(\alpha)(y+z) = 0$. Clearly, as $x \neq 0$, neither y nor z can be zero. Hence, $y+z$ is nonzero and nonnegative, which shows $\operatorname{pos} C(\alpha)$ to be strongly degenerate.

Suppose $\operatorname{pos} C(\alpha)$ is strongly degenerate. There is then an $x \neq 0$ such that $x \geq 0$ and $C(\alpha)x = 0$. We may assume $x_i = 1$. By assumption, $C(\alpha)_{\bullet i} \neq 0$. Clearly, $C(\alpha)_{\bullet i} \in \operatorname{pos} C(\alpha)$. Yet, $x - e_i \geq 0$ and $C(\alpha)(x-e_i) = -C(\alpha)_{\bullet i}$. Thus, $C(\alpha)_{\bullet i}$ is in the lineality space of $\operatorname{pos} C(\alpha)$, so the cone is not pointed. □

6.1.20 Remark. If $A \in R^{n\times p}$, then the cone $\operatorname{pos} A$ is said to be *strictly pointed* if it is pointed and if A has no zero columns. Thus, among complementary cones, the strongly degenerate cones are precisely those cones which are not strictly pointed.

We now have some understanding of index and degree for a fixed matrix M. It is natural to continue our line of inquiry and ask what happens as the matrix M changes slightly? It would be nice if we could adapt the homotopy theorems, **2.9.11** and **2.9.12**, to cover the matrix M in an LCP. This is actually easy to do. In Exercise **6.10.6** the reader is asked to show that f_M is continuous in M. Given this, the next two results follow as corollaries to Theorems **2.9.11** and **2.9.12**. The reader may wish to look back and see how Theorem **6.1.17** followed from Theorem **2.9.6**.

6.1.21 Theorem. For each $t \in [0,1]$, let $M_t \in R^{n\times n}$ be given such that the function $M_t : [0,1] \to R^{n\times n}$ is continuous. Let $\{\alpha_i\}_{i=1}^k$ be a collection of index sets such that, for any $t \in [0,1]$ and any $i \in \{1,\ldots,k\}$, the complementary cone $\operatorname{pos} C_{M_t}(\alpha_i)$ is not strongly degenerate. Take $D \subseteq R^n$ to be the open cone defined by

$$D = \operatorname{int}\Big(\bigcup_{i=1}^{k} \operatorname{pos} C_I(\alpha_i)\Big).$$

For each $t \in [0,1]$, let the function $g_t : \operatorname{cl} D \to R^n$ be the restriction of the function f_{M_t} to the cone $\operatorname{cl} D$. Suppose, for some $y \in R^n$, that the degree of y with respect to both g_0 and g_1 is well-defined (using **2.9.4**). If, for all $t \in [0,1]$, we have $y \notin g_t(\operatorname{bd} D)$, then $\deg_{g_0}(y) = \deg_{g_1}(y)$. □

6.1.22 Theorem. For each $t \in [0,1]$, let $M_t \in R^{n\times n}$ be given such that the function $M_t : [0,1] \to R^{n\times n}$ is continuous. If, for each $t \in [0,1]$, none of the complementary cones relative to M_t are strongly degenerate, then $\deg M_t$ exists and is the same for all $t \in [0,1]$. □

One might ask if Theorem **2.9.12** can be fully adapted to the LCP setting. That is, given M_0, $M_1 \in \boldsymbol{R_0} \cap R^{n\times n}$, if $\deg M_0 = \deg M_1$, is it true that there exist $\boldsymbol{R_0}$-matrices M_t, for $t \in (0,1)$, such that the function $M_t : [0,1] \to R^{n\times n}$ is continuous? The answer, in general, is no. While unfortunate, this answer should not be unexpected. Theorem **2.9.12** allows general continuous nondegenerate homogeneous functions to be used in the homotopy. Here we are more restrictive and only allow functions which are associated with an LCP, i.e., those of the form f_M as given in **6.1.1**. As an example, we have the matrices

$$M_0 = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 3 & 6 \\ -2 & 6 & 3 \end{bmatrix} \quad \text{and} \quad M_1 = \begin{bmatrix} 3 & -2 & 6 \\ 2 & -1 & 2 \\ 6 & -2 & 3 \end{bmatrix}. \tag{1}$$

These matrices are nondegenerate and, hence, are in $\boldsymbol{R_0}$. In addition, one can check that $f_{M_0}^{-1}(e) = \{\,(1,1,1), (-5,-1,-1)\,\}$ and $f_{M_1}^{-1}(e) = \{\,(1,1,1), (-1,-5,-1)\,\}$, from which one can prove that $\deg M_0 = \deg M_1 = 2$. However, it can be shown that if $M \in \boldsymbol{R_0} \cap R^{3\times 3}$ and $\deg_M = 2$, then of the eight principal minors of M exactly six must be positive and at least one must be negative. From this it follows that there is no continuous function $M_t : [0,1] \to \boldsymbol{R_0} \cap R^{3\times 3}$ with M_0 and M_1 as in (1). The reader is asked to supply the details of this argument in Exercise **6.10.20**. However, before attempting part (b) of this exercise, we suggest the reader become familiar with the material concerning $\boldsymbol{N}$-matrices in Section 6.6.

Continuing to look into what happens as M changes slightly, we might wonder how small changes in M affect the complementary cones. Let $\{M_i\}$ be a sequence in $R^{n\times n}$ such that $\lim_{i\to\infty} M_i = M$. If $q \in \operatorname{pos} C_M(\alpha)$, can we find a sequence $\{q^i\}$, with $q^i \in \operatorname{pos} C_{M_i}(\alpha)$, such that $\lim_{i\to\infty} q^i = q$? The answer is yes. If $q \in \operatorname{pos} C_M(\alpha)$, then there is an $x \geq 0$ such that $q = C_M(\alpha)x$. Letting $q^i = C_{M_i}(\alpha)x$ gives the desired sequence.

Suppose we now ask the converse of the previous question. If we have a sequence of points in a sequence of complementary cones, is the limit of the sequence of points in the limit of the sequence of complementary cones? If the limiting complementary cone is strongly degenerate, the answer could be no. However, if there is no strong degeneracy, the answer is yes. We will now prove a slightly more general result.

6.1.23 Theorem. Let $\mathcal{S} \subseteq R^n$ be a compact set. Let $\{M_i\}$ be a sequence of matrices in $R^{n\times n}$ such that $\lim_{i\to\infty} M_i = M$. Suppose, for some index set α, the cone $\operatorname{pos} C_M(\alpha)$ is not strongly degenerate. If $\mathcal{S}\cap\operatorname{pos} C_{M_i}(\alpha) \neq \emptyset$ for all i, then $\mathcal{S} \cap \operatorname{pos} C_M(\alpha) \neq \emptyset$.

Proof. Suppose $q^i \in \mathcal{S} \cap \operatorname{pos} C_{M_i}(\alpha)$, for all i. As $\mathcal{S}$ is compact, we may assume there is some $q \in \mathcal{S}$ such that $\lim_{i\to\infty} q^i = q$. We will show that $q \in \operatorname{pos} C_M(\alpha)$.

For each q^i, select some $x^i \geq 0$ such that $C_{M_i}(\alpha)x^i = q^i$. If the sequence $\{\|x^i\|\}$ is bounded, then we may assume there is some $x \in R^n$ such that $\lim_{i\to\infty} x^i = x$. It would then follow that $x \geq 0$ and that $q = \lim_{i\to\infty} C_{M_i}(\alpha)x^i = C_M(\alpha)x$. Hence, $q \in \operatorname{pos} C_M(\alpha)$.

Suppose, however, that the sequence $\{\|x^i\|\}$ is unbounded. We will show this is impossible. By assumption, if $x \geq 0$ and $C_M(\alpha)x = 0$, then $x = 0$. By Gordan's Theorem of the Alternative, there is a $y \in R^n$ such that $y^{\mathrm{T}}C_M(\alpha) > 0$. We then have

$$y^{\mathrm{T}}C_{M_i}(\alpha)x^i \;=\; y^{\mathrm{T}}q^i \tag{2}$$

for each i. Since the q^i converge to q, the right side of (2) is bounded over all i. However, as $y^{\mathrm{T}}C_M(\alpha) > 0$ and the x^i are nonnegative and unbounded, the left side of (2) is unbounded, a contradiction. □

6.1.24 Remark. The assumption that $\operatorname{pos} C_M(\alpha)$ is not strongly degenerate is critical. Consider M_1 and M_2 from **6.1.16**. We noted before that $\operatorname{pos} C_{M_2}(\{1,2\})$ was strongly degenerate. Note now that this cone does not contain $(-1,-1)$. Yet, for any $\lambda > 0$, if we let $M = M_2 + \lambda M_1$, then $(-1,-1) \in \operatorname{pos} C_M(\{1,2\})$.

Although a strongly degenerate complementary cone (as a set in R^n) is not necessarily continuous in M, the fact that it is strongly degenerate is, indeed, continuous in M. This is the essence of the next result.

6.1.25 Theorem. Let $\alpha \subseteq \{1,\ldots,n\}$ be given. The set of $M \in R^{n\times n}$ for which $\operatorname{pos} C_M(\alpha)$ is strongly degenerate is closed in $R^{n\times n}$.

Proof. Let $\{M_i\}$ be a sequence in $R^{n\times n}$ such that $\lim_{i\to\infty} M_i = M$. Suppose $\operatorname{pos} C_{M_i}(\alpha)$ is strongly degenerate for all i. We must show that $\operatorname{pos} C_M(\alpha)$ is strongly degenerate.

For each i, there must be a nonzero $x^i \in R^n$ such that $x^i \geq 0$ and $C_{M_i}(\alpha)x^i = 0$. By scaling, we may assume that $\|x^i\| = 1$ for all i. Since S^{n-1} is compact, we may assume there is an $x \in R^n$ such that $\lim_{i\to\infty} x^i = x$. Clearly, $x \geq 0$ and $C_M(\alpha)x = 0$. Also, $\|x\| = 1$ and, hence, $\operatorname{pos} C_M(\alpha)$ is strongly degenerate. □

We have been concentrating a bit on the complementary cones, the image of the map f_M. We are, of course, interested in what happens in the domain of f_M, as it is the points in the domain which correspond to solutions of the LCP. Corollary **6.1.10** indicates that for each full complementary cone containing q, there is exactly one corresponding solution to the LCP (q, M). What can be said if the complementary cone were degenerate? The following results answer this question. Notice that we came close to the answer, for the case of a strongly degenerate cone, in the proof of **6.1.23**.

6.1.26 Lemma. Let $A \in R^{n\times p}$ and $q \in R^n$ be given. If $q \in \operatorname{ri}(\operatorname{pos} A)$, then there exists a $u \in R^p$, with $u > 0$, such that $q = Au$.

Proof. As $q \in \operatorname{pos} A$, there exists an $x \geq 0$ such that $q = Ax$. For some $\alpha \subseteq \{1, \ldots, p\}$, we have $x_\alpha = 0$ and $x_{\bar{\alpha}} > 0$.

If $\alpha = \emptyset$, we are done. Otherwise, select $y \in R^p$ with $y_{\bar{\alpha}} = x_{\bar{\alpha}}$ and with $y_i = -\delta$ for some $\delta > 0$ and all $i \in \alpha$. From continuity and the fact that $Ax = q \in \operatorname{ri}(\operatorname{pos} A)$, we will have $Ay \in \operatorname{pos} A$ if we select $\delta > 0$ small enough. Thus, $Ay = Az$ for some $z \geq 0$. Clearly, $A(z - y) = 0$ and $(z - y)_\alpha > 0$. Thus, for $\lambda > 0$ small enough, we have $x + \lambda(z - y) > 0$ and $A(x + \lambda(z - y)) = Ax = q$. □

6.1.27 Theorem. Suppose we are given $M \in R^{n\times n}$, $q \in R^n$, and an index set α, such that $q \in \operatorname{pos} C_M(\alpha)$. Consider the set $\mathcal{S} = f_M^{-1}(q) \cap \operatorname{pos} C_I(\alpha)$. The set $\mathcal{S}$ is a nonempty polyhedron. If $q \in \operatorname{ri}(\operatorname{pos} C_M(\alpha))$ and if $\operatorname{pos} C_M(\alpha)$ is degenerate, then $\mathcal{S}$ has infinitely many points. The set $\mathcal{S}$ is unbounded if and only if $\operatorname{pos} C_M(\alpha)$ is strongly degenerate.

Proof. Rather than deal with $\mathcal{S}$, we will deal with $\mathcal{S}^+ = \{C_I(\alpha)x : x \in \mathcal{S}\}$. Clearly, $z \in \mathcal{S}^+$ if and only if $z \geq 0$ and $C_M(\alpha)z = q$. If the above statements are true for $\mathcal{S}^+$, then they are true for $\mathcal{S}$.

Since $q \in \operatorname{pos} C_M(\alpha)$, there is some $y \geq 0$ such that $C_M(\alpha)y = q$, i.e., $\mathcal{S}^+$ is nonempty. We see that $C_M(\alpha)(y + x) = q$ if and only if x is in the nullspace of $C_M(\alpha)$. Thus, $\mathcal{S}^+$ is the intersection of the nonnegative orthant and an affine space, hence it is a polyhedron.

If $\operatorname{pos} C_M(\alpha)$ is full, then we already know that $\mathcal{S}^+$ is a single point.

If $\operatorname{pos} C_M(\alpha)$ is strongly degenerate, then there is an $x \neq 0$ such that $x \geq 0$ and $C_M(\alpha)x = 0$. Thus, $y + \lambda x \geq 0$ and $C_M(\alpha)(y + \lambda x) = q$, for all $\lambda \geq 0$. Hence, $\mathcal{S}^+$ is unbounded.

Now, suppose $\mathcal{S}^+$ is unbounded. There is then an unbounded sequence of (nonzero) points, $\{x^i\}$, in the nullspace of $C_M(\alpha)$ such that $y + x^i$ is in $\mathcal{S}^+$ for all i. As S^{n-1} is compact, we may assume there is some $x \neq 0$ such that $\lim_{i\to\infty} x^i/\|x^i\| = x$. Clearly, we must have

$$\lim_{i\to\infty} \frac{y + x^i}{\|x^i\|} = x,$$

which implies, in view of the facts $\{y + x^i\} \subseteq \mathcal{S}^+$ and $\|x^i\| \to \infty$, that x is nonnegative and satisfies $C_M(\alpha)x = 0$. So, $\operatorname{pos} C_M(\alpha)$ is strongly degenerate.

Finally, suppose $\operatorname{pos} C_M(\alpha)$ is degenerate and that $q \in \operatorname{ri}(\operatorname{pos} C_M(\alpha))$. By Lemma **6.1.26**, there is an $x > 0$ such that $q = C_M(\alpha)x$. In addition, degeneracy implies the existence of a $y \neq 0$ with $C_M(\alpha)y = 0$. Hence, for all $\lambda > 0$ small enough we have $x + \lambda y \in \mathcal{S}^+$. Thus, $\mathcal{S}^+$ has infinitely many points. □

Before ending this section we will illustrate some of the material presented by examining the matrix

$$M = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

We wish to show that M is a $\boldsymbol{Q}$-matrix. Unfortunately, by letting $z = (0, 0, 1)$ and $w = (1, 1, 0)$, we see that the LCP $(0, M)$ has a nontrivial solution. Thus, $M \notin \boldsymbol{R_0} \supseteq \boldsymbol{R}$. This means that $\deg M$ is not defined. It also means that the material in Chapter 3 cannot be (directly) used to show $M \in \boldsymbol{Q}$. We will have to look at the structure of $K(M)$ a bit more carefully.

Let $\mathcal{C}$ be the union of the strongly degenerate complementary cones of M. If in each connected component of $R^3 \setminus \mathcal{C}$ we could find a point with a well-defined and nonzero local degree, then Theorem **6.1.17** would imply that every point with a well-defined local degree would have a nonzero local degree. It would then follow from Theorem **6.1.8** that the LCP (q, M) had a solution for every $q \notin \mathcal{K}(M)$. As $K(M)$ is closed, Theorem **6.1.12** would imply that $K(M) = R^3$, i.e., $M \in \boldsymbol{Q}$. Thus, we will now find $\mathcal{C}$ and show that each connected component of $R^3 \setminus \mathcal{C}$ contains a point with a well-defined and nonzero local degree.

The complementary cone $\operatorname{pos} C(\{3\})$ is strongly degenerate. However, all the other complementary cones are full. Therefore, $\mathcal{C} = \operatorname{pos} C(\{3\})$ and this is the hyperplane $\{x \in R^3 : x_3 = 0\}$. It follows that $R^3 \setminus \mathcal{C}$ has the sets $H^+ = \{x \in R^3 : x_3 > 0\}$ and $H^- = \{x \in R^3 : x_3 < 0\}$ as its two connected components.

Consider the point $q^1 = (1,1,1)$. It is easy to show that $f_M^{-1}(q^1)$ consists of the following three points: $(1,1,1)$, $(3,-1,2)$, and $(-1,3,2)$. From this we find that $\deg_M(q^1)$ exists and equals -1.

Consider the point $q^2 = (0,0,-1)$. Again, it is easy to show that $f_M^{-1}(q^2)$ consists of the following two points: $(3,-1,-1)$ and $(-1,3,-1)$. From this we find that $\deg_M(q^2)$ exists and equals -2.

Since $q^1 \in H^+$ and $q^2 \in H^-$, we conclude that M is a $\boldsymbol{Q}$-matrix.

We will now perturb M and see what happens. Let $M(\delta)$ be equal to the matrix M except that $m(\delta)_{33} = \delta$. Notice that $M = M(0)$. If $0 < |\delta| < 1$, then $M(\delta)$ is nondegenerate. Theorem **6.1.14** implies that $M(\delta)$ would then have a well-defined degree, but what would it be?

Consider q^1 and q^2 as given above. Suppose $q^i \in \operatorname{int}(\operatorname{pos} C_M(\alpha))$ for some i and some α. It is easy to show that if $|\delta| > 0$ is small enough, then $q^i \in \operatorname{int}(\operatorname{pos} C_{M(\delta)}(\alpha))$ and $\operatorname{sgn}(\det M_{\alpha\alpha}) = \operatorname{sgn}(\det M(\delta)_{\alpha\alpha})$. Furthermore, assuming $|\delta| > 0$ is small enough, if $\operatorname{pos} C_M(\alpha)$ is not strongly degenerate and if $q^i \notin \operatorname{pos} C_M(\alpha)$, then by **6.1.23** we have $q^i \notin \operatorname{pos} C_{M(\delta)}(\alpha)$. Thus, the only possible difference between $\deg_M(q^i)$ and $\deg_{M(\delta)}(q^i)$ is the index of $\operatorname{pos} C_{M(\delta)}(\{3\})$. It may now happen that this index appears in the degree calculation for q^i.

Suppose δ is small and positive. We find that $q^1 \notin \operatorname{pos} C_{M(\delta)}(\{3\})$. Thus, $\deg_{M(\delta)}(q^1) = \deg_M(q^1) = -1$, and hence, $\deg M(\delta) = -1$. Yet, $\deg_M(q^2) = -2$, so what happens now? As can be checked, $\operatorname{pos} C_{M(\delta)}(\{3\})$ contains q^2. We must therefore add $\operatorname{sgn}(m(\delta)_{33}) = \operatorname{sgn}(\delta) = +1$ to the local degree of q^2. Therefore, as should be the case, $\deg_{M(\delta)}(q^2) = -1$. Geometrically, we can imagine letting δ equal an infinitesimally small positive quantity. We find that the points in H^+ are not contained in $\operatorname{pos} C_{M(\delta)}(\{3\})$ and, so, they retain their local degrees of -1. However, $\operatorname{pos} C_{M(\delta)}(\{3\})$ equals H^-, if we think of $\delta > 0$ as being infinitesimally small, so the points in H^- have their local degrees increased (by $+1$) to the value -1.

Similarly, if we imagine δ as being negative and $|\delta|$ as being infinitesimally small, then we find it is the points in H^- which are not contained in $\operatorname{pos} C_{M(\delta)}(\{3\})$. Thus, these points retain their local degrees of -2. Further, we find $\operatorname{pos} C_{M(\delta)}(\{3\})$ equal to H^+. Thus, the points in H^+ have their local degrees increased by $\operatorname{sgn}(m(\delta)_{33}) = -1$ to the value -2. Indeed, for $\delta < 0$ with $|\delta|$ small enough, we can easily check that $\deg M(\delta) = -2$ using either q^1 or q^2.

6.2 Facets

In the previous section we turned our attention to degree theory and complementary cones. In this section our chief objects of study will be the facets of the complementary cones. Why do we make such a sudden change in emphasis? The answer is that our emphasis is changing only superficially. In the previous section, the union of the facets, $\mathcal{K}(M)$, was shown to have an important role in degree theory. We will study this role more deeply in this section. In fact, we will develop the basic properties of degree using just the facets. In this sense one might reverse the previous question and ask why we are repeating ourselves. The answer to this question is that in the previous section the basic concept of degree gave us a global view while in this section the geometry of the facets will give us a local view. The two views together offer more insight into the LCP than either one separately.

From **2.9.14** and **2.9.15**(a), we know that for $M \in R^{n\times n}$ the dimension of any facet of any complementary cone relative to M is no larger than $n-1$. However, only those facets with dimension equal to $n-1$ will be of interest. This is an important point and, shortly, we will explain why.

6.2.1 Notation. For $q \in R^n$ and $\delta > 0$, let $B(q, \delta)$ denote the open ball of radius δ around q. That is, $B(q, \delta) = \{\, q' \in R^n : \|q - q'\|_2 < \delta \,\}$. (See Definition **2.1.7**.)

If H is a hyperplane containing q, then the connected components of $B(q, \delta) \setminus H$ are a pair of *open hemiballs.*

6.2.2 Definition. We recall from **2.1.15** that a path is a continuous function $q(t) : [\,0, 1\,] \to R^n$. We will often use q^t to denote either the path itself or the point $q(t)$ for some $t \in [\,0, 1\,]$. The meaning of q^t will be clear from context. A path q^t is said to be *between* q^0 and q^1.

Given two paths q^t and p^t where $q^1 = p^0$, we say that the path r^t is the *composition* of the paths q^t and p^t if $r^s = q^{2s}$ for $s \in [\,0, \frac{1}{2}\,]$ and $r^s = p^{2s-1}$ for $s \in [\,\frac{1}{2}, 1\,]$.

Let $x, y \in R^n$ be given. Consider the path q^t defined by $q^t = (1-t)x + ty$. The image of this path is the line segment $\ell\,[\,x, y\,]$. We will refer to q^t as the *path* of $\ell\,[\,x, y\,]$.

The above is somewhat general in that it could easily appear in a book not devoted to the LCP. The following, in contrast, is very specific to the geometric view of the LCP.

6.2.3 Definition. Let $M \in R^{n \times n}$ be given. If $n > 2$, consider those linear subspaces of R^n which satisfy at least one of the following two conditions:

(a) The subspace equals $\{\, C_M(\alpha)_{\cdot\beta} x \,:\, x \in R^{n-2} \,\}$ for some index sets $\alpha, \beta \subseteq \{1, \ldots, n\}$ with $|\beta| = n - 2$;

(b) The subspace equals the intersection of two geometrically distinct subspaces each of the form $\{\, C_M(\alpha)_{\cdot\beta} x \,:\, x \in R^{n-1} \,\}$ where α and β are index sets in $\{1, \ldots, n\}$ and $|\beta| = n - 1$.

We define $\mathcal{L}(M)$ to be the intersection of $\mathcal{K}(M)$ with the union of these linear subspaces. If $n = 1$, we let $\mathcal{L}(M) = \emptyset$ for all M. If $n = 2$, we let $\mathcal{L}(M) = \{0\}$ for all M.

The description of $\mathcal{L}(M)$ is, admittedly, hard to take in at once. Therefore, we make a few basic observations. The subspaces satisfying **6.2.3(a)** are the affine hulls of the $n-2$ (or less) dimensional faces of the complementary cones. The additional subspaces obtained from **6.2.3(b)** are gotten by first taking the affine hulls of all the facets of all the complementary cones. These affine hulls are then intersected and any intersection which has dimension $n - 2$ or less is used for $\mathcal{L}(M)$. Thus, $\mathcal{L}(M)$ is contained in a finite union of $n-2$ (or less) dimensional subspaces. In fact, if $n > 2$, then $\mathcal{L}(M)$ is never empty as it will always contain the nonnegative parts of the coordinate axes.

It is now time to discuss why we have turned our attention to the set $\mathcal{L}(M)$. This will also explain why we will be interested only in those facets which have dimension $n-1$. We will begin our discussion by looking once again at the set $R^n \setminus \mathcal{K}(M)$.

The set $R^n \setminus \mathcal{K}(M)$ consists of those points which have a well-defined local degree. From Theorem **6.1.12** we know that almost all of the points in R^n are in this set. In fact, Theorem **6.1.17** implies that local degree is an invariant within the connected components of $R^n \setminus \mathcal{K}(M)$. Thus, if we imagine moving along a path $q^t : [0,1] \to R^n$, while keeping track of $f_M^{-1}(q^t)$ and $\deg_M(q^t)$, then the "really interesting" things happen only when we move from one component of $R^n \setminus \mathcal{K}(M)$ to another. To do this, the path must intersect $\mathcal{K}(M)$, i.e., the path must cross through a facet of a complementary cone.

If we wish to analyze the changes that take place in f_M^{-1} and in the local degree as we move between different components of $R^n \setminus \mathcal{K}(M)$, then our job will be easier if, when q^t crosses through a facet, it does so in a simple way. The question arises as to what "simple" means in this context. Part of the answer to this question is given in the next result.

6.2.4 Theorem. Let $M \in R^{n\times n}$ be given. If $q \in \mathcal{K}(M) \setminus \mathcal{L}(M)$, then there is an $(n-1)$-dimensional subspace H and a $\delta > 0$ such that if F is any facet of any complementary cone relative to M, and $B(q,\delta) \cap F \neq \emptyset$, then

$$B(q,\delta) \cap F \;=\; B(q,\delta) \cap H \;=\; B(q,\delta) \cap \mathcal{K}(M)\,. \qquad (1)$$

Further, H is the affine hull of F, $\dim F = n-1$, and $q \in \operatorname{ri} F$.

Proof. If $n = 1$, then $q = 0$ and the theorem is trivial. If $n = 2$, then q is a multiple of some column in $(I, -M)$ and $q \neq 0$. It is easy to see that the theorem is true if we let H equal the subspace of all multiples of that column. Thus, we assume that $n > 2$. (In the future, for results concerning $\mathcal{L}(M)$, we assume the reader will verify the case when $n \leq 2$.)

Since $q \in \mathcal{K}(M)$, then $q \in F$ where F is the facet of some complementary cone. As $q \notin \mathcal{L}(M)$, it follows from condition (a) of **6.2.3** that

$\dim F = n-1$ and that $q \in \operatorname{ri} F$. Let H be the $(n-1)$-dimensional subspace which is the affine hull of F. As $q \in \operatorname{ri} F$, for all small enough $\delta > 0$ we have $B(q,\delta) \cap F = B(q,\delta) \cap H$. This implies $B(q,\delta) \cap \mathcal{K}(M) \supseteq B(q,\delta) \cap H$.

Since there are finitely many facets of complementary cones, and they are all closed sets, there exists a $\delta > 0$ such that any facet which intersects $B(q,\delta)$ must contain q. If it were true that all facets which contained q were contained in H, then we could conclude from the above argument that the theorem is true for $\delta > 0$ small enough. Thus, suppose $q \in F'$ where F' is the facet of some complementary cone and F' is not contained in H. It must be that the affine hulls of F and F' are geometrically distinct. Using condition (b) of **6.2.3**, we have $q \in F \cap F' \subseteq \mathcal{L}(M)$. This is a contradiction and, hence, the theorem is valid. □

The previous theorem indicates that $\mathcal{L}(M)$ consists of points which are, in some sense, degenerate. It would probably be best if the path q^t did not contain any of the points in $\mathcal{L}(M)$. Starting with this, we will now fully answer the question of what it means for the path q^t to intersect $\mathcal{K}(M)$ in a simple (nondegenerate) manner.

6.2.5 Definition. Let $M \in R^{n \times n}$ be given. Let $q^t : [0,1] \to R^n$ be a path. Suppose for some $s \in (0,1)$ that $q^s \in \mathcal{K}(M)$. We then say that the intersection of $\mathcal{K}(M)$ and the path q^t at the point q^s is *nondegenerate* if $q^s \notin \mathcal{L}(M)$ and if there exists an $(n-1)$-dimensional subspace H and a $\delta > 0$ such that the following hold.

(a) With $q = q^s$, the conclusion of Theorem **6.2.4** is valid.

(b) For all $t \in (s-\delta, s+\delta)$, if $q^t \in H$, then $t = s$.

(c) The two open hemiballs which are the connected component s of $B(q^s,\delta) \setminus H$ each contain points q^t with t arbitrarily close to s.

6.2.6 Remark. Notice that we ignore the endpoints of a path when considering its intersections with $\mathcal{K}(M)$.

We can describe what happens in the vicinity of the nondegenerate intersection q^s using the conditions given in Definition **6.2.5**. Within $B(q^s,\delta)$ the path q^t starts on one side of H and crosses to the other side of H precisely when $t = s$. Any facet containing the point q^s is $(n-1)$-dimensional, is contained in H, and contains q^s in its relative interior.

Now that we know what it means for a path to intersect $\mathcal{K}(M)$ in a simple way, the question arises as to whether such intersections are the exception or the rule. More precisely, can any two points in R^n be connected with a path all of whose intersections with $\mathcal{K}(M)$ are nondegenerate? The answer to this question is that such a path always exists. We shall show this in a moment. First we shall prove the following result which will be needed as a lemma and which, on its own, is quite useful.

6.2.7 Theorem. Let $M \in R^{n \times n}$ and $q, q' \in R^n$ be given. For any $\varepsilon > 0$, there is a $q'' \in B(q', \varepsilon) \setminus (\mathcal{K}(M) \cup \{q\})$ such that all the intersections of $\mathcal{K}(M)$ with the path of $\ell\,[\,q, q''\,]$ are nondegenerate.

Proof. The set $\mathcal{L}(M)$ is contained in a finite union of subspaces of dimension $n-2$ or less. Let S be one of these subspaces. The affine hull of the subspace S and the point q is a subspace of dimension $n-1$ or less. Let $\mathcal{S}$ be the union of these affine hulls.

We will now expand $\mathcal{S}$ by adding to it the affine hulls of all the facets of the complementary cones. It is still true that $\mathcal{S}$ is a finite union of subspaces of dimension $n-1$ or less. It then follows from **2.9.17**, that there is a point $q'' \neq q$ such that $q'' \in B(q', \varepsilon) \setminus \mathcal{S}$.

It is clear from its construction that $\mathcal{S}$ contains $\mathcal{K}(M)$. Therefore, $q'' \notin \mathcal{K}(M)$. Let q^t be the path of $\ell\,[\,q, q''\,]$ with, say, $q^0 = q$ and $q^1 = q''$. Note, as $q'' \neq q$, if $s \neq t$, then $q^s \neq q^t$.

Suppose for some $s \in (0, 1)$ that $q^s \in \mathcal{K}(M)$. If $q^s \in \mathcal{L}(M)$, then $\mathcal{S}$ would contain the unique line through q^s and q. This would imply that $q'' \in \mathcal{S}$ which is false. Thus, $q^s \notin \mathcal{L}(M)$. We may now conclude that there is an $(n-1)$-dimensional subspace H and a $\delta > 0$ such that the conclusion of Theorem **6.2.4** holds for q^s.

We know from **6.2.4** that H is the affine hull of some $(n-1)$-dimensional facet of a complementary cone, thus $H \subseteq \mathcal{S}$. If $\ell\,[\,q^0, q^1\,] \subseteq H$, then $q'' \in \mathcal{S}$ which is false. Thus, the line segment $\ell\,[\,q^0, q^1\,]$ transversely intersects H at the point q^s. We may conclude that conditions (b) and (c) of Definition **6.2.5** are satisfied. The theorem follows. □

We now show that any two points can be joined by a path which has only nondegenerate intersections with $\mathcal{K}(M)$.

6.2.8 Theorem. Let $M \in R^{n \times n}$ be given. For any $q, q' \in R^n$, there exists a path between q and q' such that all the intersections of $\mathcal{K}(M)$ with the path are nondegenerate.

Proof. Let $\mathcal{S}$ be the union of the affine hulls of all the facets of the complementary cones. Clearly, $\mathcal{K}(M) \subseteq \mathcal{S}$. Since there are only finitely many facets, there is a $\delta > 0$ such that if the affine hull of a facet intersects the open ball $B(q, \delta)$, then that affine hull contains q. The set $R^n \setminus \mathcal{S}$ is open and Proposition **2.9.17** implies it is also dense. Thus, there is a $q'' \in B(q, \delta)$ and an $\varepsilon > 0$ such that $B(q'', \varepsilon) \subseteq B(q, \delta)$ and $B(q'', \varepsilon) \cap \mathcal{S} = \emptyset$.

Suppose, for some point $\bar{q} \in B(q'', \varepsilon)$, that $\ell\,[\,q, \bar{q}\,] \cap \mathcal{S}$ contains a point which is not q. It must be that this point is in one of the subspaces comprising $\mathcal{S}$ and, as $\ell\,[\,q, \bar{q}\,] \in B(q, \delta)$, this subspace must contain q. Therefore, $\ell\,[\,q, \bar{q}\,] \subseteq \mathcal{S}$ which is false as $\bar{q} \notin \mathcal{S}$. We conclude that $\ell\,[\,q, \bar{q}\,] \cap \mathcal{S} \subseteq \{q\}$.

From Theorem **6.2.7**, we know that there is some $\bar{q} \in B(q'', \varepsilon)$ such that all the intersections of $\mathcal{K}(M)$ with the path of $\ell\,[\,\bar{q}, q'\,]$ are nondegenerate. As $\ell\,[\,q, \bar{q}\,]$ has no intersections with $\mathcal{K}(M)$, it is vacuously true that all such intersections are nondegenerate. Since $\bar{q} \notin \mathcal{K}(M)$, the composition of the paths of $\ell\,[\,q, \bar{q}\,]$ and $\ell\,[\,\bar{q}, q'\,]$ is the desired path between q and q'. □

Since we may "move" from any point in R^n to any other point via a path which intersects $\mathcal{K}(M)$ only nondegenerately, then we need only study what happens when a path crosses through the relative interior of an $(n-1)$-dimensional facet of a complementary cone. Thus, as stated at the beginning of this section, we will only be interested in such facets.

6.2.9 Definition. Given $M \in R^{n \times n}$, $\alpha \subseteq \{1, \ldots, n\}$, and $i \in \{1, \ldots, n\}$, we say that the complementary cones $\operatorname{pos} C_M(\alpha)$ and $\operatorname{pos} C_M(\alpha \triangle \{i\})$ are *adjacent*. In addition, we refer to the facet $\operatorname{pos} C_M(\alpha)_{\cdot \bar{\imath}}$ as the *common* facet between these two cones, and we consider it to be *adjacent* to both cones. Notice, $C_M(\alpha)_{\cdot \bar{\imath}} = C_M(\alpha \triangle \{i\})_{\cdot \bar{\imath}}$.

The geometry surrounding the common facet between two complementary cones depends to a great extent on the geometry of the two cones. We will first assume that both complementary cones are full. This will be relaxed later. With this assumption, it turns out there are only two cases to consider.

6.2.10 Definition. Given $M \in R^{n \times n}$, $\alpha \subseteq \{1, \ldots, n\}$, and $i \in \{1, \ldots, n\}$, consider the product

$$(\det M_{\alpha\alpha})(\det M_{\beta\beta}), \tag{2}$$

where $\beta = \alpha \triangle \{i\}$. We say that the common facet $\operatorname{pos} C_M(\alpha)_{\bullet\bar{i}}$ is *proper* if (2) is positive. We say that the facet is *reflecting* if (2) is negative.

6.2.11 Remark. Notice, as $\det C_M(\alpha) = (-1)^{|\alpha|} \det M_{\alpha\alpha}$, that (2) is equal to

$$-\big(\det C_M(\alpha)\big)\big(\det C_M(\beta)\big). \tag{3}$$

Hence, if (2) is nonzero, then both $\operatorname{pos} C_M(\alpha)$ and $\operatorname{pos} C_M(\beta)$ are full complementary cones and, so, the facet $\operatorname{pos} C_M(\alpha)_{\bullet\bar{i}}$ has dimension $n-1$.

Let $M \in R^{n \times n}$, $\alpha \subseteq \{1, \ldots, n\}$, and $i \in \{1, \ldots, n\}$ be given. Define $\beta = \alpha \triangle \{i\}$. Suppose that both $\operatorname{pos} C_M(\alpha)$ and $\operatorname{pos} C_M(\beta)$ are full complementary cones. Consider a path q^t in which, for some $s \in (0,1)$, the point $q^s \in \operatorname{pos} C_M(\alpha)_{\bullet\bar{i}}$ is a nondegenerate intersection of the path with $\mathcal{K}(M)$. Let H and δ be as described in Definition **6.2.5**. Let B^+ and B^- be the two open hemiballs which are the connected components of $B(q^s, \delta) \setminus H$. We now have the tools to describe what happens as the path q^t crosses the facet $\operatorname{pos} C_M(\alpha)_{\bullet\bar{i}}$ at q^s.

6.2.12 Theorem. Assume the conditions and notations in the preceding paragraph. It follows that each of $\operatorname{pos} C_M(\alpha)$ and $\operatorname{pos} C_M(\beta)$ contain exactly one of B^+ and B^-. Further, each of the two cones is disjoint from the hemiball it does not contain. If the facet $\operatorname{pos} C_M(\alpha)_{\bullet\bar{i}}$ is reflecting, then both complementary cones contain the same hemiball. If the facet $\operatorname{pos} C_M(\alpha)_{\bullet\bar{i}}$ is proper, then the complementary cones contain different hemiballs. In either case, the total contribution to the local degree made by the two complementary cones is constant for all points in $B^+ \cup B^-$.

Proof. We know from (1) that the intersection of the facet $\operatorname{pos} C(\alpha)_{\bullet\bar{i}}$ with $B(q^s, \delta)$ equals $B(q^s, \delta) \cap H$. This facet is part of the boundary of the full cone $\operatorname{pos} C(\alpha)$ and, again via (1), we know that no other part of the boundary of $\operatorname{pos} C(\alpha)$ intersects $B(q^s, \delta)$. Thus, $\operatorname{pos} C(\alpha)$ contains exactly one of B^+ and B^-. Further, $\operatorname{pos} C(\alpha)$ is disjoint from the hemiball it does not contain. A similar argument applies to $\operatorname{pos} C(\beta)$.

The columns of $C(\alpha)_{\bullet\bar{\imath}}$ form a basis for the hyperplane H. Thus, $\operatorname{pos} C(\alpha)$ is entirely on one side of H, i.e., $\operatorname{pos} C(\alpha)$ is entirely contained in one of the two closed halfspaces for which H is the common boundary. The same is true for $\operatorname{pos} C(\beta)$. From (3), we see that if $\operatorname{pos} C(\alpha)_{\bullet\bar{\imath}}$ is reflecting, then the two complementary cones are on the same side of H and, if $\operatorname{pos} C(\alpha)_{\bullet\bar{\imath}}$ is proper, then the two complementary cones are on opposite sides of H. From this, all but the last sentence of the theorem follows.

Since $B^+ \cup B^-$ does not intersect $\mathcal{K}(M)$, all points in this union have a well-defined local degree. If $\operatorname{pos} C(\alpha)_{\bullet\bar{\imath}}$ is proper, then (2) implies that the complementary cones have the same index. Since one contains B^+ and the other contains B^-, the total contribution they make to the local degree is the same over $B^+ \cup B^-$. Similarly, if $\operatorname{pos} C(\alpha)_{\bullet\bar{\imath}}$ is reflecting, then (2) implies that the complementary cones have opposite indexes. Since they both contain one of B^+ and B^-, the total contribution they make to the local degree is the same over $B^+ \cup B^-$. □

6.2.13 Remark. The invariance of the local degree (Theorem **6.1.14**), for the case when M is nondegenerate, now follows as a corollary to Theorems **6.2.8** and **6.2.12**.

So far we have explored only the nondegenerate case. It is time to begin introducing degeneracy into the picture. In the previous theorem we described what took place as a path crossed the common facet between two full cones. We now describe what would have taken place if one or both of those cones had been degenerate.

6.2.14 Theorem. Assume the conditions and notations in the paragraph preceding Theorem **6.2.12**, except that we will now assume $\operatorname{pos} C_M(\alpha)$ is a degenerate complementary cone. It follows that $B(q^s, \delta) \cap \operatorname{pos} C_M(\alpha) = B(q^s, \delta) \cap H$.

If we assume $\operatorname{pos} C_M(\beta)$, like $\operatorname{pos} C_M(\alpha)$, is degenerate, it follows that $B(q^s, \delta) \cap \operatorname{pos} C_M(\beta) = B(q^s, \delta) \cap H$. In addition the total contribution to the local degree, made by $\operatorname{pos} C_M(\alpha)$ and $\operatorname{pos} C_M(\beta)$, is constant for all points in $B^+ \cup B^-$.

If we assume $\operatorname{pos} C_M(\beta)$ is a full cone, then $\operatorname{pos} C_M(\beta)$ contains exactly one of B^+ and B^- and, further, is disjoint from the one it does not contain. In addition, the total contribution to the local degree made by $\operatorname{pos} C_M(\alpha)$

and $\operatorname{pos} C_M(\beta)$ is constant for all points in B^+ and for all points in B^-, but differs by one between the two hemiballs.

Proof. We know from (1) that

$$B(q^s, \delta) \cap H \;=\; B(q^s, \delta) \cap \operatorname{pos} C(\alpha)_{\bullet\bar{\imath}} \;\subseteq\; B(q^s, \delta) \cap \operatorname{pos} C(\alpha)\,.$$

Theorem **6.2.4** goes on to say that $\dim\left(\operatorname{pos} C(\alpha)_{\bullet\bar{\imath}}\right) = n - 1$ and that H is the affine hull of this facet. Therefore, if the vector $C(\alpha)_{\bullet i}$ does not lie in H, then $\det C(\alpha) \neq 0$ and $\operatorname{pos} C(\alpha)$ is nondegenerate. Thus, $C(\alpha)_{\bullet i}$ lies in H and, hence, $\operatorname{pos} C(\alpha) \subseteq H$. We may now conclude that $B(q^s, \delta) \cap \operatorname{pos} C_M(\alpha) = B(q^s, \delta) \cap H$.

If $\operatorname{pos} C(\beta)$ is degenerate, then $B(q^s, \delta) \cap \operatorname{pos} C_M(\beta) = B(q^s, \delta) \cap H$ via the above reasoning.

If $\operatorname{pos} C(\beta)$ is full, then the argument given at the beginning of the proof of **6.2.12** shows that $\operatorname{pos} C(\beta)$ contains exactly one of B^+ and B^- and is disjoint from the other.

Lastly, if both $\operatorname{pos} C(\alpha)$ and $\operatorname{pos} C(\beta)$ are degenerate, then neither one intersects $B^+ \cup B^-$ and, so, the total contribution of these cones to the local degree of points in the union is zero. If $\operatorname{pos} C(\beta)$ is full, then it will contribute a $+1$ or a -1 to the local degree of points in either B^+ or B^-, whichever one it contains. It will contribute nothing to the local degree in the hemiball it does not contain. As before, $\operatorname{pos} C(\alpha)$ will contribute nothing to the local degree of points in either hemiball. The last assertion of the theorem now follows. □

The previous theorem is a good start, but we wish to take a closer look at what happens when a path crosses the facet of a degenerate cone. As might be expected, there is a difference between what happens when the path crosses a weakly degenerate cone and what happens when it crosses a strongly degenerate cone. As might be surprising, the case of strong degeneracy is much simpler to analyze. For this reason, we will consider it first. It turns out that strongly degenerate complementary cones share a key property with full complementary cones which weakly degenerate cones do not have. This property is given in the following lemma.

6.2.15 Lemma. Let $M \in R^{n \times n}$, $q \in R^n$, and $\alpha \subseteq \{1, \ldots, n\}$ be given. Suppose that $\operatorname{pos} C_M(\alpha)$ is a strongly degenerate complementary cone and that $q \in \operatorname{pos} C_M(\alpha) \setminus \mathcal{L}(M)$. It follows that q is contained in exactly one facet of $\operatorname{pos} C_M(\alpha)$.

Proof. Since $\operatorname{pos} C(\alpha)$ is degenerate, the rank of $C(\alpha)$ is no bigger than $n-1$. If $C(\alpha)$ had a rank of $n-2$ or less, then the affine hull of $\operatorname{pos} C(\alpha)$ could be generated by $n-2$ or fewer column vectors of $C(\alpha)$. Condition (a) of **6.2.3** would then imply that $\operatorname{pos} C(\alpha) \subseteq \mathcal{L}(M)$. Thus, for q to exist, the rank of $C(\alpha)$ must equal $n-1$.

Since $q \in \operatorname{pos} C(\alpha)$, there is an $x \geq 0$ such that $q = C(\alpha)x$. As $C(\alpha)$ has rank $n-1$, the null space of $C(\alpha)$ has dimension equal to one. Let $y \in R^n$ be a basis for this null space. Let $L \equiv \{x + \lambda y : \lambda \in R\}$. Thus, L is a line in R^n, i.e., a 1-dimensional affine space. Further, $L = \{z \in R^n : q = C(\alpha)z\}$.

For $i \in \{1, \ldots, n\}$, it is easy to see that $q \in \operatorname{pos} C(\alpha)_{\bullet \bar{i}}$ if and only if L intersects the facet $\operatorname{pos} I_{\bullet \bar{i}}$ of R^n_+. As $q \notin \mathcal{L}(M)$, condition (a) of **6.2.3** implies that L can only intersect a facet of R^n_+ in the facet's relative interior, i.e., each point in L is contained in at most one facet of R^n_+.

If the facet $\operatorname{pos} I_{\bullet \bar{i}}$ contained two distinct points in L, then L is contained in the affine hull of $\operatorname{pos} I_{\bullet \bar{i}}$. Since $\operatorname{pos} I_{\bullet \bar{i}}$ contains no lines, L must intersect the relative boundary of $\operatorname{pos} I_{\bullet \bar{i}}$. This violates our previous conclusion that L can intersect a facet of R^n_+ only in the facet's relative interior. Hence, each facet of R^n_+ contains at most one point of L.

As L and R^n_+ are convex, and as $q \in \operatorname{pos} C(\alpha)$, then $L \cap R^n_+$ is nonempty and convex. We know L can intersect a facet of R^n_+ only in the facet's relative interior, thus $L \cap R^n_+$ is not a single point. Since R^n_+ contains no lines, we conclude that $L \cap R^n_+$ is either an infinite ray with a single endpoint or a finite line segment with two distinct endpoints.

As $\operatorname{pos} C(\alpha)$ is strongly degenerate, then either $y \geq 0$ or $y \leq 0$. In either case, $L \cap R^n_+$ will be an infinite ray which intersects, at its single endpoint, exactly one facet of R^n_+. The lemma now follows. □

6.2.16 Remark. If $\operatorname{pos} C_M(\alpha)$ had been full instead of strongly degenerate, it would still be true that if $q \in \operatorname{pos} C_M(\alpha)_{\bullet \bar{i}} \setminus \mathcal{L}(M)$, then q is contained in exactly one facet of $\operatorname{pos} C_M(\alpha)$. (It follows as $C_M(\alpha)^{-1}q$ is well-defined and would be zero in exactly one component.) This is the key property that full and strongly degenerate complementary cones have in common: points in a facet, but not in $\mathcal{L}(M)$, must be in a unique facet.

In the case where there are no weakly degenerate complementary cones relative to M, Theorems **6.2.12** and **6.2.14** completely describe nondegenerate intersections of $\mathcal{K}(M)$ with a path. For this case, suppose q^t is a path crossing $\mathcal{K}(M)$ at the point q^s, and the intersection is nondegenerate. As always, each facet containing q^s is adjacent to exactly two complementary cones containing q^s. From the key property (Remark **6.2.16**), each complementary cone containing q^s has at most one facet containing q^s. Thus, the situation at q^s splits into several distinct cases (one for each facet), with each case covered by either Theorem **6.2.12** or Theorem **6.2.14**. One can deduce that if q^t has only nondegenerate intersections with $\mathcal{K}(M)$, and if $\deg(q^0)$ and $\deg(q^1)$ are well-defined, then $\deg(q^0)$ and $\deg(q^1)$ have the same parity if and only if the path q^t intersects an even number of strongly degenerate complementary cones. In fact, later in the section, we will prove this without the assumption that there are no weakly degenerate complementary cones relative to M. First, we must tackle the final case in which a path crosses a weakly degenerate complementary cone. It turns out that the weakly degenerate cones can be ignored, but it takes a little work to show this. The following property is what makes weak degeneracy more complicated.

6.2.17 Lemma. Let $M \in R^{n \times n}$, $q \in R^n$, and $\alpha \subseteq \{1, \dots, n\}$ be given. Suppose that $\operatorname{pos} C_M(\alpha)$ is a weakly degenerate complementary cone and that $q \in \operatorname{pos} C_M(\alpha) \setminus \mathcal{L}(M)$. It follows that q is contained in exactly two facets of $\operatorname{pos} C_M(\alpha)$.

Proof. The proof of this lemma is basically the same as the proof of Lemma **6.2.15**. The only difference is in the final paragraph. In this case, since $\operatorname{pos} C(\alpha)$ is weakly degenerate, the vector y must have at least one positive component and at least one negative component. Thus, $L \cap R^n_+$ cannot be an infinite ray, so it must be a finite line segment with two distinct endpoints. It follows that L intersects, at each endpoint, a distinct facet of R^n_+ and, further, these two facets are the only facets of R^n_+ which L intersects. The lemma now follows. □

Again, let q^s be a nondegenerate intersection of the path q^t with $\mathcal{K}(M)$. Before, when we assumed no weak degeneracy existed, we could analyze the situation around q^s on a facet-by-facet basis. Now, if q^s is contained in a weakly degenerate complementary cone, then q^s is in two facets of this cone. Thus, we would count this cone twice if we analyzed things on a facet-by-facet basis. This seems to present a problem in our approach to studying what happens as a path crosses $\mathcal{K}(M)$.

This problem can be overcome and, in fact, more general definitions of reflecting and proper can be given which apply to any facet of any complementary cone. We will state these definitions after the following proposition which justifies them.

6.2.18 Proposition. Let $M \in R^{n\times n}$ and $q \in R^n$ be given. Suppose for some $\alpha \subseteq \{1,\ldots,n\}$ and some $i \in \{1,\ldots,n\}$ that $q \in \operatorname{pos} C_M(\alpha)_{\cdot\bar{i}} \setminus \mathcal{L}(M)$. It follows that there exists a sequence $j_0,\ldots,j_m \in \{1,\ldots,n\}$ and a sequence $\beta_0,\ldots,\beta_{m+1} \subseteq \{1,\ldots,n\}$, for some $m \geq 0$, such that:

(a) $\beta_k \bigtriangleup \beta_{k+1} = \{j_k\}$, for $k = 0,\ldots,m$;

(b) $j_{k-1} \neq j_k$, for $k = 1,\ldots,m$;

(c) $\operatorname{pos} C_M(\beta_k)$ is weakly degenerate for $k = 1,\ldots,m$;

(d) $\dim(\operatorname{pos} C_M(\beta_k)_{\cdot\overline{j_k}}) = n-1$ and $q \in \operatorname{pos} C_M(\beta_k)_{\cdot\overline{j_k}}$, for $k = 0,\ldots,m$;

(e) $\alpha = \beta_k$ and $i = j_k$ for some $k \in \{0,\ldots,m\}$.

Further, exactly one of the five following cases holds:

(1) $(\det M_{\beta_0\beta_0})(\det M_{\beta_{m+1}\beta_{m+1}}) > 0$;

(2) $(\det M_{\beta_0\beta_0})(\det M_{\beta_{m+1}\beta_{m+1}}) < 0$;

(3) one of $\operatorname{pos} C_M(\beta_0)$ and $\operatorname{pos} C_M(\beta_{m+1})$ is full and the other is strongly degenerate;

(4) both $\operatorname{pos} C_M(\beta_0)$ and $\operatorname{pos} C_M(\beta_{m+1})$ are strongly degenerate;

(5) $\operatorname{pos} C_M(\beta_0)$ is weakly degenerate with $\beta_0 = \beta_{m+1}$ and $j_0 \neq j_m$.

In addition, for all the above cases, the index sets $\beta_0,\ldots,\beta_{m+1}$ are all distinct, with the single exception that in case (5) we have $\beta_0 = \beta_{m+1}$. Also, in cases (1)–(4), the m and the sequences of j_k and β_k are unique. In case (5), if we specify that $\alpha = \beta_0$ and $i = j_0$, then m and the sequences of j_k and β_k are unique.

Proof. For the moment, assume that $\operatorname{pos} C(\alpha)$ is full or strongly degenerate. Thus, by condition (e), we need to have $\beta_0 = \alpha$ and $j_0 = i$. By condition (a), we have $\beta_1 = \beta_0 \bigtriangleup \{j_0\}$.

If $\operatorname{pos} C(\beta_1)$ is full or strongly degenerate, then one of cases (1)–(4) has occurred. Since $\beta_0 \neq \beta_1$, the proposition follows in this simple case.

Suppose $\operatorname{pos} C(\beta_1)$ is weakly degenerate. Lemma **6.2.17** implies that there are exactly two possible values of j_1 which satisfy condition (d). We already know one of these values, which is j_0, but condition (b) forbids having $j_1 = j_0$. Thus, we must set j_1 to the (unique) value satisfying both conditions (b) and (d). This forces β_2 to equal $\beta_1 \triangle \{j_1\}$.

We now repeat the above procedure in which, as long as we encounter weakly degenerate cones, conditions (b) and (d) uniquely determine the next j_k and from j_k condition (a) uniquely determines the next β_{k+1}. Since there are only finitely many complementary cones, we will eventually either encounter a cone which is not weakly degenerate or we will encounter the same cone twice. We now consider, in turn, these two distinct possibilities.

Suppose, before repeating any complementary cone, we encounter β_{m+1} such that $\operatorname{pos} C(\beta_{m+1})$ is not weakly degenerate. From the construction of the sequences of j_k and β_k it is clear that the β_k are distinct, that one of cases (1)–(4) has occurred, and that conditions (a)–(e) are satisfied. Further, the construction shows the sequences are unique in this regard. Thus, the proposition is valid in this case.

Suppose, instead, that we encounter a repeated cone. Specifically, suppose we encounter β_{m+1} which equals β_k where $0 \leq k \leq m$. We may assume that this is the first repetition and, so, $\beta_0, \ldots, \beta_m$ are distinct.

If $\operatorname{pos} C(\beta_{m+1})$ is not weakly degenerate, then $k = 0$. Yet, by **6.2.15** and **6.2.16**, only one facet of $\operatorname{pos} C(\beta_0)$ can contain q. As

$$C(\beta_m)_{\bullet \overline{j_m}} = C(\beta_{m+1})_{\bullet \overline{j_m}} = C(\beta_0)_{\bullet \overline{j_m}},$$

it follows that $j_0 = j_m$. Thus, $\beta_m = \beta_1$. Since we encountered our first repetition with β_{m+1}, then we must have $m = 1$. However, this implies that $j_0 = j_1$ which violates condition (b) and, so, we must have violated the above procedure for constructing the sequence of j_k. We conclude that $\operatorname{pos} C(\beta_{m+1})$ must be weakly degenerate.

Now, assuming $\operatorname{pos} C(\beta_{m+1})$ is weakly degenerate, we know that $k \neq 0$. By **6.2.17**, only two facets of $\operatorname{pos} C(\beta_k)$ can contain q. As

$$C(\beta_m)_{\bullet \overline{j_m}} = C(\beta_{m+1})_{\bullet \overline{j_m}} = C(\beta_k)_{\bullet \overline{j_m}},$$

it follows that j_m equals either j_k or j_{k-1}. Thus, either $\beta_m = \beta_{k+1}$ or

$\beta_m = \beta_{k-1}$. Since we encountered our first repetition with β_{m+1}, then we must have $m = k + 1$. However, this implies that $j_m = j_{m-1}$ which violates condition (b) and, so, we must have violated the above procedure for constructing the sequence of j_k. We conclude that the sequence of β_k cannot repeat. This completes the proof for the case in which $\text{pos}\, C(\alpha)$ is not weakly degenerate.

We now assume that $\text{pos}\, C(\alpha)$ is weakly degenerate. We may still, momentarily, let $\beta_0 = \alpha$ and $j_0 = i$. The procedure for generating the sequences of j_k and β_k, as given above, is still valid and still produces unique sequences.

Suppose, in producing these sequences, we encounter a cone twice. Specifically, suppose we encounter β_{m+1} which equals β_k where $0 \leq k \leq m$ and, as above, this is the first repetition we encounter. In this case, we know that $\text{pos}\, C(\beta_{m+1})$ is weakly degenerate for otherwise it would not be a repetition. If $k \neq 0$, the argument given above is still valid and still leads to a contradiction. Thus, if there is a repetition, then $\beta_0 = \beta_{m+1}$ and, except for this, all the β_k are distinct. In addition, since β_1 and β_m are distinct, it must be that $j_0 \neq j_m$. We see that case (5) has occurred and that the proposition is valid in this case.

Suppose, before repeating any complementary cone, we encounter β_{m+1} such that $\text{pos}\, C(\beta_{m+1})$ is not weakly degenerate. We may now repeat our argument, for the case in which $\text{pos}\, C(\alpha)$ is not weakly degenerate, only this time we will take α to equal β_{m+1} and i to equal j_m. We will obtain for these new values of α and i the appropriate and unique sequences specified in the proposition and, further, one of cases (1)–(4) will occur. By reversing the order of the sequences obtained, we will have sequences as specified in the proposition for the original values of α and i. □

6.2.19 Remark. There is a natural graph-theoretic way of viewing Proposition **6.2.18**. From this viewpoint, each complementary cone relative to M corresponds to a vertex of a graph. In this graph, an edge exists between two vertices if and only if the two corresponding complementary cones are adjacent and their common facet contains q. It is not hard to see that if a vertex of the graph is adjacent to an edge, then either it is adjacent to exactly two edges and the corresponding cone is weakly degenerate, or it is adjacent to exactly one edge and the corresponding cone is full or strongly degenerate. The proposition can thus be deduced from the fact that if no vertex of a simple graph is adjacent to more than two edges, then the connected components of the graph consist of cycles, paths, and isolated vertices.

6.2.20 Definition. Let $M \in R^{n\times n}$, $q \in R^n$, $\alpha \subseteq \{1,\ldots,n\}$, and $i \in \{1,\ldots,n\}$ be given such that $q \in \operatorname{pos} C_M(\alpha)_{\bullet\bar{\imath}} \setminus \mathcal{L}(M)$. There exist unique sequences $j_0,\ldots,j_m \in \{1,\ldots,n\}$ and $\beta_0,\ldots,\beta_{m+1} \subseteq \{1,\ldots,n\}$ as described in Proposition **6.2.18**. We will refer to $\{\operatorname{pos} C_M(\beta_k)_{\bullet\overline{j_k}}\}_{k=0}^m$ as the *family* of facets around q which contains $\operatorname{pos} C_M(\alpha)_{\bullet\bar{\imath}}$. There are five distinct *classes* into which each family falls: *proper, reflecting, absorbing, isolated*, and *cyclic*. A family is said to be proper, reflecting, absorbing, isolated, or cyclic, if case (1), (2), (3), (4), or (5), respectively, of Proposition **6.2.18** holds. At times we may refer to $\operatorname{pos} C_M(\alpha)_{\bullet\bar{\imath}}$ as being proper, reflecting, absorbing, isolated, or cyclic around a point q, by which we mean that the family around q which contains $\operatorname{pos} C_M(\alpha)_{\bullet\bar{\imath}}$ is in that class.

6.2.21 Remark. If $\operatorname{pos} C_M(\beta_k)$ is not weakly degenerate, then $k = 0$ or $k = m+1$. Thus, either $\operatorname{pos} C_M(\beta_0)_{\bullet\overline{j_0}}$ or $\operatorname{pos} C_M(\beta_{m+1})_{\bullet\overline{j_m}}$ will be the cone's only facet containing q. If $\operatorname{pos} C_M(\beta_k)$ is weakly degenerate, then $\operatorname{pos} C_M(\beta_k)_{\bullet\overline{j_{k-1}}}$ and $\operatorname{pos} C_M(\beta_k)_{\bullet\overline{j_k}}$ are the cone's only two facets containing q. (If the class is cyclic, interpret j_{-1} as j_m and j_{m+1} as j_0.) Thus, no complementary cone is associated with more than one family around q. Therefore, we say that $\{\operatorname{pos} C_M(\beta_k)\}_{k=0}^{m+1}$ are the cones *associated* with the facet family.

With regards to our inquiries as to what happens as a path crosses through a facet, it will turn out that facet families are the important objects to study and that a family's class is the key fact we need to know.

Looking back on Definition **6.2.10** we see that it corresponds to the special case in Proposition **6.2.18** where $m = 0$ and both $\operatorname{pos} C(\beta_0)$ and $\operatorname{pos} C(\beta_1)$ are full. A proper facet, as defined in **6.2.10**, is just a proper family consisting of a single facet. Similarly, a reflecting facet is a reflecting family consisting of a single facet.

As the reader is asked to prove in Exercise **6.10.12**, the fact that Definition **6.2.10** only deals with single facet families allows us to classify facets without reference to a q. In general, a given facet may (depending on q) belong to more than one facet family and, further, the different families to which the facet belongs do not all have to be in the same class. This is demonstrated by the following example.

6.2.22 Example. Consider the matrix

$$M = \begin{bmatrix} -1 & -2 & 1 \\ -1 & 1 & -2 \\ 3 & 0 & 0 \end{bmatrix}.$$

It is not difficult to check that M has no strongly degenerate complementary cones. (The reader can also check that $\mathrm{SOL}(q, M) = \emptyset$ if we let $q = (0, 0, -1)$ and, hence, $\deg M = 0$.)

Consider the facet $\operatorname{pos} C(\{1,2,3\})_{\bullet\bar{1}}$. Letting $q^1 = (1, 2, 0)$, we find that $q^1 \in \operatorname{pos} C(\{1,2,3\})_{\bullet\bar{1}} \setminus \mathcal{L}(M)$. It is simple to check that the family of facets around q^1 containing $\operatorname{pos} C(\{1,2,3\})_{\bullet\bar{1}}$, in order, consists of $\operatorname{pos} C(\{1,2,3\})_{\bullet\bar{1}}$, $\operatorname{pos} C(\{2,3\})_{\bullet\bar{2}}$, and $\operatorname{pos} C(\{3\})_{\bullet\bar{3}}$. Thus, as $\beta_0 = \{1,2,3\}$ and $\beta_{m+1} = \beta_3 = \emptyset$, we conclude that the facet $\operatorname{pos} C(\{1,2,3\})_{\bullet\bar{1}}$ is proper around q^1.

Letting $q^2 = (-1, 3, 0)$, we find that $q^2 \in \operatorname{pos} C(\{1,2,3\})_{\bullet\bar{1}} \setminus \mathcal{L}(M)$. One can check that the family of facets around q^2 containing $\operatorname{pos} C(\{1,2,3\})_{\bullet\bar{1}}$, in order, consists of $\operatorname{pos} C(\{1,2,3\})_{\bullet\bar{1}}$, $\operatorname{pos} C(\{2,3\})_{\bullet\bar{2}}$, and $\operatorname{pos} C(\{3\})_{\bullet\bar{1}}$. Thus, as $\beta_0 = \{1,2,3\}$ and $\beta_{m+1} = \beta_3 = \{1,3\}$, we conclude that the facet $\operatorname{pos} C(\{1,2,3\})_{\bullet\bar{1}}$ is reflecting around q^2.

The following is another example the reader may find useful in thinking about the different classes of facet families.

6.2.23 Example. Consider the matrix

$$M = \begin{bmatrix} -1 & -2 & -1 \\ -2 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

As in Example **6.2.22**, M has no strongly degenerate complementary cones, $\mathrm{SOL}(q, M) = \emptyset$ for $q = (0, 0, -1)$, and $\deg M = 0$.

Letting $q = (2, -1, 0)$, we find that $q \in \operatorname{pos} C(\{1,2,3\})_{\bullet\bar{1}} \setminus \mathcal{L}(M)$. One may check that the family of facets around q containing $\operatorname{pos} C(\{1,2,3\})_{\bullet\bar{1}}$, in order, consists of $\operatorname{pos} C(\{1,2,3\})_{\bullet\bar{1}}$, $\operatorname{pos} C(\{2,3\})_{\bullet\bar{2}}$, $\operatorname{pos} C(\{3\})_{\bullet\bar{1}}$, and

$\operatorname{pos} C(\{1,3\})_{\bullet\bar{2}}$. Thus, as $\beta_0 = \beta_4 = \beta_{m+1} = \{1,2,3\}$, we conclude that the facet $\operatorname{pos} C(\{1,2,3\})_{\bullet\bar{1}}$ is cyclic around q.

Example **6.2.23** has another interesting property. Since $K(M)$ is closed, it is always true that $K(M) \supseteq \operatorname{cl}\big(\operatorname{int} K(M)\big)$. Exercise **6.10.9** asks the reader to show $K(M) = \operatorname{cl}\big(\operatorname{int} K(M)\big)$ if M is nondegenerate. Yet, both Example **6.2.23** and any solution to Exercise **6.10.13** give a matrix M for which $K(M) \neq \operatorname{cl}\big(\operatorname{int} K(M)\big)$. In addition, any solution to Exercise **6.10.13** gives an example of an isolated facet.

In Definition **6.2.20** we finally reach a classification scheme for facets which does not restrict the type of complementary cones that M may have. We can now usefully describe the facet structure of the complementary cones for any matrix M. As one might expect, the facet structure of the complementary cones is invariant under principal pivoting. We make this statement precise in the following theorem, which is left to the reader as Exercise **6.10.14**.

6.2.24 Theorem. Let $\alpha, \beta \subseteq \{1, \ldots, n\}$, $i \in \{1, \ldots, n\}$, $M \in R^{n\times n}$, and $q \in R^n$ be given. Suppose $\det M_{\alpha\alpha} \neq 0$ and let (q', M') be the pivotal transform of (q, M) with pivot $M_{\alpha\alpha}$ as given by (2.3.10) and (2.3.11).

(a) If the complementary cone $\operatorname{pos} C_M(\beta)$ is full, weakly degenerate, or strongly degenerate, then the complementary cone $\operatorname{pos} C_{M'}(\alpha \triangle \beta)$ is, respectively, full, weakly degenerate, or strongly degenerate.

(b) If $q \in \mathcal{K}(M)$, then $q' \in \mathcal{K}(M')$. If $q \in \mathcal{L}(M)$, then $q' \in \mathcal{L}(M')$.

(c) If the facet $\operatorname{pos} C_M(\beta)_{\bullet\bar{i}}$ is proper, reflecting, absorbing, isolated, or cyclic, around q, then the facet $\operatorname{pos} C_{M'}(\alpha \triangle \beta)_{\bullet\bar{i}}$ will be, respectively, proper, reflecting, absorbing, isolated, or cyclic, around q'. □

For this classification scheme to be useful in understanding what happens as a path crosses a facet, we must extend Theorems **6.2.12** and **6.2.14** to deal explicitly with the facet classes of Definition **6.2.20**. This is done in the next result.

6.2.25 Theorem. Let $M \in R^{n\times n}$, $\alpha \subseteq \{1, \ldots, n\}$, and $i \in \{1, \ldots, n\}$ be given. Consider a path $q^t : [0,1] \to R^n$ in which, for some $s \in (0,1)$, the point $q^s \in \operatorname{pos} C_M(\alpha)_{\bullet\bar{i}}$ is a nondegenerate intersection of the path with

$\mathcal{K}(M)$. Let H and δ be as described in Definition **6.2.5**. Let B^+ and B^- be the two open hemiballs which are the connected components of $B(q^s, \delta) \setminus H$. Using the notation in Proposition **6.2.18**, let $\{\text{pos}\, C_M(\beta_k)_{\bullet \overline{\jmath_k}}\}_{k=0}^{m}$ be the family around q^s containing $\text{pos}\, C_M(\alpha)_{\bullet \bar{\imath}}$. The following hold.

(a) For $k = 0$ or $k = m+1$, if $\text{pos}\, C_M(\beta_k)$ is a full cone, then it will contain exactly one of B^+ and B^-. Further, the full cone will be disjoint from the hemiball it does not contain.

(b) For $k \in \{0, \ldots, m+1\}$, if $\text{pos}\, C_M(\beta_k)$ is a degenerate cone, then $B(q^s, \delta) \cap \text{pos}\, C_M(\beta_k) = B(q^s, \delta) \cap H$.

(c) If $\text{pos}\, C_M(\alpha)_{\bullet \bar{\imath}}$ is proper around q^s, then each of B^+ and B^- is contained by one of the full cones $\text{pos}\, C_M(\beta_0)$ and $\text{pos}\, C_M(\beta_{m+1})$.

(d) If $\text{pos}\, C_M(\alpha)_{\bullet \bar{\imath}}$ is reflecting around q^s, then the full cones $\text{pos}\, C_M(\beta_0)$ and $\text{pos}\, C_M(\beta_{m+1})$ either both contain B^+ or both contain B^-.

(e) If $\text{pos}\, C_M(\alpha)_{\bullet \bar{\imath}}$ is not absorbing, then the total contribution to the local degree made by $\{\text{pos}\, C_M(\beta_k)\}_{k=0}^{m+1}$ is constant for all points in $B^+ \cup B^-$.

(f) If $\text{pos}\, C_M(\alpha)_{\bullet \bar{\imath}}$ is absorbing, then the total contribution to the local degree made by $\{\text{pos}\, C_M(\beta_k)\}_{k=0}^{m+1}$ is constant for all points in B^+ and for all points in B^-, but differs by one between the two hemiballs.

Proof. As the facets $\text{pos}\, C(\beta_0)_{\bullet \overline{\jmath_0}}$ and $\text{pos}\, C(\beta_{m+1})_{\bullet \overline{\jmath_m}}$ contain q^s, part (a) can be shown by using the argument at the beginning of the proof of Theorem **6.2.12**. Likewise, part (b) can be shown by using the argument at the beginning of the proof of Theorem **6.2.14**.

Suppose that $\text{pos}\, C(\beta_0)$ is full. For $k = 0, \ldots, m$, define $C^k \in R^{n \times n}$ as follows: $C^k_{\bullet \overline{\jmath_k}} = C(\beta_k)_{\bullet \overline{\jmath_k}}$ and $C^k_{\bullet j_k} = C(\beta_0)_{\bullet j_0}$. We claim

$$\text{sgn}\left((\det C(\beta_0))(\det C^k)\right) = (-1)^k \qquad \text{for } k = 0, \ldots, m. \tag{4}$$

This is obvious for $k = 0$ as $C^0 = C(\beta_0)$. For $k = 1, \ldots, m$, we will show that $(\det C^{k-1})(\det C^k) < 0$. By induction, the claim will follow.

From **6.2.5(a)**, we know $\text{pos}\, C(\beta_0)_{\bullet \overline{\jmath_0}} \subseteq H$. Thus, as $\det C(\beta_0) \neq 0$, we have $C(\beta_0)_{\bullet j_0} \notin H$. As $\text{pos}\, C(\beta_k)$ is degenerate, we deduce from part (b) that $\text{pos}\, C(\beta_k) \subseteq H$. (Also, see the beginning of the proof of Theorem **6.2.14**.) This implies that $\text{pos}\, C(\beta_k)_{\bullet \overline{\jmath_{k-1}}}$ and $\text{pos}\, C(\beta_k)_{\bullet \overline{\jmath_k}}$ are contained in

H. Let $\gamma = \{j_{k-1}, j_k\}$ and note that $C(\beta_{k-1})_{\bullet\bar{\gamma}} = C(\beta_k)_{\bullet\bar{\gamma}} = C^{k-1}_{\bullet\bar{\gamma}} = C^k_{\bullet\bar{\gamma}}$ and that $C(\beta_{k-1})_{\bullet\overline{j_{k-1}}} = C(\beta_k)_{\bullet\overline{j_{k-1}}}$. Thus, if we let H' be the subspace which is the affine hull of $\operatorname{pos} C^{k-1}_{\bullet\overline{j_k}}$ and L be the subspace which is the affine hull of $\operatorname{pos} C(\beta_k)_{\bullet\bar{\gamma}}$, then we may conclude that $\dim L = n-2$, $\dim H' = n-1$, and $L = H \cap H'$.

As $q^s \notin \mathcal{L}(M)$, we see that $q^s \notin L$. Since q^s is in both $\operatorname{pos} C(\beta_k)_{\bullet\overline{j_{k-1}}}$ and $\operatorname{pos} C(\beta_k)_{\bullet\overline{j_k}}$, and since $C(\beta_{k-1})_{\bullet j_k} = C(\beta_k)_{\bullet j_k}$ and $\operatorname{pos} C(\beta_k)_{\bullet\bar{\gamma}} \subseteq L$, then $C(\beta_k)_{\bullet j_{k-1}}$ and $C(\beta_{k-1})_{\bullet j_k}$ must both be in the component (open halfplane) of $H \setminus L$ containing q^s. Thus, $C(\beta_k)_{\bullet j_{k-1}}$ and $C(\beta_{k-1})_{\bullet j_k}$ must be in the same component (open halfspace) of $R^n \setminus H'$. Therefore, the sign of the determinant of C^{k-1} does not change if we replace column j_k, which is currently $C(\beta_{k-1})_{\bullet j_k}$, with the column $C(\beta_k)_{\bullet j_{k-1}}$. However, the matrix we would get after this replacement would simply be the matrix C^k with columns j_{k-1} and j_k switched. Thus, $(\det C^{k-1})(\det C^k) < 0$, which proves our original claim that (4) is valid.

Remark **6.2.11**, condition (a) of Proposition **6.2.18**, and (4), imply

$$\begin{aligned} &\operatorname{sgn}\left((\det M_{\beta_0\beta_0})(\det M_{\beta_{m+1}\beta_{m+1}})\right) \\ &= (-1)^{m+1} \operatorname{sgn}\left((\det C(\beta_0))(\det C(\beta_{m+1}))\right) \qquad (5) \\ &= -\operatorname{sgn}\left((\det C^m)(\det C(\beta_{m+1}))\right). \end{aligned}$$

Note that $C^m_{\bullet\overline{j_m}} = C(\beta_m)_{\bullet\overline{j_m}} = C(\beta_{m+1})_{\bullet\overline{j_m}}$. If $\operatorname{pos} C(\alpha)_{\bullet\bar{\imath}}$ is proper, then (5) implies $\left(\det C^m\right)\left(\det C(\beta_{m+1})\right) < 0$. This means $C(\beta_0)_{\bullet j_0}$ and $C(\beta_{m+1})_{\bullet j_m}$ are in different components of $R^n \setminus H$; thus, so are $\operatorname{pos} C(\beta_0)$ and $\operatorname{pos} C(\beta_{m+1})$. Hence, part (c) is valid. If $\operatorname{pos} C(\alpha)_{\bullet\bar{\imath}}$ is reflecting, then (5) implies $\left(\det C^m\right)\left(\det C(\beta_{m+1})\right) > 0$. This means $C(\beta_0)_{\bullet j_0}$ and $C(\beta_{m+1})_{\bullet j_m}$ are in the same component of $R^n \setminus H$ and, thus, so are $\operatorname{pos} C(\beta_0)$ and $\operatorname{pos} C(\beta_{m+1})$. Hence, part (d) is valid.

For parts (e) and (f) notice that none of the points in $B^+ \cup B^-$ are in $\mathcal{K}(M)$. Hence, local degree is well-defined throughout $B^+ \cup B^-$. Also, all the cones $\{\operatorname{pos} C(\beta_k)\}_{k=1}^m$ are degenerate, so they contribute nothing to the local degree of any point in R^n.

If $\operatorname{pos} C(\alpha)_{\bullet\bar{\imath}}$ is cyclic or isolated, then $\operatorname{pos} C(\beta_0)$ and $\operatorname{pos} C(\beta_{m+1})$ are degenerate. Thus, they contribute nothing to the local degree of any point in R^n. It follows that part (e) is true in this case.

If $\mathrm{pos}\, C(\alpha)_{\bullet\bar{\imath}}$ is proper, then using part (c) we find that each point of $B^+ \cup B^-$ is in exactly one of $\mathrm{pos}\, C(\beta_0)$ and $\mathrm{pos}\, C(\beta_{m+1})$. Since we have $(M_{\beta_0\beta_0})(M_{\beta_{m+1}\beta_{m+1}}) > 0$, then the two complementary cones have the same index. It follows that part (e) is true if $\mathrm{pos}\, C(\alpha)_{\bullet\bar{\imath}}$ is proper.

If $\mathrm{pos}\, C(\alpha)_{\bullet\bar{\imath}}$ is reflecting, then using part (d) we find that each point of $B^+ \cup B^-$ is either in both $\mathrm{pos}\, C(\beta_0)$ and $\mathrm{pos}\, C(\beta_{m+1})$ or in neither. Since we have $(M_{\beta_0\beta_0})(M_{\beta_{m+1}\beta_{m+1}}) < 0$, then the two complementary cones have opposite indexes. It follows that part (e) is true if $\mathrm{pos}\, C(\alpha)_{\bullet\bar{\imath}}$ is reflecting. Thus, part (e) is true entirely.

If $\mathrm{pos}\, C(\alpha)_{\bullet\bar{\imath}}$ is absorbing, one of $\mathrm{pos}\, C(\beta_0)$ and $\mathrm{pos}\, C(\beta_{m+1})$ is full and the other degenerate. The degenerate cone contributes nothing to the local degree of any point in R^n. The full cone contributes ± 1 to all points in the hemiball it contains and nothing to all points in the other hemiball. Part (f) now follows. □

6.2.26 Remark. Notice that H and δ (and, hence, B^+ and B^-) are obtained via Theorem **6.2.4** and Definition **6.2.5** and, thus, they are determined only by the point q^s. Therefore, the same H, δ, B^+, and B^- may be used for any facet containing q^s, not just $\mathrm{pos}\, C_M(\alpha)_{\bullet\bar{\imath}}$.

Theorem **6.2.25** is, at last, the basic tool we need to examine what happens at the nondegenerate intersection of a path and a facet. As an appropriate end to this section, we will now prove two theorems which are important in understanding the geometric nature of the LCP and both may be considered corollaries of Theorem **6.2.25**. The first theorem, to which we have been alluding throughout this section, concerns the parity of the set $\mathrm{SOL}(q; M)$ as q varies. The second theorem is a stronger version of Theorem **6.1.17**.

6.2.27 Theorem. Let $M \in R^{n\times n}$ be given. Let q^t be a path such that all the intersections of q^t with $\mathcal{K}(M)$ are nondegenerate. Suppose $q^0, q^1 \notin \mathcal{K}(M)$ and, so, $\deg_M(q^0)$ and $\deg_M(q^1)$ are well-defined. The following are then equivalent:

(a) $\deg_M(q^0)$ and $\deg_M(q^1)$ have the same parity,

(b) $|\,\mathrm{SOL}(q^0, M)|$ and $|\,\mathrm{SOL}(q^1, M)|$ have the same parity,

(c) There are an even number of pairs (s, C), where $s \in (0, 1)$, C is a strongly degenerate complementary cone relative to M, and $q^s \in C$.

(d) There are an even number of pairs (s, F), where $s \in (0,1)$ and F is an absorbing facet family around q^s.

Proof. (a) ⇔ (b). By **6.1.4**, $\deg(q^0)$ is the sum of the indexes of all (w, z) which solve (q^0, M). Since $|\operatorname{ind}(w, z)| = 1$ for any solution, then $\deg(q^0)$ and $|\operatorname{SOL}(q^0, M)|$ have the same parity. Similarly, $\deg(q^1)$ and $|\operatorname{SOL}(q^1, M)|$ have the same parity.

(c) ⇔ (d). By Corollary **6.1.9** and Lemma **6.2.15**, if q^s is in a strongly degenerate cone, then it must be in a unique facet of the cone. By Proposition **6.2.18**, every facet containing q^s belongs to a unique facet family around q^s. If the family is absorbing, then exactly one of the facets in the family is part of a strongly degenerate cone. If the family is isolated, then exactly two of the facets in the family are part of a strongly degenerate cone and, from **6.2.18** these two strongly degenerate cones are distinct. In all other cases, no facets of the family are from strongly degenerate cones. Thus, for each $s \in (0,1)$, the number of strongly degenerate cones containing q^s has the same parity as the number of absorbing families containing q^s. In the next paragraph we will show that q^s is contained in $\mathcal{K}(M)$ for only finitely many values of $s \in (0,1)$. Thus, (c) and (d) are equivalent.

(a) ⇔ (d). From **6.2.5**, if $q^s \in \mathcal{K}(M)$, then there exists an open interval around s such that if $q^t \in \mathcal{K}(M)$ for t in the interval, then $t = s$. Because $\mathcal{K}(M)$ is closed, if $q^s \notin \mathcal{K}(M)$, then there exists an open interval around s such that $q^t \notin \mathcal{K}(M)$ for t in the interval. Thus, by virtue of its compactness, the interval $[0,1]$ may be covered by finitely many open intervals each of which contains at most one point s for which $q^s \in \mathcal{K}(M)$. Hence, there are finitely many such points. Let $\{q^{s_k}\}_{k=1}^m$ be the points of the path q^t which intersect $\mathcal{K}(M)$. It is not restrictive to assume that $0 < s_1 < \cdots < s_m < 1$.

Fix $k \in \{2, \ldots, m\}$ and consider those points of the path q^t for which $t \in (s_{k-1}, s_k)$. None of the points are in $\mathcal{K}(M)$. By continuity, all of the points are in the same set of complementary cones. Thus, all of the points have the same well-defined local degree. A similar statement is true if we had taken $t \in [0, s_1)$ or $t \in (s_m, 1]$. We must now take a look at what happens to the local degree in the neighborhood of a q^{s_k}.

Fix $k \in \{1, \ldots, m\}$. Every facet containing q^{s_k} is in a unique facet family around q^{s_k}. In light of Remark **6.2.26**, consider the hemiballs B^+ and B^- associated with q^{s_k}. Let $\operatorname{pos} C(\alpha)$ be a complementary cone.

If $\operatorname{pos} C(\alpha)$ contains all of $B^+ \cup B^-$, or none of $B^+ \cup B^-$, then the contribution to the local degree made by $\operatorname{pos} C(\alpha)$ is constant over all points in $B^+ \cup B^-$.

If $\operatorname{pos} C(\alpha)$ contains some, but not all of the points in $B^+ \cup B^-$, then Theorem **6.2.4** implies that $\operatorname{pos} C(\alpha)$ has a facet containing q^{s_k}. Thus, $\operatorname{pos} C(\alpha)$ is associated with a facet family around q^{s_k}. Therefore, we deduce from parts (e) and (f) of Theorem **6.2.25** that the local degree is constant for all points in B^+ and for all points in B^- and, further, the parity of the local degree is the same for the two hemiballs if and only if there are an even number of absorbing facet families around q^{s_k}. In light of everything we now know, the theorem follows. □

6.2.28 Theorem. Let $M \in R^{n\times n}$ be given. Let $\mathcal{S}$ be the set of all points $q \notin \mathcal{L}(M)$ for which there exists, relative to M, a family of facets absorbing around q. We then have $\deg_M(q) = \deg_M(q')$ for any $q, q' \in R^n \setminus \mathcal{K}(M)$ that belong to the same connected component of $R^n \setminus \operatorname{cl} \mathcal{S}$.

Proof. Suppose there were a path q^t in $R^n \setminus \operatorname{cl} \mathcal{S}$ between q and q' such that all intersections of the path and $\mathcal{K}(M)$ were nondegenerate. The theorem would then follow by using the proof of (a) $\Leftrightarrow$ (d) in Theorem **6.2.27**. The only change needed is in the last paragraph of that proof. Since no facet family could be absorbing around q^{s_k}, only part (e) of Theorem **6.2.25** is needed. We would deduce that the local degree is constant for all points in $B^+ \cup B^-$ and, so, Theorem **6.2.28** would follow. Therefore, we now show that such a path exists.

Since $R^n \setminus \operatorname{cl} \mathcal{S}$ is open, the connected components are path connected. Thus, a path q^t between q and q' does exist within $R^n \setminus \operatorname{cl} \mathcal{S}$. However, it might be that not all the intersections of the path with $\mathcal{K}(M)$ are nondegenerate. However, we can find an acceptable path using q^t.

The path q^t is compact, thus there exists a finite collection of open balls within $R^n \setminus \operatorname{cl} \mathcal{S}$ such that the union of these open balls contains the path. It is not hard to see that, within this collection, there exists a sequence of open balls, $B_1, \ldots, B_m$, such that $q \in B_1$, $q' \in B_m$, and $B_{k-1} \cap B_k \neq \emptyset$ for all $k = 2, \ldots, m$. Invoking Theorem **6.1.12**, we may assume that for each $k \in \{1, \ldots, m\}$ there is a $p^k \in B_k \setminus \mathcal{K}(M)$. In fact, we will let $p^1 = q$ and $p^m = q'$. We will now show that for each $k \in \{2, \ldots, m\}$ there exists a path in $R^n \setminus \operatorname{cl} \mathcal{S}$ between p^{k-1} and p^k such that all intersections of the path and $\mathcal{K}(M)$ are nondegenerate. It follows that if we sequentially take the composition of all these paths we will arrive at a path between q and q' with the properties we desire.

Fix $k \in \{2, \ldots, m\}$. We may find an $r \in R^n$ and an $\varepsilon > 0$ such that $B(r, \varepsilon) \subseteq (B_{k-1} \cap B_k) \setminus \mathcal{K}(M)$. By Theorem **6.2.7**, there exists $r', r'' \in B(r, \varepsilon)$ such that all the intersections of the paths associated with $\ell\,[p^{k-1}, r'] \subseteq B_{k-1}$ and $\ell\,[r'', p^k] \subseteq B_k$ are nondegenerate. As $\ell\,[r', r'']$ is contained in $B(r, \varepsilon)$, its path does not intersect $\mathcal{K}(M)$. Thus, if we sequentially take the composition of the paths of $\ell\,[p^{k-1}, r']$, $\ell\,[r', r'']$, and $\ell\,[r'', p^k]$, we obtain a path from p^{k-1} to p^k in which all intersections with $\mathcal{K}(M)$ are nondegenerate. □

If there exists a family of facets absorbing around q, then q is contained in a strongly degenerate cone. Since complementary cones are closed, the set $\mathrm{cl}\,\mathcal{S}$ given in Theorem **6.2.28** is contained in the union of the strongly degenerate complementary cones. Thus, Theorem **6.2.28** is a stronger version of Theorem **6.1.17**.

An important point to notice is that, aside from the definitions, we have not used any of the results from degree theory to prove Theorem **6.2.28**. In fact, we have not used any degree-theoretic results in this section. Yet, we have come full circle and shown a key degree-theoretic result via a different path.

6.3 The Geometric Side of Lemke's Method

In Section 4.4 we studied Lemke's method for solving the LCP. In that section, the method was presented from the point of view of pivotal algebra. In this section, we will look at the geometric side of Lemke's method.

Suppose we wish to use Lemke's method to solve the LCP (q, M). Looking back at Section 4.4, we see that the essential idea was to select a fixed vector d such that for some suitably large value of z_0 the vector $q + dz_0$ is nonnegative. This gives us the trivial solution $(w, z) = (q + dz_0, 0)$ for the LCP $(q + dz_0, M)$. The object of the method is to reduce z_0 to zero while, at all times, keeping track of a solution to the LCP $(q + dz_0, M)$. Of course, we would wish this process of keeping track of a solution to be relatively simple and quick. The process used is basic pivotal algebra but, as nothing is free, the corresponding penalty for simplicity and quickness is that we cannot guarantee that z_0 will decrease monotonically or even that the method, in general, will find a solution if one exists. Thus, as seen in Section 4.4, we will do well to study the behavior of Lemke's method.

From the above description, one is immediately struck by the similarity of Lemke's method to homotopy and path-following methods. In these latter methods, one works with a mathematical problem which has the key property that if the parameters of the problem change only slightly, then a solution to the problem before the change will be close to a solution of the problem after the change. One can therefore start with the parameters set to values for which a solution is known and then gradually change the parameters to the values desired. If the parameters are changed in small increments, it should be possible to keep track of a solution since, by the aforementioned key property, after each change of parameters a new solution will be close to the old solution. In this way we start with a simple version of the problem and continuously (homotopically) change it to the version we wish to solve. At the same time, the solution continuously (homotopically) changes. If there is only a single parameter, then the continuously changing sequences of problems and solutions each form a path and, so, the term *path-following* is often applied to these methods.

In Lemke's method, the path of problems which we follow is particularly simple. If we consider the ray $\{q + dz_0 : z_0 \geq 0\}$, then the path is just those linear complementarity problems (q', M) with q' somewhere along the ray. On the other hand, we do not uniformly move along the ray in one direction since the parameter z_0 is not guaranteed to monotonically decrease. In this sense, Lemke's method is not a path-following method. Yet, while the sequence of problems which Lemke's method follows may repeat, the corresponding sequence of solutions it follows does not repeat. That is, under appropriate nondegeneracy assumptions, if Lemke's method backtracks and repeats a value of z_0, then the current solution for the LCP $(q + dz_0, M)$ will be different from any previous solution encountered for that value of z_0. In this sense, Lemke's method does follow a path, but it is the path of solutions, not the path of problems.

We will now describe, using geometry, a procedure for solving the LCP (q, M) which we will later show is actually Lemke's method. To develop this procedure we will use the previously mentioned ideas which underlie homotopy methods. Since the homotopy paradigm is inherently a parametric approach, the following development can be seen to parallel and extend the subsection on the parametric form of Lemke's method in Section 4.5. Indeed, the algorithm we will develop (**6.3.1**) is just a geometric way of describing Algorithm **4.5.4**. Of course, both of these algorithms are equivalent to Lemke's method (**4.4.5**).

Geometrically, we solve the LCP (q, M) by determining a complementary cone relative to M which contains q. We do not initially know of such a cone but, following the paradigm of the homotopy methods, we will start with a $\bar{q}$ such that a complementary cone containing $\bar{q}$ will be obvious. If we take $d \in R^n$ to be any fixed positive vector, it is clear that the nonnegative orthant will contain $q + dz_0$ if $z_0 > 0$ is large enough. Thus, letting z_0 take on some appropriate large positive value, the vector $q + dz_0 \in \text{pos}\, C(\emptyset)$ can be used as $\bar{q}$.

We will now designate the nonnegative orthant as the *distinguished complementary cone* and attempt to decrease z_0 until it becomes zero. If we can decrease z_0 to zero without causing $q + dz_0$ to leave the nonnegative orthant, then we have solved the LCP (q, M). Otherwise, we stop decreasing z_0 at the point where $q + dz_0$ would leave the nonnegative orthant if z_0 were decreased any further.

We have now reached the boundary of the distinguished complementary cone. As in Section 4.4, we assume nondegeneracy. In our current context, this means that $q + dz_0$ is never in $\mathcal{L}(M)$ for any $z_0 > 0$. Thus, as $q + dz_0$ is now in the boundary of the distinguished cone, it must be in a unique facet of the cone.

Any further change in z_0 will cause $q + dz_0$ to leave the distinguished cone. Following the paradigm of the homotopy methods, we wish always to know of a complementary cone containing $q + dz_0$. Therefore, a convenient way of continuing is to realize that the *(distinguished) facet* containing $q + dz_0$ must be common to the distinguished cone and some other complementary cone. This other complementary cone is unique and easily determined. We will now let this other cone be the new distinguished cone. Assuming, for the moment, that the new distinguished cone is full, there is some direction (increasing or decreasing) in which we can slightly change z_0 and have $q + dz_0$ remain in the new distinguished cone. We now move

z_0 as far as we can in this direction under the constraint that $q + dz_0$ must not leave the new distinguished cone. If z_0 can be decreased to zero, then the new distinguished cone contains q and we have solved the LCP (q, M). If z_0 can be increased indefinitely, then our procedure terminates without finding a solution to the LCP (q, M). Otherwise, z_0 reaches some finite positive value such that any further change in z_0 would cause $q + dz_0$ to leave the distinguished cone. We have now reached the position we were in at the beginning of this paragraph and the whole procedure repeats.

We have now described the essential step in the geometric procedure for solving the LCP (q, M). Although we made the assumption that the new distinguished complementary cone was nondegenerate, this was only for ease of exposition and allowed us to concentrate on the key ideas. We now formally describe the procedure without making this restriction.

6.3.1 Algorithm. (Lemke)

Step 0. *Initialization.* Given the LCP (q, M), we select $d > 0$ so that all the intersections of $\mathcal{K}(M)$ with the open ray $\{q + dz_0 : z_0 > 0\}$ are nondegenerate. If $q \geq 0$, then stop: $z = 0$ solves (q, M). Otherwise, set z_0 equal to the unique (positive) value for which $q + dz_0$ is in a facet of the nonnegative orthant. Suppose $\operatorname{pos} C(\emptyset)_{\bullet\bar{r}}$ is the (unique) facet containing $q + dz_0$. Set the distinguished cone to be $\operatorname{pos} C(\{r\})$ and set the distinguished facet to be $\operatorname{pos} C(\{r\})_{\bullet\bar{r}}$.

Step 1. *Moving through the distinguished cone.* There is a distinguished cone $\operatorname{pos} C(\alpha)$. This cone has a distinguished facet $\operatorname{pos} C(\alpha)_{\bullet\bar{i}}$. This facet contains the point $q + dz_0$ for the current value of z_0.

- If the distinguished cone is strongly degenerate, then stop. The procedure ends on a secondary ray without yielding a solution.
- If the distinguished cone is weakly degenerate, then there is a unique $j \neq i$ such that $\operatorname{pos} C(\alpha)_{\bullet\bar{j}}$ contains $q + dz_0$ for the current value of z_0. (See Lemma **6.2.17**.) Set the distinguished cone to be $\operatorname{pos} C(\alpha \triangle \{j\})$ and set the distinguished facet to be $\operatorname{pos} C(\alpha \triangle \{j\})_{\bullet\bar{j}}$. Return to the beginning of Step 1.

- If the distinguished cone is full, then there is a unique direction (either increasing or decreasing) in which we can slightly change z_0 and have $q + dz_0$ remain in the cone. Set the value of z_0 as far as possible in this direction with the constraint that $q + dz_0$ must remain in the distinguished cone.

Step 2. *Changing cones.* The only thing which has changed since the beginning of Step 1 is the value of z_0.

- If $z_0 = \infty$, then stop. The procedure ends on a secondary ray without yielding a solution.
- If $z_0 \leq 0$, then the distinguished cone contains q. Thus, a solution to (q, M) has been found.
- If $0 < z_0 < \infty$, then $q + dz_0$ is currently in a facet of the distinguished cone. Suppose this facet is $\operatorname{pos} C(\alpha)_{\bullet \bar{j}}$. Notice that we must have $j \neq i$. Set the distinguished cone to be $\operatorname{pos} C(\alpha \triangle \{j\})$ and set the distinguished facet to be $\operatorname{pos} C(\alpha \triangle \{j\})_{\bullet \bar{j}}$. Return to the beginning of Step 1.

As it happens, for each almost complementary basis generated by Algorithm **4.4.5** we can associate a complementary cone and a facet of that cone. First, for $i = 1, \ldots, n$, associate the variable z_i with the vector $-M_{\bullet i}$, and associate the variable w_i with the vector $I_{\bullet i}$. Now, given an almost complementary basis generated by Algorithm **4.4.5**, consider the $n - 1$ basic variables other than z_0. The columns associated with these variables generate a facet. Now consider the column associated with the driving variable. This column, together with the columns of the facet, generates a complementary cone. We will associate this cone and facet with the basis. In this way, Algorithm **4.4.5** can be thought of as generating a sequence of cones and facets as it generates almost complementary bases. The following shows that Algorithm **6.3.1** is just a geometric description of Algorithm **4.4.5**.

6.3.2 Theorem. Given an LCP (q, M) and a $d > 0$ as specified in Step 0 of Algorithm **6.3.1**, the sequence of distinguished cones and facets generated by Algorithm **6.3.1** will be identical to the sequence of cones and facets generated (as specified above) by Algorithm **4.4.5**. In addition, the sequence of values taken by z_0 is the same in both algorithms.

Proof. Each tableau generated by Algorithm **4.4.5** represents the system $w = q + dz_0 + Mz$. Also, the nondegeneracy assumption on d, made in Step 0 of **6.3.1**, implies that a distinguished facet will contain $q + dz_0$ for exactly one value of z_0. Thus, the value given to z_0 by setting to zero all the nonbasic variables of a tableau must be the unique value of z_0 for which the facet associated with the tableau contains $q + dz_0$. The last assertion of the theorem will now follow from the rest of the theorem.

If $q \geq 0$, then both algorithms stop in Step 0 with the solution $z = 0$. If $q \not\geq 0$, let $\bar{z}_0$ be the smallest value of z_0 for which $q+dz_0$ is nonnegative. We have $\bar{z}_0 > 0$ and a unique index r such that $(q + d\bar{z}_0)_r = 0$. In Algorithm **6.3.1**, in Step 0, we set the distinguished cone and facet to be $\operatorname{pos} C(\{r\})$ and $\operatorname{pos} C(\{r\})_{\boldsymbol{\cdot}\bar{r}}$, respectively. Analogously, in Algorithm **4.4.5**, the first pivot will be $\langle z_0, w_r \rangle$. The basis is now $w_1, \ldots, w_{r-1}, z_0, w_{r+1}, \ldots, w_n$ and the driving variable is z_r. Thus, both algorithms start in the same fashion.

We now proceed inductively. Suppose the cone and facet associated with the current tableau of **4.4.5** is also the current distinguished cone and facet of **6.3.1**. We must show that both algorithms, at the end of the current iteration, either stop in the same fashion or produce the same cone and facet for the next iteration. We will go through the possible ways in which Algorithm **4.4.5** can end the current iteration and, for each one, see what the corresponding action of Algorithm **6.3.1** would be. For notation, let the current distinguished cone and facet be $\operatorname{pos} C(\alpha)$ and $\operatorname{pos} C(\alpha)_{\boldsymbol{\cdot}\bar{\imath}}$, respectively. We may assume the current value of z_0 is positive. Without loss of generality, we may assume the driving variable is z_i. Let $\bar{m}_{0i}$ represent the element in the current tableau in the row of z_0 and the column of z_i.

Suppose z_i is unblocked in Step 1 of **4.4.5** and suppose $\bar{m}_{0i} = 0$. In this case, as we increase z_i to infinity, we obtain from the tableau an infinite ray of solutions to the LCP $(q + dz_0, M)$, where the value of z_0 remains fixed. As $q + dz_0 \in \operatorname{pos} C(\alpha)$ for this value of z_0, Theorem **6.1.27** implies that $\operatorname{pos} C(\alpha)$ is strongly degenerate. We see that both algorithms end with a secondary ray.

Suppose z_i is unblocked in Step 1 of **4.4.5** and suppose $\bar{m}_{0i} > 0$. In this case, by increasing z_i, we may obtain from the tableau a solution to the

LCP $(q + dz_0, M)$ for arbitrarily large z_0. These solutions are contained in $\operatorname{pos} C(\alpha)$. Thus, we may make z_0 arbitrarily large and $q+dz_0$ will remain in $\operatorname{pos} C(\alpha)$. The nondegeneracy assumption on d implies that no two points of the ray $\{q + dz_0 : z_0 \geq 0\}$ can be in the same facet of $\operatorname{pos} C(\alpha)$, thus increasing z_0 must move $q+dz_0$ into the interior of $\operatorname{pos} C(\alpha)$ and, hence, the cone must be full. Therefore, **6.3.1** will terminate in Step 2 with $z_0 = \infty$. In this case, we see that both algorithms end with a secondary ray.

Suppose z_i is blocked in Step 1 of **4.4.5** by z_0. Thus, z_0 starts out positive and decreases to zero as we increase z_i. Therefore, starting with $q + dz_0$ contained in $\operatorname{pos} C(\alpha)_{\bullet\bar{\imath}}$, we may decrease z_0 slightly and remain in $\operatorname{pos} C(\alpha)$. As before, we can show that $\operatorname{pos} C(\alpha)$ must be full. Thus, in Step 1 of **6.3.1**, we would end up decreasing z_0. Further, we will find that z_0 can be decreased to a nonpositive value and still have $q + dz_0$ remain in $\operatorname{pos} C(\alpha)$. Therefore, **6.3.1** will terminate in Step 2 with $z_0 \leq 0$. In this case, we see that both algorithms end with a solution to (q, M).

Suppose z_i is blocked in Step 1 of **4.4.5** by a variable other than z_0. We may assume without loss of generality that this other variable is w_j. It follows that the ray $\{q + dz_0 : z_0 \geq 0\}$ intersects $\operatorname{pos} C(\alpha)$ not only in the facet $\operatorname{pos} C(\alpha)_{\bullet\bar{\imath}}$ but also in the facet $\operatorname{pos} C(\alpha)_{\bullet\bar{\jmath}}$ and, moreover, the value of z_0 for which $\operatorname{pos} C(\alpha)_{\bullet\bar{\jmath}}$ contains $q + dz_0$ is positive. By convexity and by the nondegeneracy assumption on d, the ray $\{q + dz_0 : z_0 \geq 0\}$ intersects only these two facets of $\operatorname{pos} C(\alpha)$. Furthermore, the intersection of the ray and $\operatorname{pos} C(\alpha)$ must be a (possibly degenerate) line segment with one endpoint in $\operatorname{pos} C(\alpha)_{\bullet\bar{\imath}}$ and the other endpoint in $\operatorname{pos} C(\alpha)_{\bullet\bar{\jmath}}$. Clearly, the value of z_0 is always positive along this line segment.

Keeping this geometry in mind, we examine the behavior of Algorithm **6.3.1**. If $\bar{m}_{0i} \neq 0$, then the intersection of $\operatorname{pos} C(\alpha)$ and $\{q+dz_0 : z_0 \geq 0\}$ is a line segment of positive length. As before, we may conclude that $\operatorname{pos} C(\alpha)$ is full. Thus, in Step 1 of **6.3.1**, we would move z_0 in some direction. If $\bar{m}_{0i} > 0$, then we would increase z_0 until $q + dz_0$ was in $\operatorname{pos} C(\alpha)_{\bullet\bar{\jmath}}$. If $\bar{m}_{0i} < 0$, then we would decrease z_0 until $q + dz_0$ was in $\operatorname{pos} C(\alpha)_{\bullet\bar{\jmath}}$. In both cases we reach Step 2 of **6.3.1** with $0 < z_0 < \infty$. In addition, the distinguished cone and facet become $\operatorname{pos} C(\alpha \triangle \{j\})$ and $\operatorname{pos} C(\alpha \triangle \{j\})_{\bullet\bar{\jmath}}$, respectively.

If $\bar{m}_{0i} = 0$, then the point $q + dz_0$ does not change as we increase z_i. Therefore, this point is contained in both $\operatorname{pos} C(\alpha)_{\bullet\bar{\imath}}$ and $\operatorname{pos} C(\alpha)_{\bullet\bar{\jmath}}$ and, by the nondegeneracy assumption on d, this point is in the relative interior of each facet. We conclude that $\text{pos}\, C(\alpha)$ must be degenerate and, by Lemma **6.2.15**, it is weakly degenerate. Thus, in Step 1 of **6.3.1** we would set the distinguished cone and facet to be $\text{pos}\, C(\alpha \triangle \{j\})$ and $\text{pos}\, C(\alpha \triangle \{j\})_{\cdot \bar{j}}$, respectively.

In Algorithm **4.4.5**, in all cases, we would pivot on $\langle z_i, w_j \rangle$ to obtain $\text{pos}\, C(\alpha \triangle \{j\})$ as the new cone and $\text{pos}\, C(\alpha \triangle \{j\})_{\cdot \bar{j}}$ as the new facet. Hence, we have shown that Algorithms **4.4.5** and **6.3.1** follow the same sequence of cones and facets. □

Earlier in this section it was mentioned that Lemke's method follows a path of solutions. This path can be seen within the proof of Theorem **6.3.2**, but we will now explicitly describe it.

In Algorithm **4.4.5**, we view Step 0 as follows. We start with $z_0 = \infty$ and all other nonbasic variables equal to zero. Clearly, all basic variables acquire the value of infinity. We drive down the value of z_0 until it is blocked, i.e., until z_0 or a basic variable reaches zero. As we decrease z_0, for each value of z_0, the basic variables give us a solution to the LCP $(q+dz_0, M)$. In this initial tableau the solution given for each z_0 is $(w, z) = (q+dz_0, 0)$, and this ray of solutions is the beginning of the path of solutions.

As we discussed earlier in this section, we view Step 0 in Algorithm **6.3.1** as follows. The nonnegative orthant is the initial distinguished cone, and we start by decreasing z_0 from infinity. In this way the algorithm follows the solutions of the LCP $(q + dz_0, M)$ corresponding to the nonnegative orthant. This is analogous to the way in which we just described Step 0 of **4.4.5**. We see that the path of solutions starts out with the same primary ray of solutions in both **4.4.5** and **6.3.1**.

In general, for Algorithm **4.4.5**, we have a tableau and a driving variable. As we increase the driving variable from zero, for each value it takes on, the values acquired by the basic variables (including z_0) and the value of the driving variable give a solution to the LCP $(q + dz_0, M)$. From the relationship between tableaus and distinguished cones, this solution corresponds to the distinguished cone associated with the tableau. Thus, Lemke's method follows a path of solutions; not surprisingly, Algorithms **4.4.5** and **6.3.1** follow the same path.

6.3.3 Example. Consider

$$M = \begin{bmatrix} -2 & 4 \\ -4 & 2 \end{bmatrix}, \quad q = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \quad \text{and} \quad d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The complementary cones and the ray $\{q+dz_0 : z_0 \geq 0\}$ are given in Figure 6.1. As in Section 1.3, we label the (column) vectors $I_{\bullet 1}$, $I_{\bullet 2}$, $-M_{\bullet 1}$, and $-M_{\bullet 2}$ as 1, 2, $\overline{1}$, and $\overline{2}$, respectively. Again, the complementary cones are indicated by arcs around the origin. The path of solutions which Lemke's method follows is shown in Figure 6.2. Each point along the path represents the vector $w - z$ where (w, z) is a solution to $(q + dz_0, M)$ for some value of $z_0 \geq 0$. In both algorithms we imagine that z_0 starts at positive infinity.

When Algorithm **4.4.5** processes the LCP (q, M) it starts with the initial pivot $\langle z_0, w_1 \rangle$. We find that $z_0 = 3$, $z = (0, 0)$, and $w = (0, 2)$. We are now at point s in Figures 6.1 and 6.2. The next pivot is $\langle z_1, w_2 \rangle$. We find that $z_0 = 5$, $z = (1, 0)$, and $w = (0, 0)$. We are now at point r in Figures 6.1 and 6.2. The final pivot, which brings us to a solution, is $\langle z_0, z_2 \rangle$. We find that $z_0 = 0$, $z = (\frac{1}{6}, \frac{5}{6})$, and $w = (0, 0)$. We are now at point q in Figures 6.1 and 6.2.

When Algorithm **6.3.1** processes the LCP (q, M), it starts by setting $z_0 = 3$. We are at the point s in Figures 6.1 and 6.2. The distinguished cone is $\operatorname{pos} C(\{1\})$ and the distinguished facet is $\operatorname{pos} C(\{1\})_{\bullet\overline{1}}$. We next find the largest value of z_0 such that $q+dz_0$ remains in $\operatorname{pos} C(\{1\})$. This occurs when $z_0 = 5$. We are at the point r in Figures 6.1 and 6.2. The distinguished cone is $\operatorname{pos} C(\{1, 2\})$ and the distinguished facet is $\operatorname{pos} C(\{1, 2\})_{\bullet\overline{2}}$. We next find the smallest value of z_0 such that $q + dz_0$ remains in $\operatorname{pos} C(\{1, 2\})$. This occurs when $z_0 = -1$, thus we set $z_0 = 0$. We are at the point q in Figures 6.1 and 6.2. We have found that the complementary cone $\operatorname{pos} C(\{1, 2\})$ contains q and from this we may obtain a solution to (q, M).

6.3.4 Example. Consider

$$M = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad \text{and} \quad d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The complementary cones and the ray $\{q + dz_0 : z_0 \geq 0\}$ are given in Figure 6.3. The path of solutions which Lemke's method follows is shown in Figure 6.4. (Figures 6.1 thru 6.6 are not scaled identically.)

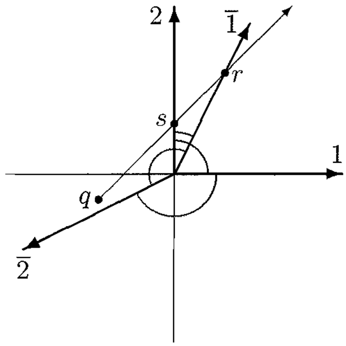

Figure 6.1

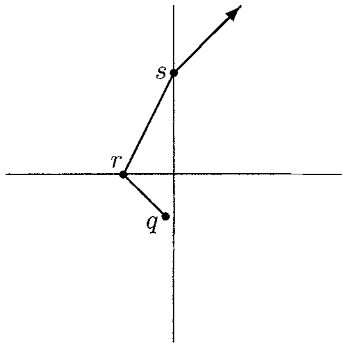

Figure 6.2

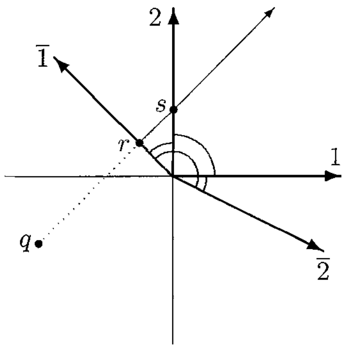

Figure 6.3

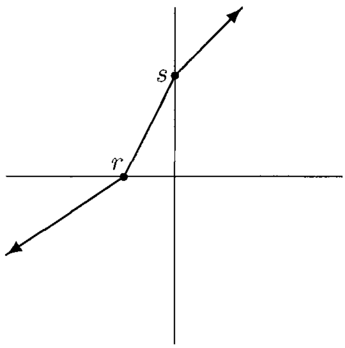

Figure 6.4

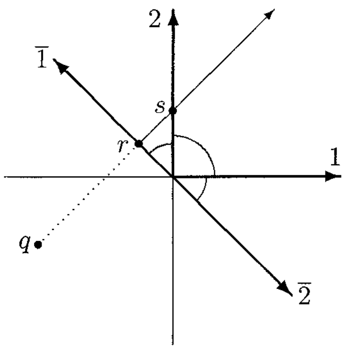

Figure 6.5

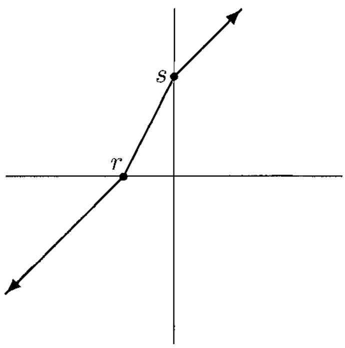

Figure 6.6

When Algorithm **4.4.5** processes the LCP (q, M) it starts with the initial pivot $\langle z_0, w_1\rangle$. We find that $z_0 = 2$, $z = (0, 0)$, and $w = (0, 1)$. We are now at point s in Figures 6.3 and 6.4. The next pivot is $\langle z_1, w_2\rangle$. We find that $z_0 = \frac{3}{2}$, $z = (\frac{1}{2}, 0)$, and $w = (0, 0)$. We are now at point r in Figures 6.3 and 6.4. The driving variable is now z_2 and it is unblocked. Lemke's method terminates without producing a solution. (In this example, the LCP has no solution.)

When Algorithm **6.3.1** processes the LCP (q, M) it starts by setting $z_0 = 2$. We are at the point s in Figures 6.1 and 6.2. The distinguished cone is $\operatorname{pos} C(\{1\})$ and the distinguished facet is $\operatorname{pos} C(\{1\})_{\bullet\bar{1}}$. We next find the smallest value of z_0 such that $q + dz_0$ remains in $\operatorname{pos} C(\{1\})$. This occurs when $z_0 = \frac{3}{2}$. We are at the point r in Figures 6.1 and 6.2. The distinguished cone is $\operatorname{pos} C(\{1, 2\})$ and the distinguished facet is $\operatorname{pos} C(\{1, 2\})_{\bullet\bar{2}}$. We next find the largest value of z_0 such that $q + dz_0$ remains in $\operatorname{pos} C(\{1, 2\})$. There is no largest value, i.e., $z_0 = \infty$. Thus, as seen in Figure 6.4, the algorithm ends with a secondary ray.

6.3.5 Example. Consider

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad \text{and} \quad d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The complementary cones and the ray $\{q + dz_0 : z_0 \geq 0\}$ are given in Figure 6.5. The path of solutions which Lemke's method follows is shown in Figure 6.6.

The description of what happens when Algorithm **4.4.5** processes this LCP (q, M) is exactly the same as in Example **6.3.4**.

The description of what happens when Algorithm **6.3.1** processes this LCP (q, M) is the same as in Example **6.3.4** except for the very end. When the point r is reached we find that the new distinguished cone, $\operatorname{pos} C(\{1, 2\})$, is strongly degenerate. As before, Algorithm **6.3.1** halts on a secondary ray. This ray cannot be seen in Figure 6.5 as the value of z_0 does not change once the strongly degenerate cone is reached. However, for this particular value of z_0, we see in Figure 6.6 the ray of solutions to $(q+dz_0, M)$ that are associated with the strongly degenerate cone $\operatorname{pos} C(\{1, 2\})$.

In Section 4.4 we introduced the almost complementary path of feasible solutions that Lemke's method follows as it processes the LCP (q, M). The path of solutions discussed here is just a different geometric realization of the almost complementary path. For example, in Figures 6.2, 6.4, and 6.6, each point x along the path corresponds to the solution $(w, z) = (x^+, x^-)$ of the LCP $(q + dz_0, M)$ for some value of z_0. As such, we may map x onto the point (x^+, z_0, x^-) of the almost complementary path. This mapping is a bijection between the path of solutions and the almost complementary path. Assuming d is selected as in Step 0 of **6.3.1**, the reader may verify that this mapping is a continuous bijection for any LCP (q, M).

In the proof of Theorem **4.4.4**, we noted that the almost complementary path never encounters the same point twice. Thus, the same must be true for the path of solutions. This means that the geometry seen in Figures 6.1, 6.3, and 6.5 is quite different from the geometry seen in Figures 6.2, 6.4, and 6.6. For example, in Figure 6.1, as Lemke's algorithm processes the LCP, z_0 takes on every value in the open interval $(3, 5)$ three times. In Figure 6.2 there is no repetition. Each time z_0 attains a given value, the current solution to $(q + dz_0, M)$ is different. The geometry in Figure 6.1 depicts the linear complementarity problems $(q + dz_0, M)$ which are encountered during Lemke's algorithm. These may repeat. The geometry in Figure 6.2 depicts solutions to the linear complementarity problems $(q + dz_0, M)$ which are encountered during Lemke's algorithm. These do not repeat.

Notice that Figures 6.4 and 6.6 are almost identical. Both figures show Algorithm **6.3.1** terminating with a secondary ray. Yet, Figures 6.3 and 6.5 are quite different. In Figure 6.3 we have z_0 increasing without bound as $q + dz_0$ moves to infinity along a ray within the full cone $\text{pos}\, C(\{1, 2\})$. In Figure 6.5 we have z_0 staying fixed at the value $\frac{3}{2}$. In some sense $q + dz_0$ is "absorbed" by the strongly degenerate cone $\text{pos}\, C(\{1, 2\})$. Thus, Algorithm **6.3.1** makes the distinction of whether a terminating secondary ray is associated with a full cone or a strongly degenerate cone. This is not a distinction made by Algorithm **4.4.5**.

In Step 0 of Algorithm **6.3.1** we impose a nondegeneracy requirement on d. This implies the nondegeneracy of all basic solutions of (4.4.5) having $z_0 > 0$. Thus, if we require Lemke's method to select z_0 as the blocking variable when that is possible, then all the results in Section 4.4 concerning **4.4.1** and **4.4.5** will still hold under the nondegeneracy requirement of **6.3.1**. In theory, this nondegeneracy assumption is not at all restrictive.

6.3.6 Proposition. Given an LCP (q, M), a vector $d > 0$, and a $\varepsilon > 0$, there exists a $\bar{d} > 0$, with $\|\bar{d} - d\| \leq \varepsilon$, such that all the intersections of $\mathcal{K}(M)$ with the open ray $\{q + \bar{d}z_0 : z_0 > 0\}$ are nondegenerate.

Proof. The affine hull of any facet of any complementary cone has dimension at most $n-1$ (Proposition **2.9.14**). Thus, the complement of the union of all such affine hulls is an open and dense set in R^n (Proposition **2.9.17**). Hence, there is a closed ball $\bar{B}$ in this complement such that $\bar{B} \subseteq R^n_{++}$ and $\|\bar{d} - d\| \leq \varepsilon$ for all $\bar{d} \in \bar{B}$. Since $\bar{B}$ is in the complement, for any $\bar{d} \in \bar{B}$, the ray $\{q + \bar{d}z_0 : z_0 > 0\}$ intersects each facet in at most one point.

It now follows that for some $\bar{z}_0 > 0$ the set $S \equiv \{q + \bar{d}z_0 : z_0 \geq \bar{z}_0, \bar{d} \in \bar{B}\}$ does not intersect any facet of any complementary cone. Clearly, the set S has a nonempty interior and, so, Theorem **6.2.7** implies there is a point $\bar{q} \in S$ such that all intersections of the line segment $\ell[q, \bar{q}]$ with $\mathcal{K}(M)$ are nondegenerate. If we take $\bar{d}$ to be the point in $\bar{B}$ for which the ray $\{q + \bar{d}z_0 : z_0 > 0\}$ contains $\bar{q}$, then all the intersections of this ray with $\mathcal{K}(M)$ are nondegenerate. □

In practice, one would not know in advance whether a particular d satisfied the nondegeneracy requirement of Step 0. However, Proposition **6.3.6** indicates that degeneracy can be resolved using techniques that perturb d.

Algorithm **6.3.1** and Theorem **6.3.2** may give the reader a sense of *déjà vu*. This is because several of the concepts involved have already been used in the material of the previous section when we were discussing the classification and properties of the different types of facets. In fact, historically, the behavior of Lemke's method has been an impetus and a guide in much of the work on the classification of facets.

The next result describes the connection between Lemke's method and the classification of facets. The reader should find that the statement and proof of the following theorem follow quite naturally from the material in Section 6.2 and our discussions here.

6.3.7 Theorem. Let the LCP (q, M) be given along with a positive vector d satisfying the nondegeneracy condition of Step 0 in Algorithm **6.3.1**. Suppose we process this LCP using Algorithm **6.3.1** and that at some point the full cone $\text{pos}\, C(\alpha)$ is the current distinguished complementary cone at the beginning of Step 1. We will find:

(a) Step 1 decreases z_0 if $\text{ind}(\text{pos}\, C(\alpha)) = 1$, and Step 1 increases z_0 if $\text{ind}(\text{pos}\, C(\alpha)) = -1$.

Let $\bar{z}_0$ be the value z_0 is set to in Step 1. Suppose that $0 < \bar{z}_0 < \infty$. Let $\text{pos}\, C(\alpha)_{\bullet \bar{\imath}}$ be the next distinguished facet obtained in Step 2. We will find:

(b) The facet $\text{pos}\, C(\alpha)_{\bullet \bar{\imath}}$ is proper, reflecting, or absorbing around $q + d\bar{z}_0$.

(c) If $\text{pos}\, C(\alpha)_{\bullet \bar{\imath}}$ is absorbing around $q + d\bar{z}_0$, then the value z_0 remains fixed at $\bar{z}_0$ throughout the rest of the algorithm and, further, the algorithm eventually encounters a strongly degenerate cone and terminates with a secondary ray.

(d) If $\text{pos}\, C(\alpha)_{\bullet \bar{\imath}}$ is proper around $q + d\bar{z}_0$, then the value of z_0 will be changed at least once more by the algorithm. If, in Step 1, z_0 increased (decreased), then the next time z_0 is changed it will again be an increase (decrease).

(e) If $\text{pos}\, C(\alpha)_{\bullet \bar{\imath}}$ is reflecting around $q + d\bar{z}_0$, then the value of z_0 will be changed at least once more by the algorithm. If, in Step 1, z_0 increased (decreased), then the next time z_0 is changed it will instead be a decrease (increase).

Proof. From the nondegeneracy assumption, we know $q + d\bar{z}_0$ is not in $\mathcal{L}(M)$. Thus, with $q + d\bar{z}_0$ taking the place of q, there exist sequences $j_0, \ldots, j_m$, and $\beta_0, \ldots, \beta_{m+1}$, as described in Proposition **6.2.18**. In addition, from Remark **6.2.21**, we know $\alpha = \beta_0$ and $i = j_0$.

Using properties (a)–(e) of Proposition **6.2.18**, we make the key observation that Algorithm **6.3.1** will generate at least $m + 1$ additional distinguished complementary cones after $\text{pos}\, C(\alpha)$ and these cones will be $\text{pos}\, C(\beta_1), \ldots, \text{pos}\, C(\beta_{m+1})$ in exactly that order. We will now see that the theorem is a simple consequence of this key observation.

Since $\text{pos}\, C(\beta_1), \ldots, \text{pos}\, C(\beta_m)$ are weakly degenerate, we note that Algorithm **6.3.1** will keep $z_0 = \bar{z}_0$ while it moves along this sequence of distinguished cones. Thus, the next possible time for when the algorithm will change the value of z_0 is when $\text{pos}\, C(\beta_{m+1})$ is the distinguished complementary cone.

Since $\text{pos}\, C(\alpha)$ is full, $\text{pos}\, C(\alpha)_{\bullet \bar{\imath}}$ cannot be isolated or cyclic around $q + d\bar{z}_0$. If $\text{pos}\, C(\alpha)_{\bullet \bar{\imath}}$ is absorbing, then $\text{pos}\, C(\beta_{m+1})$ must be strongly degenerate. Hence, Algorithm **6.3.1** keeps $z_0 = \bar{z}_0$ until $\text{pos}\, C(\beta_{m+1})$ becomes the distinguished cone. At this point the algorithm terminates with a secondary ray.

If $\text{pos}\, C(\alpha)_{\bullet\bar{\imath}}$ is proper (reflecting), then $\text{pos}\, C(\alpha)$ and $\text{pos}\, C(\beta_{m+1})$ lie on opposite sides (the same side) of $\text{pos}\, C(\alpha)_{\bullet\bar{\imath}}$. Thus, for $q + dz_0$ to remain in $\text{pos}\, C(\beta_{m+1})$, z_0 must move in the same direction (the opposite direction) as it had been moved when $\text{pos}\, C(\alpha)$ was the distinguished cone. All parts of the theorem have now been shown except for part (a).

As mentioned previously in this section, the nonnegative orthant is the initial distinguished cone and we start by decreasing z_0 from infinity. Note, $\text{ind}(\text{pos}\, C(\emptyset)) = 1$. By definition, if $\text{pos}\, C(\alpha)_{\bullet\bar{\imath}}$ is proper (reflecting), then $\text{pos}\, C(\alpha)$ and $\text{pos}\, C(\beta_{m+1})$ will have the same index (opposite indexes). Thus, using parts (d) and (e), part (a) follows by induction. □

6.3.8 Remark. Considering the behavior of Lemke's method, as described in Theorem **6.3.7**, the reader should now understand the meaning behind the terminology of *proper*, *reflecting*, and *absorbing* facets.

6.3.9 Corollary. Let the LCP (q, M) be given along with a positive vector d satisfying the nondegeneracy condition of Step 0 in Algorithm **6.3.1**. Suppose we process this LCP using Algorithm **6.3.1**. If $M \in \boldsymbol{P_0}$, then the value of z_0 is never increased by the algorithm.

Proof. In **6.3.1**, if the current distinguished cone is degenerate, then the value of z_0 does not change. If the index of the current distinguished cone is 1, then part (a) of Theorem **6.3.7** implies that Algorithm **6.3.1** will decrease z_0 in Step 1. These are the only two possibilities if $M \in \boldsymbol{P_0}$. □

As can be seen from Theorem **6.3.7**, there are strong ties between Algorithm **6.3.1** and the material presented in Sections 6.1 and 6.2. Theorem **6.3.2** shows that Lemke's method provides a strong connection between the algebraic and geometric aspects of the LCP. We will now use some of the ideas we have developed to obtain some further results and generalizations concerning Lemke's method.

A natural question to ask is when may one be certain that a solution to the LCP will be found by Lemke's method. Indeed, this question is addressed in Section 4.4. Looking at the problem geometrically, a fairly obvious answer would be that if there were no strongly degenerate cones and if the nonnegative orthant were the only complementary cone which contained $q + dz_0$ for arbitrarily large values of z_0, then Lemke's method must find a solution to the LCP (q, M). This leads us to the next result.

6.3.10 Theorem. If $M \in R^{n\times n}$ is d-regular for some $d > 0$, then for any $q \in R^n$ there is a $\bar{d} > 0$ arbitrarily close to d such that Algorithm **6.3.1**, using $\bar{d}$, will find a solution to the LCP (q, M).

Proof. Any regular matrix is in $\boldsymbol{R_0}$. Thus, M has no strongly degenerate complementary cones. Hence, if Algorithm **6.3.1** does not find a solution, it must be that at some point we have $z_0 = \infty$ and the algorithm terminates on a secondary ray in Step 2. If z_0 is set to infinity, then Theorem **6.3.7** implies that the index of the current distinguished complementary cone is -1. In addition, in order to have $z_0 = \infty$, this current distinguished complementary cone must contain $q + \bar{d}z_0$ for arbitrarily large values of z_0. Since $\text{ind}(\text{pos}\, C(\emptyset)) = 1$, the theorem will follow if we can show that $\text{pos}\, C(\emptyset)$, i.e., the nonnegative orthant, is the only full complementary cone which contains $q + \bar{d}z_0$ for z_0 arbitrarily large.

By Proposition **6.3.6**, we may select $\bar{d} > 0$ arbitrarily close to d such that $\bar{d}$ satisfies the nondegeneracy assumption in Step 0 of **6.3.1**. In addition, as complementary cones are closed, if we select $\bar{d}$ close enough to d, then $d \notin \text{pos}\, C(\alpha)$ will imply $\bar{d} \notin \text{pos}\, C(\alpha)$ for any complementary cone $\text{pos}\, C(\alpha)$. Since M is d-regular, $\text{pos}\, C(\emptyset)$ is the only complementary cone containing d. Thus, $\text{pos}\, C(\emptyset)$ is the only complementary cone containing $\bar{d}$. Therefore, suppose some full complementary cone, $\text{pos}\, C(\alpha)$, contains $q + \bar{d}z_0$ for arbitrarily large z_0. This means $C(\alpha)^{-1}(q + \bar{d}z_0) \geq 0$ for arbitrarily large z_0, hence $C(\alpha)^{-1}\bar{d} \geq 0$. Thus, $\bar{d} \in \text{pos}\, C(\alpha)$ and, hence, $\alpha = \emptyset$. □

An obvious corollary of the previous theorem is that $\boldsymbol{R} \subseteq \boldsymbol{Q}$. This result is already known to us, see (3.9.9). It is interesting that it came up again as a natural consequence of the geometry of Lemke's method.

In examining the proof of Theorem **6.3.10** we find that the requirement that M be d-regular was not fully utilized. What we actually used was the fact that M was pseudo-regular and that d was not contained in any complementary cone with index -1. Thus, we can immediately state a slight extension of **6.3.10**.

6.3.11 Corollary. Let $M \in \boldsymbol{R_0} \cap R^{n \times n}$ be given. Suppose there is a $d > 0$ such that no complementary cone containing d has index -1. It follows that for any $q \in R^n$ there is a $\bar{d} > 0$ arbitrarily close to d such that Algorithm **6.3.1**, using $\bar{d}$, will find a solution to the LCP (q, M). □

We can observe that if M and d satisfy the hypotheses of Corollary **6.3.11** then $M \in \boldsymbol{Q}$. Notice, we do not have to use the corollary itself to make this observation as it is easy to see that such an M must have a well-defined positive degree.

There is a further generalization of Theorem **6.3.10** that we can make. So far, we have required the covering vector d to be positive. In essence, this was to insure that $q + dz_0$ would be contained in the nonnegative orthant for all z_0 large enough. This, in turn, was to insure that we would have a distinguished complementary cone with which to start Algorithm **6.3.1**. However, both Algorithms **4.4.5** and **6.3.1** may be used with a covering vector d which is not positive, so long as we know of a full complementary cone which contains d in its interior. We would then only have to slightly change the way we initialize the algorithms.

To be specific, suppose we have a (not necessarily positive) vector d which satisfies the nondegeneracy assumption in Step 0 of **6.3.1**. If $\text{pos}\, C(\alpha)$ is a full complementary cone and if $d \in \text{int}(\text{pos}\, C(\alpha))$, then by an argument similar to the one given near the end of the proof of Theorem **6.3.10** we know that $q + dz_0$ is contained in $\text{pos}\, C(\alpha)$ for all z_0 large enough. Thus, we could make $\text{pos}\, C(\alpha)$ the initial distinguished cone with $z_0 = \infty$ and begin in Step 1 of **6.3.1** by decreasing z_0.

The corresponding change in Algorithm **4.4.5** is to begin by block pivoting on $M_{\alpha\alpha}$. We now use this new tableau as the initial tableau and begin with Step 0 of **4.4.5**. (Of course, we must remember that if **4.4.5** refers to variable z_i, for some $i \in \alpha$, then we must substitute variable w_i, and vice versa.)

Henceforth, if we refer to using Algorithms **4.4.5** or **6.3.1** with $d \not> 0$ as a covering vector, then we shall implicitly mean that the algorithms be amended in the manner just described.

By using a more general covering vector in the manner described above, we have not changed any essential part of Lemke's method. All the results in this section still hold. The only change to note is that part (a) of Theorem **6.3.7** should read that z_0 is decreasing (increasing) in Step 1 if the index of the current distinguished cone is the same as (different from) the index of the initial distinguished cone. When we assumed $d > 0$, the initial distinguished cone was $\operatorname{pos} C(\emptyset)$, and $\operatorname{ind}(\operatorname{pos} C(\emptyset)) = 1$.

Since we now know how to use Lemke's method with a more general covering vector, we may extend Corollary **6.3.11**.

6.3.12 Corollary. Let $M \in \boldsymbol{R_0} \cap R^{n\times n}$ be given. Suppose there is a d such that $d \in \operatorname{int}(\operatorname{pos} C(\alpha))$ for some full complementary cone $\operatorname{pos} C(\alpha)$ and, further, no complementary cone with index equal to $-\operatorname{ind}(\operatorname{pos} C(\alpha))$ contains d. It follows that for any $q \in R^n$ there is a $\bar{d} \in \operatorname{int}(\operatorname{pos} C(\alpha))$ arbitrarily close to d such that Algorithm **6.3.1**, using $\bar{d}$, will find a solution to the LCP (q, M). □

It is natural to ask if, given a pseudo-regular matrix M, it is always possible to find a vector d satisfying the hypothesis of Corollary **6.3.12**. The answer is no because, if such a d existed, the corollary would then imply that $M \in \boldsymbol{Q}$. However, a pseudo-regular matrix need not be in $\boldsymbol{Q}$. (For example, $-I \in \boldsymbol{R_0} \setminus \boldsymbol{Q}$.)

We can also determine that the answer is no by using degree theory and, as this will lead to some additional insights, we will now do so. From the proof of Theorem **6.3.10** we see that if d satisfies the hypothesis of Corollary **6.3.12**, then any vector $\bar{d}$ close enough to d will also satisfy the hypothesis. Thus, as $R^n \setminus \mathcal{K}(M)$ is dense (see **6.1.12**), we may assume that $d \notin \mathcal{K}(M)$ and, so, $\deg(d)$ is well-defined (see **6.1.8**). Therefore, our question concerning the existence of a vector d satisfying the hypothesis of Corollary **6.3.12** can be stated succinctly as: Given $M \in \boldsymbol{R_0}$, does there exist a $d \notin \mathcal{K}(M)$ such that $|\deg(d)| = |\operatorname{SOL}(d, M)| > 0$? Clearly, if $\deg M = 0$, the answer must be no. As we have seen, such matrices exist as any matrix in $\boldsymbol{R_0} \setminus \boldsymbol{Q}$ must have degree equal to zero.

In some sense the above discussion is not satisfying. We are asking whether a $d \notin \mathcal{K}(M)$ exists such that, one, $|\deg(d)| = |\operatorname{SOL}(d, M)|$ and, two, $\deg(d) \neq 0$. Naturally, the second condition fails if $\deg M = 0$. Yet, the first condition is the more interesting condition. If we restrict our attention to matrices with nonzero degree, then we need only worry about the first condition. Since, clearly, we have $|\deg M| = |\deg(d)| \leq |\operatorname{SOL}(d, M)|$

for all $d \notin \mathcal{K}(M)$, we are led to the question: Given $M \in \boldsymbol{R_0}$, is it true that for all $d \notin \mathcal{K}(M)$ we have $|\deg M| < |\operatorname{SOL}(d, M)|$? If the answer to this question is yes, we say that the matrix M is *superfluous*.

At this point, the reader may wish to consider several simple examples of pseudo-regular matrices to see if any are superfluous. In this way, the reader will be convinced that it is not at all obvious that superfluous matrices exist. One might hope that there are no superfluous matrices as then Corollary **6.3.12** would imply that for every matrix M with nonzero degree there is some covering vector d which can be used with Lemke's method to find a solution to the LCP (q, M) for any $q \in R^n$. However, as we shall see in Section 6.7, superfluous matrices exist and examples can be found with arbitrarily large degrees. After all our investigation, it seems that in a very real sense it is not always possible to find a vector d satisfying the hypothesis of Corollary **6.3.12**.

The common assumption in Theorem **6.3.10** and its corollaries is that $M \in \boldsymbol{R_0}$. This seems to be a necessary assumption because otherwise strongly degenerate cones would exist, and there would likely be a q for which Lemke's method would terminate with a strongly degenerate cone before z_0 reaches zero. However, this problem is serious only if there is a solution to the LCP (q, M). If we knew that Lemke's method terminates on a secondary ray only for those q in which the LCP (q, M) has no solution, then we could still claim that Lemke's method properly processes (q, M) for all vectors q. With this in mind, we may extend Corollary **6.3.12** as follows.

6.3.13 Theorem. Let $M \in \boldsymbol{Q_0} \cap R^{n \times n}$ be given. Suppose there is a d such that $d \in \operatorname{int}(\operatorname{pos} C(\alpha))$ for some full complementary cone $\operatorname{pos} C(\alpha)$ and, further, no complementary cone with index equal to $-\operatorname{ind}(\operatorname{pos} C(\alpha))$ contains d. If, in addition, no point in the interior of $K(M)$ is in a strongly degenerate cone, then for any $q \in R^n$ there is a $\bar{d} \in \operatorname{int}(\operatorname{pos} C(\alpha))$ arbitrarily close to d such that Algorithm **6.3.1**, using $\bar{d}$, will find a solution to the LCP (q, M), if a solution exists.

Proof. By an argument similar to the one given in the proof of Theorem **6.3.10**, we may select $\bar{d}$ arbitrarily close to d such that $\bar{d}$ satisfies the nondegeneracy assumption in Step 0 of Algorithm **6.3.1** and also satisfies, like d itself, the conditions in the hypothesis of this theorem. Further, as in the proof of Theorem **6.3.10**, it follows that Algorithm **6.3.1**, using $\bar{d}$ as covering vector, cannot terminate on a secondary ray in Step 2. Thus, if the algorithm does not find a solution to (q, M), then it must encounter a strongly degenerate distinguished cone and terminate on a secondary ray in Step 1. Suppose this occurs.

If $\bar{z}_0 > 0$ is the value of z when the algorithm terminates, then Theorem **6.3.7** implies that $q + \bar{d}\bar{z}_0$ is contained in an absorbing facet. As the ray $\{q + \bar{d}z_0 : z_0 > 0\}$ has only nondegenerate intersections with $\mathcal{K}(M)$, there is a hyperplane H and an open ball $B(q + \bar{d}\bar{z}_0, \delta)$ as described in Definition **6.2.5**. As usual, let B^+ and B^- represent the two open hemiballs which are the connected components of $B(q + \bar{d}\bar{z}_0, \delta) \setminus H$. The argument at the beginning of the proof of Theorem **6.2.12** shows that if B^+ intersects a full complementary cone then it must be contained in the full cone. A similar statement is true for B^-.

If both B^+ and B^- were contained in full cones, then $q + \bar{d}\bar{z}_0$ would be in the interior of $K(M)$. This is not possible as $q + \bar{d}\bar{z}_0$ is in a strongly degenerate cone. Thus, as $q+\bar{d}\bar{z}_0$ is in an absorbing facet, **6.2.25(f)** implies that exactly one of the hemiballs is contained in $K(M)$. We may assume $B^+ \subseteq K(M)$ and $B^- \cap K(M) = \emptyset$.

From **6.2.5(b)**, we know that $q + \bar{d}\bar{z}_0$ is the only point on the ray $\{q + \bar{d}z_0 : z_0 > 0\}$ contained in H. Thus, by slightly increasing z_0 from $\bar{z}_0$, or slightly decreasing z_0, we will cause $q + \bar{d}z_0$ to be contained in B^-. As $M \in \boldsymbol{Q_0}$, it follows from **3.2.1** that either $\{q+\bar{d}z_0 : z_0 > \bar{z}_0\} \cap K(M) = \emptyset$ or $\{q+\bar{d}z_0 : z_0 < \bar{z}_0\} \cap K(M) = \emptyset$. It clearly cannot be the former because, as in the proof of Theorem **6.3.10**, we know $q + \bar{d}z_0 \in \operatorname{pos} C(\alpha)$ for arbitrarily large z_0. Thus, $\{q + \bar{d}z_0 : z_0 < \bar{z}_0\} \cap K(M) = \emptyset$. Hence, $\mathrm{SOL}(q, M) = \emptyset$. □

We will end this section by proving that any $\boldsymbol{L}$-matrix satisfies the hypothesis of Theorem **6.3.13** with $d = e$. Hence, the $\boldsymbol{L}$-matrices can be processed by Lemke's method without ambiguity in the outcome. We first need to prove a lemma which is interesting enough to be given as a separate theorem. We will find additional use for this next result in Section 6.4.

6.3.14 Theorem. Let $M \in R^{n \times n}$ be an $\boldsymbol{L}$-matrix. If q is in a strongly degenerate complementary cone, then q is in the boundary of $K(M)$.

Proof. Let $\operatorname{pos} C(\alpha)$ be a strongly degenerate complementary cone containing q. Since $\operatorname{pos} C(\alpha)$ is strongly degenerate, there is a $v \in R^n$ such that $0 \neq v \in \mathrm{SOL}(0, M)$ with $v_{\bar{\alpha}} = 0$ and $(Mv)_\alpha = 0$. Since $q \in \operatorname{pos} C(\alpha)$, there is a $z \in \mathrm{SOL}(q, M)$ with $z_{\bar{\alpha}} = 0$ and $(Mz + q)_\alpha = 0$. Clearly, $q \in K(M)$ as it is in a complementary cone. We must show $q \notin \operatorname{int} K(M)$.

As $M \in \boldsymbol{L}$, we have $v \geq x \geq 0$ and $Mv \geq -M^{\mathrm{T}}x \geq 0$ for some $x \neq 0$. (Using Definition **3.9.18**, let $x = D_2 v$ and notice we may assume that D_1 and D_2 have been scaled so that no element is larger than one.) Immediately we see $x_{\bar{\alpha}} = 0$ and $(M^{\mathrm{T}}x)_\alpha = 0$. Now, as $Mv + M^{\mathrm{T}}x \geq 0$ and $z \geq 0$, we have $z^{\mathrm{T}}Mv + x^{\mathrm{T}}Mz \geq 0$. However, $z^{\mathrm{T}}Mv = 0$, so $x^{\mathrm{T}}Mz \geq 0$. Yet, as $z \geq 0$ and $-M^{\mathrm{T}}x \geq 0$, we have $x^{\mathrm{T}}Mz \leq 0$, so $x^{\mathrm{T}}Mz = 0$. Thus, as we must have $x^{\mathrm{T}}(Mz + q) = 0$, it follows that $x^{\mathrm{T}}q = 0$.

Suppose $\mathrm{SOL}(\bar{q}, M) \neq \emptyset$. Thus, there is a $\bar{z} \geq 0$ such that $M\bar{z} + \bar{q} \geq 0$. As $x \geq 0$, we have $x^{\mathrm{T}}M\bar{z} + x^{\mathrm{T}}\bar{q} \geq 0$. We see that $x^{\mathrm{T}}M\bar{z} \leq 0$, hence $x^{\mathrm{T}}\bar{q} \geq 0$. Therefore, $x^{\mathrm{T}}(\bar{q} - q) \geq 0$. As $x \neq 0$, this cannot be true for all $\bar{q}$ in an open ball around q. Thus, $q \notin \operatorname{int} K(M)$. □

6.3.15 Corollary. If $M \in R^{n \times n}$ is an $\boldsymbol{L}$-matrix, then M satisfies the hypothesis of Theorem **6.3.13** with $d = e$ and $\alpha = \emptyset$.

Proof. We know $\boldsymbol{L} \subseteq \boldsymbol{Q}_0$ from Corollary **3.9.19**. Clearly, e is in the interior of $\operatorname{pos} C(\emptyset)$ and, as $\boldsymbol{L} \subseteq \boldsymbol{E}_0$, we know $|\mathrm{SOL}(e, M)| = 1$ so no other complementary cone contains e (see **3.9.3**). Theorem **6.3.14** now completes the proof. □

6.4 LCP Existence Theorems

In this section, we will highlight and illustrate the material we have so far developed in this chapter. To this end we will show that the augmented LCP, given in (3.7.1) and (3.7.2), and the bimatrix game LCP, given in (4.4.16) and (4.4.17), always have solutions. In addition, we will prove an existence result which will be used in Chapter 7.

Given any LCP (q, M) of order n, and given any $d \in R^n$ and $\lambda \in R$ with $d > 0$ and $\lambda \geq 0$, we can construct the augmented LCP $(\tilde{q}, \tilde{M})$ as indicated in (3.7.1) and (3.7.2). Theorem **3.7.3** guarantees that this augmented LCP will have a solution. (See also Theorem **4.4.4**.) Our next task will be to prove Theorem **3.7.3** using geometric techniques.

First, let us pictorially understand what it is that Theorem **3.7.3** says. To do this, notice that if $\lambda < 0$, then the augmented LCP $(\tilde{q}, \tilde{M})$ cannot have a solution as the last row of $\tilde{M}$ is nonpositive. Thus, $K(\tilde{M})$ is contained in the closed halfspace

$$H_+ = \{x \in R^{n+1} : x_{n+1} \geq 0\}.$$

Theorem **3.7.3** goes on to say that $K(\tilde{M}) = H_+$.

6.4.1 Theorem. The augmented LCP $(\tilde{q}, \tilde{M})$, as given in (3.7.1) and (3.7.2), has a solution. ($K(\tilde{M}) = H_+$.)

Proof. We will start by giving a geometric description of the ideas behind the proof. The first thing to note is that the complementary cone $\operatorname{pos} C(\{n+1\})$ is simply the hyperplane

$$H_0 = \{x \in R^{n+1} : x_{n+1} = 0\},$$

which is the boundary of H_+. Clearly, this complementary cone is strongly degenerate. A key observation is that none of the other complementary cones are strongly degenerate. To see this, note that $(I, -\tilde{M})$ has no zero column. Theorem **6.1.19** then implies that a complementary cone will be strongly degenerate if and only if it contains a line. Yet, as $K(\tilde{M}) \subseteq H_+$, if a complementary cone contains a line, then that line must lie in H_0. We now note that $C(\{n+1\})_{\bullet \bar{\imath}}$ has rank n for all $i = 1, \ldots, n+1$. Therefore, as all the column vectors of the complementary matrix $C(\{1, \ldots, n\})$ lie in the interior of H_+, it follows that no complementary cone with any of these columns as generators can contain a line in H_0. Thus, $\operatorname{pos} C(\{n+1\})$ is the only strongly degenerate complementary cone.

We can now imagine keeping track of the parity of the number of solutions to the LCP $(\tilde{q}, \tilde{M})$ as $\tilde{q}$ moves into H_+ from the outside. Initially, when $\tilde{q}$ is outside H_+, the LCP has no solution and, hence, the parity is even. The parity changes to odd when H_0, the boundary of H_+, is crossed. This happens because $\tilde{q}$ passes through exactly one strongly degenerate complementary cone (see Theorem **6.2.27**). The point $\tilde{q}$ can now move anywhere within H_+ without crossing H_0 again. Thus, it can move anywhere within H_+ without intersecting a strongly degenerate cone, and so the parity of the number of solutions will remain odd. We conclude that at least one solution must exist for any $\tilde{q}$ in H_+.

Now that the ideas behind the proof have been discussed, we can fill in the details. The complementary cone $\operatorname{pos} C(\{n+1\})$ is strongly degenerate as, letting $x_i = d_i$ for $i = 1, \ldots, n$, and $x_{n+1} = 1$, we have $C(\{n+1\})x = 0$ and $x > 0$.

Next, consider some $\alpha \neq \{n+1\}$ such that there is an $x \in R^{n+1}$ with $C(\alpha)x = 0$ and $x \geq 0$. Since the last row of $C(\alpha)$ is nonnegative, x_i must be zero for all i such that $C(\alpha)_{n+1,i} > 0$. Thus, letting $\beta = \alpha \setminus \{n+1\}$, we must have $x_\beta = 0$. If $n+1 \notin \alpha$, then $\beta = \alpha$ and, as $C(\alpha)_{\bullet\bar{\alpha}} = I_{\bullet\bar{\alpha}}$, we must have $x = 0$. If $n+1 \in \alpha$, then $i \in \alpha$ for some $i = 1, \ldots, n$, as $\alpha \neq \{n+1\}$. Thus, the only nonzero element in row i of $C(\alpha)_{\bullet\bar{\beta}}$ is $C(\alpha)_{i,n+1} = -d_i < 0$. Hence, $x_{n+1} = 0$. As before, since we know $x_\alpha = 0$, we conclude $x = 0$ from the fact that $C(\alpha)_{\bullet\bar{\alpha}} = I_{\bullet\bar{\alpha}}$. We have shown that, except for $\operatorname{pos} C(\{n+1\})$, no complementary cone is strongly degenerate.

For $\delta \in R$, we define the matrix $\tilde{M}(\delta) \in R^{(n+1)\times(n+1)}$ as

$$\tilde{M}(\delta) \;=\; \tilde{M} + \delta e_{n+1} e_{n+1}^{\mathrm{T}} \,.$$

By Theorem **6.1.25**, we know that for all $\delta > 0$ small enough, none of the complementary cones of $\tilde{M}(\delta)$, except for (possibly) $\operatorname{pos} C(\{n+1\})$, will be strongly degenerate. As for $\operatorname{pos} C(\{n+1\})$, it is nondegenerate when $\delta > 0$ as $\det C(\{n+1\}) = -\delta$. Thus, $\tilde{M}(\delta)$ has no strongly degenerate complementary cones, hence it has a well-defined degree. We will now calculate this degree.

Let α be any index set not equal to $\{n+1\}$. As $\operatorname{pos} C_{\tilde{M}}(\alpha)$ is not strongly degenerate, and as it does not contain $-e_{n+1}$, then by Theorem **6.1.23** we know that $-e_{n+1} \notin \operatorname{pos} C_{\tilde{M}(\delta)}(\alpha)$ for all small enough $\delta > 0$. However, letting $x_{n+1} = 1/\delta$ and letting $x_i = d_i/\delta$ for $i = 1, \ldots, n$, we have $x > 0$ and $C_{\tilde{M}(\delta)}(\{n+1\})x = -e_{n+1}$. Thus, $-e_{n+1}$ is in exactly one complementary cone. Furthermore, that one cone is full and $-e_{n+1}$ is in its interior. Thus, for all $\delta > 0$ small enough, $\deg \tilde{M}(\delta) = \operatorname{sgn}(\det \tilde{M}(\delta)_{n+1,n+1}) = 1$.

Consider $\tilde{q} \in R^{n+1}$ with $\tilde{q}_{n+1} > 0$. Clearly, $\tilde{q} \notin \operatorname{pos} C_{\tilde{M}(\delta)}(\{n+1\})$ for $\delta > 0$. Thus, as $\deg \tilde{M}(\delta) = 1$ for $\delta > 0$ small enough, $\tilde{q}$ must be contained in at least one of the other complementary cones. In fact, since there are only finitely many complementary cones, there is some $\alpha \neq \{n+1\}$ such that $\tilde{q} \in \operatorname{pos} C_{\tilde{M}(\delta)}(\alpha)$ for arbitrarily small $\delta > 0$. As $\operatorname{pos} C_{\tilde{M}}(\alpha)$ is not strongly degenerate, Theorem **6.1.23** implies $\tilde{q} \in \operatorname{pos} C_{\tilde{M}}(\alpha)$. Therefore, $\operatorname{int} H_+ \subseteq K(\tilde{M})$ and so, being closed, $K(\tilde{M})$ must contain H_+. □

We now turn our attention to proving that the bimatrix game LCP has a solution. (See also Theorem **4.4.22**.)

6.4.2 Theorem. For any positive $m \times n$ matrices A and B, the bimatrix game LCP, as given in (4.4.16) and (4.4.17), has a solution.

Proof. The first thing we must do is determine which complementary cones are strongly degenerate. Let $q \in R^{m+n}$ and $M \in R^{(m+n)\times(m+n)}$ be defined as in (4.4.17). It will be convenient to define the index sets $\beta = \{1, \ldots, m\}$ and $\bar{\beta} = \{m+1, \ldots, m+n\}$. Consider an index set $\alpha \subseteq \{1, \ldots, m+n\}$. If $\alpha = \emptyset$, then $\operatorname{pos} C(\alpha)$ is, of course, not strongly degenerate. Suppose $\emptyset \neq \alpha \subseteq \beta$. Define $x \in R^{m+n}$ by letting $x_\alpha = e$, $x_{\beta\setminus\alpha} = 0$, and $x_{\bar{\beta}} = (B^{\mathrm{T}}_{\bullet\alpha})e$. We have $0 \neq x \geq 0$ and $C(\alpha)x = 0$. Thus, $\operatorname{pos} C(\alpha)$ is strongly degenerate. A similar argument shows that $\operatorname{pos} C(\alpha)$ is strongly degenerate if $\emptyset \neq \alpha \subseteq \bar{\beta}$.

We now assume $\alpha \cap \beta \neq \emptyset$ and $\alpha \cap \bar{\beta} \neq \emptyset$. Suppose $C(\alpha)x = 0$ for some $x \geq 0$. If $i \in \alpha \cap \beta$, then $C(\alpha)_{ij} < 0$ if $j \in \alpha \cap \bar{\beta}$ and $C(\alpha)_{ij} = 0$ if $j \notin \alpha \cap \bar{\beta}$. Thus, $x_{\alpha\cap\bar{\beta}} = 0$. Similarly, if $i \in \alpha \cap \bar{\beta}$, then $C(\alpha)_{ij} < 0$ if $j \in \alpha \cap \beta$ and $C(\alpha)_{ij} = 0$ if $j \notin \alpha \cap \beta$. Thus, $x_{\alpha\cap\beta} = 0$, hence $x_\alpha = 0$. Since $C(\alpha)_{\bullet\bar{\alpha}} = I_{\bullet\bar{\alpha}}$, this implies $x = 0$. Therefore, $\operatorname{pos} C(\alpha)$ is not strongly degenerate. We have now determined exactly which complementary cones are strongly degenerate.

For $\delta \geq 0$, define $M(\delta) \in R^{(m+n)\times(m+n)}$ as

$$M(\delta) = \begin{bmatrix} \delta I & A \\ B^{\mathrm{T}} & \delta I \end{bmatrix}.$$

For all $\delta > 0$ small enough, Theorem **6.1.25** implies that $\operatorname{pos} C_{M(\delta)}(\alpha)$ is not strongly degenerate if $\alpha = \emptyset$ or if $\alpha \cap \beta \neq \emptyset \neq \alpha \cap \bar{\beta}$. If $\emptyset \neq \alpha \subseteq \beta$, or if $\emptyset \neq \alpha \subseteq \bar{\beta}$, then $\det M(\delta)_{\alpha\alpha} = \delta^{|\alpha|} > 0$, so $\operatorname{pos} C_{M(\delta)}(\alpha)$ is not strongly degenerate. Thus, for all $\delta > 0$ small enough, $M(\delta)$ has no strongly degenerate complementary cones. Therefore, $\deg M(\delta)$ is well-defined.

It turns out that this degree is easy to calculate. Since $M(\delta) \geq 0$, the LCP $(e, M(\delta))$ has exactly one solution. This solution is $(w, z) = (e, 0)$ and it corresponds to the complementary cone $\operatorname{pos} C_{M(\delta)}(\emptyset)$. Since $q = e$ is in the interior of this cone, $\deg M(\delta) = \operatorname{sgn}(\det C_{M(\delta)}(\emptyset)) = \operatorname{sgn}(\det I) = 1$.

The question arises as to which complementary cones of $M(\delta)$ contain $-e$. We will show that, for all $\delta > 0$ small enough, $-e \in \operatorname{pos} C_{M(\delta)}(\beta)$. Define $x \in R^{m+n}$ by

$$x_i = \begin{cases} 1/\delta & \text{if } i \in \beta, \\ -1 + (B^{\mathrm{T}}e)_{i-m}/\delta & \text{if } i \in \bar{\beta}. \end{cases}$$

One can easily check that $C_{M(\delta)}(\beta)x = -e$, and that $x > 0$ for all $\delta > 0$ small enough. Thus, $-e$ is in the interior of $\operatorname{pos} C_{M(\delta)}(\beta)$ for all $\delta > 0$ small enough. By a similar argument, it can be shown that $-e$ is in the interior of $\operatorname{pos} C_{M(\delta)}(\bar{\beta})$ for all $\delta > 0$ small enough.

If $\alpha \subset \beta$, then $-e \notin \operatorname{pos} C_{M(\delta)}(\alpha)$. This can be seen from the fact that $C_{M(\delta)}(\alpha)_{i\cdot} \geq 0$ for any $i \in \beta \setminus \alpha$. By similar reasoning, if $\alpha \subset \bar{\beta}$, then $-e \notin \operatorname{pos} C_{M(\delta)}(\alpha)$.

If the only complementary cones containing $-e$ were $\operatorname{pos} C_{M(\delta)}(\beta)$ and $\operatorname{pos} C_{M(\delta)}(\bar{\beta})$, then

$$\deg M(\delta) = \operatorname{sgn}(\det M(\delta)_{\beta\beta}) + \operatorname{sgn}(\det M(\delta)_{\bar{\beta}\bar{\beta}}) = 2\,.$$

However, $\deg M(\delta) = 1$. Thus, for each $\delta > 0$ small enough, there exists an α such that $\alpha \cap \beta \neq \emptyset \neq \alpha \cap \bar{\beta}$ and, in addition, $-e \in \operatorname{pos} C_{M(\delta)}(\alpha)$.

As in the proof of Theorem **6.4.1**, since there are only finitely many complementary cones, there is some α, where $\alpha \cap \beta \neq \emptyset \neq \alpha \cap \bar{\beta}$, such that $-e \in \operatorname{pos} C_{M(\delta)}(\alpha)$ for arbitrarily small $\delta > 0$. As $\operatorname{pos} C_M(\alpha)$ is not strongly degenerate, Theorem **6.1.23** implies $-e \in \operatorname{pos} C_M(\alpha)$. Thus, the bimatrix game LCP has a solution. □

We will now prove a general perturbation theorem concerning local degree. Afterwards we will specialize this to the case of $\boldsymbol{L}$-matrices. The same result will also be used in Section 7.5.

6.4.3 Theorem. Let $M \in R^{n\times n}$ be given. Let $\mathcal{S}$ be the set of all points $q \in R^n$ for which $\deg_M(q)$ is well-defined and nonzero. If $q \in \operatorname{cl} \mathcal{S}$ and if q is not in any strongly degenerate complementary cone, then there exists an $\varepsilon > 0$ such that $\mathrm{SOL}(q', M') \neq \emptyset$ for all $M' \in R^{n\times n}$ and all $q' \in R^n$ such that $\|M - M'\| + \|q - q'\| \leq \varepsilon$.

Proof. We will assume throughout that $q' \in R^n$ and $M' \in R^{n\times n}$ with $\|q - q'\| + \|M - M'\| \leq \varepsilon$.

Consider the collection of index sets α for which $q \in \operatorname{pos} C_M(\alpha)$. Denote this collection as $\{\alpha_i\}_{i=1}^k$. By assumption, for all $i = 1, \ldots, k$, the complementary cone $\operatorname{pos} C_M(\alpha_i)$ is not strongly degenerate. Let

$$D = \operatorname{int}\Big(\bigcup_{i=1}^{k} \operatorname{pos} C_I(\alpha_i) \Big) .$$

Suppose, for some $x \in R^n$, that $f_M(x) = q$. If $\operatorname{pos} C_I(\alpha)$ is any of the orthants containing x, then $q \in \operatorname{pos} C_M(\alpha)$ and, so, $\alpha = \alpha_i$ for some $i = 1, \ldots, k$. It follows that $x \in D$. Hence, $f_M^{-1}(q) \subseteq D$.

We claim there is an $\varepsilon > 0$ which guarantees that $q' \notin f_{M'}(\operatorname{bd} D)$. If our claim is false, then there is a sequence of matrices, $\{M_i\}$, with limit M, and two sequences of vectors, $\{q^i\}$ and $\{x^i\}$, with $\lim_{i \to \infty} q^i = q$, such that $x^i \in \operatorname{bd} D$ and $f_{M_i}(x^i) = q^i$ for all i. There are two possibilities depending on whether or not the sequence $\{x^i\}$ has a bounded subsequence.

Suppose $\{x^i\}$ has a bounded subsequence. Thus, $\{x^i\}$ will have a convergent subsequence. Hence, we may assume that $\lim_{i \to \infty} x^i = x$ for some $x \in R^n$. It follows that $f_M(x) = q$ (see Exercise **6.10.6**). However, like all boundaries, $\operatorname{bd} D$ is closed. Thus, $x \in \operatorname{bd} D$ which contradicts the fact that $f_M^{-1}(q) \subseteq D = \operatorname{int} D$.

Suppose $\{x^i\}$ has no bounded subsequence. We may then assume that $\lim_{i \to \infty} \|x^i\| = \infty$ and that $\lim_{i \to \infty} x^i/\|x^i\| = x$ where $\|x\| = 1$. Notice, since $x^i \in \operatorname{cl} D$, that $x^i/\|x^i\| \in \operatorname{cl} D$ and, so, $x \in \operatorname{cl} D$. As $f_{M_i}(x^i/\|x^i\|) = q^i/\|x^i\|$, we take limits and obtain $f_M(x) = 0$. Since $0 \neq x \in \operatorname{cl} D$, this would imply that one of the complementary cones containing q is strongly degenerate. This is a contradiction and it follows that our above claim is true.

We now define the maps $g_M, g_{M'} : \operatorname{cl} D \to R^n$ to be the restrictions, respectively, of f_M and $f_{M'}$ to $\operatorname{cl} D$. From Theorem **6.1.25** we may deduce that if $\varepsilon > 0$ is small enough, then $\operatorname{pos} C_{M'}(\alpha_i)$ is not strongly degenerate, for any $i = 1, \ldots, k$. Our previous claim showed that ε can be selected so that $q' \notin g_{M'}(\operatorname{bd} D)$. We may now invoke Theorem **6.1.21** to show that if q' has a well-defined local degree under both g_M and $g_{M'}$, then the two local degrees have the same value.

Since complementary cones are closed, we may take $\varepsilon > 0$ to be so small that any complementary cone (relative to M) containing q' must contain q. This implies that q' is not contained in any strongly degenerate complementary cone relative to M. Thus, we may use Theorem **6.1.17** to show that $\deg_M(q')$, if it exists, does not depend on the specific q' we select, as long as we have $\|q - q'\| < \varepsilon$. Furthermore, since $q \in \operatorname{cl} \mathcal{S}$, we then know $\deg_M(q') \neq 0$, if it exists.

In addition to the above, by taking $\varepsilon > 0$ small enough to guarantee that any complementary cone (relative to M) containing q' must contain q, we have $f_M^{-1}(q') = g_M^{-1}(q')$. This, means that, if they exist, the local degree of g_M at q' equals $\deg_M(q')$.

Using Proposition **2.9.17** as in Theorem **6.1.12**, we find that the complement of the union $\mathcal{K}(M) \cup \mathcal{K}(M')$ is dense in R^n. This is the set of points that have a well-defined local degree for both f_M and $f_{M'}$. These points also have a well-defined local degree for both, g_M and $g_{M'}$. (From Definition **2.9.4** it is clear that if f_M and $f_{M'}$ have well-defined local degrees at a point, so do the restrictions g_M and $g_{M'}$.)

We can now bring things together to finish the proof. We know that if $\varepsilon > 0$ is small enough, then the local degrees, if they exist, of f_M, g_M, and $g_{M'}$ at q' are all equal and nonzero. We know that the set of q' for which these degrees exist is dense among all q' for which $\|q - q'\| < \varepsilon$. If the local degree of $g_{M'}$ at q' exists and is not zero, then $g_{M'}^{-1}(q') \neq \emptyset$, so $q' \in K(M')$. Since $K(M')$ is closed, we conclude that $q' \in K(M')$ for all q' such that $\|q - q'\| < \varepsilon$. □

To conclude this section, we will apply **6.4.3** to the class $\boldsymbol{L}$.

6.4.4 Theorem. Let $M \in R^{n \times n}$ and $q \in R^n$ be given. If $M \in \boldsymbol{L}$ and if $\mathrm{SOL}(q, M)$ is nonempty and bounded, then there exists an $\varepsilon > 0$ such that for all $M' \in R^{n \times n}$ and all $q' \in R^n$ for which $\|M - M'\| + \|q - q'\| \leq \varepsilon$, we have $\mathrm{SOL}(q', M') \neq \emptyset$.

Proof. In light of Corollary **3.9.19**, we know that $K(M) = \operatorname{pos}(I, -M)$ is a closed convex cone. We know, from Theorem **6.3.14**, all strongly degenerate complementary cones are contained in the boundary of $K(M)$. Thus, if $\mathcal{C}$ is the union of the strongly degenerate complementary cones, then $\operatorname{int} K(M)$ is contained in a connected component of $R^n \setminus \mathcal{C}$. It now follows from Theorem **6.1.17** that the local degree is invariant over all points in $\operatorname{int} K(M)$ for which a local degree is defined.

Since $e \in \operatorname{int} R^n_+ \subseteq K(M)$, we have $e \in \operatorname{int} K(M)$. We will now show that $\deg(e) = 1$ and, hence, that the local degree is nonzero at all points in int $K(M)$ where it is defined. Since $\boldsymbol{L} \subseteq \boldsymbol{E}_0$, it follows from Theorem **3.9.3** that $(w, z) = (e, 0)$ is the only solution of the LCP (e, M). We conclude that $\deg(e) = 1$.

We now know the local degree is nonzero at all points in int $K(M)$ where it is defined. Since local degree is well-defined on a dense set (see **6.1.12**), any point in int $K(M)$ is in the closure of the set of all points with a well-defined nonzero local degree. Since $K(M)$ is convex, all of $K(M)$ is in this closure.

As SOL(q, M) is nonempty, we have $q \in K(M)$. In addition SOL(q, M) is bounded, so Theorem **6.1.27** implies that q is not in any strongly degenerate complementary cone relative to M. We can now invoke Theorem **6.4.3** to finish the proof. □

6.4.5 Remark. Within the above proof we showed that if $M \in \boldsymbol{L}$ and if $q \in K(M) \setminus \mathcal{K}(M)$, then $\deg(q) = 1$. Since $\boldsymbol{L} \subseteq \boldsymbol{Q}_0$, we know that $K(M)$ will be convex. Thus, Theorem **6.1.17** can be used to show that every point in the boundary of $K(M)$ is in a strongly degenerate complementary cone. Therefore, if $M \in \boldsymbol{L}$ and if $q \in \operatorname{bd} K(M)$, then SOL$(q, M)$ must be unbounded; as a matter of fact, the latter solution set must contain a ray, which is called a *solution ray*. A proof of this last statement is not difficult; we refer the reader to Section 7.5 under the heading "Solution rays" for further discussion on this subject.

6.5 Local Analysis

In previous sections we used degree theory to obtain several results which show that, under various conditions, the LCP (q, M) has a solution. If $q \notin \mathcal{K}(M)$, we can even say something about exactly how many solutions (q, M) has. It would be nice if we could drop the assumption that $q \notin \mathcal{K}(M)$. One might argue that, since almost all $q \in R^n$ satisfy this assumption, it is relatively mild. Still, we have made heavy use of this assumption and, perhaps, we can find some way around it. Furthermore, we really do not have much to say about $q \in \mathcal{K}(M)$. In general, since the complementary cones are closed, if we know that SOL$(q, M) \neq \emptyset$ for all $q \notin \mathcal{K}(M)$, then we know that SOL$(q, M) \neq \emptyset$ for all $q \in R^n$. It is this one fact which allows us to include those q in $\mathcal{K}(M)$ into some of our previous results. One wonders if better results can be had.

To get around the requirement that $q \notin \mathcal{K}(M)$, we must ask what it is doing for us. In essence, it is a nondegeneracy assumption. It enables us to assume that q has a well-defined local degree and, thus, allows us to use degree theory to analyze the LCP (q, M). Difficulties arising from degeneracy in mathematical programming can frequently be circumvented if one may work within a smaller, but nondegenerate, subspace of the original problem. If $q \in \mathcal{K}(M)$, perhaps there is an LCP related to (q, M), but with a smaller dimension, such that degree-theoretic information from this related LCP will apply to (q, M). It turns out that such a related LCP exists and, if it is nondegenerate, will provide us with the required information concerning (q, M). To prove all this, we will need to study what happens in the local neighborhood around points in $f_M^{-1}(q)$. The purpose of this section is to perform this local analysis. Incidentally, the analysis touches on the issue of sensitivity in the LCP. A more comprehensive treatment of the latter subject is given in Chapter 7.

We now introduce some additional definitions and notations that will be used in the rest of this section.

6.5.1 Notation. Let $x \in R^n$ and $\alpha \subseteq \{1, \ldots, n\}$ be given. Define x_α^0 to be that element of R^n for which $(x_\alpha^0)_\alpha = x_\alpha$ and $(x_\alpha^0)_{\bar{\alpha}} = 0$. In other words, x_α^0 is obtained by expanding x_α out to be an n-vector using zeroes. Notice that it is not necessary for $x_{\bar{\alpha}}$ to be defined in order to define x_α^0.

6.5.2 Notations. For $x \in R^n$, we define the following subsets of R^n.

$$S_x = \{y \in R^n : x_i y_i \geq 0 \text{ for all } i = 1, \ldots, n\},$$

$$Q_x = \{y \in S_x : y_i = 0 \text{ for all } i \notin \operatorname{supp} x\},$$

$$V_x = \{y \in R^n : y_i = 0 \text{ for all } i \in \operatorname{supp} x\}.$$

In other words, S_x is the union of all orthants containing x, Q_x is the intersection of all orthants containing x, and V_x is the lineality space of S_x. The set S_x is called a *semiorthant* and the set V_x is called the *spine* of S_x.

6.5.3 Remark. We note the following facts which are immediate consequences of the above.

(a) $x \in \operatorname{ri} Q_x \subseteq \operatorname{int} S_x$.

(b) $S_x = \{q + v : q \in Q_x \text{ and } v \in V_x\}$.

(c) S_x, Q_x, and V_x are closed and convex cones.

(d) V_x is the orthogonal subspace to the affine hull of Q_x.

We begin our discussion with the following somewhat technical lemma.

6.5.4 Lemma. Let $M \in R^{n \times n}$ and $q \in R^n$ be given. Let $\alpha = \operatorname{supp} q$ and let $\beta = \bar{\alpha}$. If $q \geq 0$ and if $M_{\beta\beta}$ is nondegenerate, then:

(a) for all $x \in S_q$, we have $(f_M(x))_\alpha = x_\alpha + (f_M(x^0_\beta))_\alpha$ and $(f_M(x))_\beta = f_{M_{\beta\beta}}(x_\beta)$;

(b) for all u in the affine hull of Q_q, for all $v \in V_q$, and for all $\lambda > 0$ we have $|f_M^{-1}(u + \lambda v) \cap S_q| \leq |f_{M_{\beta\beta}}^{-1}(v_\beta)|$;

(c) for all $u \in \operatorname{ri} Q_q$ and for all $v \in V_q$ there is a $\bar{\lambda} > 0$ such that if $0 < \lambda \leq \bar{\lambda}$, then $|f_M^{-1}(u + \lambda v) \cap S_q| = |f_{M_{\beta\beta}}^{-1}(v_\beta)|$.

Proof. As $q \geq 0$, we have $x_\alpha \geq 0$ for all $x \in S_q$. Thus, given $x \in S_q$, there is some $\gamma \subseteq \beta$, for which $x \in \operatorname{pos} C_I(\gamma)$. Hence, $f_M(x) = C_{-M}(\gamma)x$. Noting that $C_{-M}(\gamma)_{\bullet\alpha} = I_{\bullet\alpha}$, we may deduce part (a).

Suppose $x \in f_M^{-1}(u + \lambda v) \cap S_q$. Using part (a), we have

$$f_{M_{\beta\beta}}(x_\beta) = (f_M(x))_\beta = (u + \lambda v)_\beta = \lambda v_\beta.$$

Hence, $\lambda^{-1} x_\beta \in f_{M_{\beta\beta}}^{-1}(v_\beta)$. In addition, using part (a), we have $x_\alpha = (u + \lambda v)_\alpha - (f_M(x^0_\beta))_\alpha$. Therefore, if y is in $f_M^{-1}(u + \lambda v) \cap S_q$ and $y \neq x$, then $y_\beta \neq x_\beta$. We may now deduce part (b).

As $M_{\beta\beta}$ is nondegenerate, we know $|f_{M_{\beta\beta}}^{-1}(v_\beta)| \leq 2^{|\beta|} < \infty$. Thus, there is a $\bar{\lambda} > 0$ such that if $0 < \lambda \leq \bar{\lambda}$ and if $x_\beta \in f_{M_{\beta\beta}}^{-1}(v_\beta)$, then we will have $u + \lambda(x^0_\beta - (f_M(x^0_\beta))^0_\alpha) \in S_q$. Letting $y = u + \lambda(x^0_\beta - (f_M(x^0_\beta))^0_\alpha)$, and using part (a), gives us $(f_M(y))_\beta = f_{M_{\beta\beta}}(\lambda x_\beta) = \lambda v_\beta$ and

$$(f_M(y))_\alpha = u_\alpha - \lambda(f_M(x^0_\beta))_\alpha + (f_M(\lambda x^0_\beta))_\alpha = u_\alpha.$$

Thus, $y \in f_M^{-1}(u + \lambda v)$. As $y_\beta = \lambda x_\beta$, a different x_β implies a different y_β. Part (c) of the lemma now follows. □

The above lemma gives a relationship between solutions to an LCP with matrix M and solutions to an LCP with matrix $M_{\beta\beta}$. If $q > 0$, then $(w, z) = (q, 0)$ is a nondegenerate solution to (q, M). In this case, $\beta = \emptyset$, $V_q = \{0\}$, $Q_q = S_q = R^n_+$, and the lemma is trivially true with $M_{\emptyset\emptyset}$ taken to be the identity matrix. If $q \geq 0$ contains zero components, then $(w, z) = (q, 0)$ is a degenerate solution to (q, M). In this case, the lemma allows us to investigate the behavior of f_M in the vicinity of q by studying the behavior of a lower dimensional LCP. This is just the approach outlined at the beginning of this section. Note that the assumptions $q \geq 0$ and $M_{\beta\beta}$ is nondegenerate imply that $z = 0$ is a locally unique solution of the LCP (q, M) (see Theorem **3.6.5**). The next result is basic to the local analysis we are developing, and it follows almost as a corollary to the lemma.

6.5.5 Theorem. Let $M \in R^{n \times n}$ and $q \in R^n$ be given. Let $\alpha = \operatorname{supp} q$ and let $\beta = \bar{\alpha}$. If $q \geq 0$ and if $M_{\beta\beta}$ is nondegenerate, then $f_M(S_q)$ covers a neighborhood of q if and only if $M_{\beta\beta} \in \boldsymbol{Q}$. Hence, if $\deg M_{\beta\beta} \neq 0$, then $f_M(S_q)$ covers a neighborhood of q.

Proof. Necessity. Suppose $M_{\beta\beta} \notin \boldsymbol{Q}$. We may then find a vector v_β such that $f^{-1}_{M_{\beta\beta}}(v_\beta) = \emptyset$. Clearly, $v_\beta \neq 0$. As $v^0_\beta \in V_q$, Lemma **6.5.4(b)** implies that for $\lambda > 0$ we have $f^{-1}_M(q + \lambda v^0_\beta) \cap S_q = \emptyset$. Thus, $f_M(S_q)$ does not contain a neighborhood of q.

Sufficiency. As $M_{\beta\beta}$ is nondegenerate, there exists a $\delta \in (0, 1)$ such that for all $x \in R^n$ and for all $\gamma \subseteq \beta$ we have $\|C_{-M}(\gamma)x\| \geq \delta\|x\|$. As $q \in \operatorname{int} S_q$, there exists an $\varepsilon > 0$ for which $\|y\| \leq \varepsilon$ implies $q + y \in S_q$.

Given $x \in S_q$, there is a $\gamma \subseteq \beta$ such that for all $t \geq 0$, if $q+t(x-q) \in S_q$, then $q + t(x - q) \in \operatorname{pos} C_I(\gamma)$ and, so,

$$f_M(q + t(x - q)) \;=\; C_{-M}(\gamma)(q + t(x - q)) \;=\; q + tC_{-M}(\gamma)(x - q)\,. \quad (1)$$

In addition, $\|f_M(q + t(x - q)) - q\| = \|tC_{-M}(\gamma)(x - q)\| \geq \delta\|t(x - q)\|$. We further note that $q + t(x - q) \in S_q$ if $\|t(x - q)\| \leq \varepsilon$. From this and from (1) we may conclude that if $f^{-1}_M(q + v) \cap S_q \neq \emptyset$ and if $0 \leq t < \delta\varepsilon/\|v\|$, then $f^{-1}_M(q + tv) \cap S_q \neq \emptyset$.

If $f_M(S_q)$ does not cover a neighborhood of q, then there is some $v \in R^n$ such that $\|v\| < \delta\varepsilon/2$, and $f^{-1}_M(q + v) \cap S_q = \emptyset$. Let $\bar{q} = (q + v)^0_\alpha$ and $\bar{v} = (q + v)^0_\beta = v^0_\beta$. Thus, $\bar{q} \geq 0$ and $\operatorname{supp} \bar{q} = \alpha$. The above discussion then holds true if q is replaced with $\bar{q}$ and ε is replaced with $\varepsilon/2$. As $f^{-1}_M(\bar{q} + \bar{v}) \cap S_q = f^{-1}_M(q + v) \cap S_q = \emptyset$ and as $\|\bar{v}\| < \delta\varepsilon/2$, the end of the previous paragraph implies that $f_M^{-1}(\bar{q} + \lambda\bar{v}) \cap S_q = \emptyset$ for all $\lambda > 0$. Thus, by Lemma **6.5.4(c)**, $f^{-1}_{M_{\beta\beta}}(\bar{v}_\beta) = \emptyset$ and, so, $M_{\beta\beta} \notin \boldsymbol{Q}$. □

6.5.6 Corollary. Let $M \in R^{n\times n}$ and $q \in R^n$ be given. Let $\alpha = \operatorname{supp} q$ and let $\beta = \bar{\alpha}$. If $q \geq 0$ and if $M_{\beta\beta}$ is a nondegenerate $\boldsymbol{Q}$-matrix, then for every $\varepsilon > 0$ the set $f_M(B(q,\varepsilon))$ contains an open ball around q.

Proof. We may assume ε is small enough so that the open ball $B(q,\varepsilon)$ is contained in S_q. Let $\delta = \min\{\|q - f_M(q+x)\| : \|x\| = \varepsilon\}$ and notice that since f_M is continuous the minimum is well-defined and will be attained at some vector $\bar{x}$. If $\delta = 0$, then $f_M(q+\bar{x}) = q$. However, as $q+\bar{x} \in S_q$, some orthant in S_q contains both $q+\bar{x}$ and q. Since $M_{\beta\beta}$ is nondegenerate, f_M is injective on any orthant contained in S_q. But $f_M(q) = q$, a contradiction. It must be that $\delta > 0$.

From the beginning of the sufficiency part of the proof of **6.5.5**, we see that (1) holds here. Thus, for any $x \in S_q$, f_M maps the line segment between x and q to the line segment between $f_M(x)$ and q. Using this fact we conclude that if $\|y\| < \delta$ and if $f_M^{-1}(q + y) \cap B(q,\varepsilon) = \emptyset$, then $f_M^{-1}(q + y) \cap S_q = \emptyset$. Theorem **6.5.5** implies that if we take $\delta' > 0$ small enough, then $f_M^{-1}(q+y) \cap S_q \neq \emptyset$ if $\|y\| < \delta'$. As we may select δ' less than δ, the corollary follows. □

The conclusion of the above corollary has two implications: first, the LCP (q', M) is solvable for all q' sufficiently close to q; and second,

$$\inf\{\|z\| : z \in \mathrm{SOL}(q', M)\} \to 0 \qquad \text{as} \qquad q' \to q\,.$$

The former consequence is a solvability property for an LCP which is a slight perturbation (in the constant vector) of the given one, whereas the latter is a kind of continuity of the solutions of the perturbed problems (recall that $0 \in \mathrm{SOL}(q, M)$). Generalizations of these results are given in Section 7.3.

Theorem **6.5.5** characterizes when f_M is surjective around q. Building on this, the next theorem gives a condition for when f_M is bijective around q. Since f_M is bijective on R^n if and only if $M \in \boldsymbol{P}$, it should not come as a surprise that the condition given below for f_M to be bijective around q is that a certain principal submatrix of M must be a $\boldsymbol{P}$-matrix.

6.5.7 Theorem. Let $M \in R^{n\times n}$ and $q \in R^n$ be given. Let $\alpha = \operatorname{supp} q$ and let $\beta = \bar{\alpha}$. If $q \geq 0$ and if $M_{\beta\beta} \in \boldsymbol{P}$, then f_M is injective on S_q. That is, if $x, y \in S_q$ and if $f_M(x) = f_M(y)$, then $x = y$.

Proof. Select $x \in S_q$. Notice $f_M(x)^0_\alpha$ is in the affine hull of Q_q and $f_M(x)^0_\beta$ is in V_q. Lemma **6.5.4(b)** implies $|f_M^{-1}(f_M(x)) \cap S_q| \leq |f^{-1}_{M_{\beta\beta}}(f_M(x)_\beta)|$. Since $M_{\beta\beta} \in \boldsymbol{P}$, we know from Theorem **3.3.7** that $|f^{-1}_{M_{\beta\beta}}(f_M(x)_\beta)| = 1$. Thus, if $y \in S_q$ and if $f_M(y) = f_M(x)$, then $y = x$. $\Box$.

The above theorem is related to a strong stability property of an LCP at a given solution; see the subsection under the heading "Strong stability" in Section 7.3.

We now have a reasonable understanding of the behavior of f_M in the vicinity of a point $q \geq 0$, even if $\operatorname{ind}_M(q)$ does not exist. However, the requirement that q be nonnegative seems too restrictive. Fortunately, it turns out that this requirement is not an obstacle. The reason is that by using an appropriate principal pivotal transform of (q, M) we may assume that q is nonnegative. This technique is demonstrated in the proof of the next two results.

6.5.8 Lemma. Let $M \in R^{n\times n}$ and $\alpha \subseteq \{1, \ldots, n\}$ be given. Suppose $M_{\alpha\alpha}$ is nonsingular and let $\bar{M}$ be the principal pivotal transform of M with respect to α. It then follows that for all $x \in R^n$ we have $f_{\bar{M}}(C_I(\alpha)x) = C_{\bar{M}}(\alpha) f_M(x)$.

Proof. Select $x \in R^n$ and suppose $q = f_M(x)$. Thus, $x^+ = q + Mx^-$. Let $\beta = \{1, \ldots, n\} \setminus \alpha$. We may write the last equation as $(x^+_\alpha, x^+_\beta) = q + M(x^-_\alpha, x^-_\beta)$ where we have represented some of the vectors in a convenient partitioned manner. From (2.3.8), (2.3.9), (2.3.10), and (2.3.11), if we let $\bar{q} = C_{\bar{M}}(\alpha)q$, then $(x^-_\alpha, x^+_\beta) = \bar{q} + \bar{M}(x^+_\alpha, x^-_\beta)$. This means that $f_{\bar{M}}(C_I(\alpha)x) = \bar{q}$ and, so, the lemma follows. $\Box$

6.5.9 Remark. In the above lemma, notice that $C_I(\alpha)x$ is just the vector x with the signs of the entries indexed by α reversed. Also, notice that the two complementary matrices used in the lemma are nonsingular. This is clear for $C_I(\alpha)$. For $C_{\bar{M}}(\alpha)$, we note that $\det C_{\bar{M}}(\alpha) = (-1)^{|\alpha|} \det M^{-1}_{\alpha\alpha}$.

6.5.10 Theorem. Let $M \in R^{n\times n}$, $q \in R^n$, and $x \in R^n$ be given. If $f_M(x) = q$ and if the indexes of all the orthants in S_x exist and are identical, then f_M is injective on S_x and, for all $\varepsilon > 0$, the set $f_M(B(x, \varepsilon))$ contains an open ball around q.

Proof. As $f_M(x) = q$, then $x^+ = q + Mx^-$. Let $\gamma = \operatorname{supp} x^-$. Since every orthant containing x has a well-defined index, and as $x \in \operatorname{pos} C_I(\gamma)$, it must be that $M_{\gamma\gamma}$ is nonsingular. Let $(\bar{q}, \bar{M})$ be the principal pivotal transform of (q, M) with respect to γ. It is clear, from (2.3.8) and (2.3.9), that $\bar{q} = x^+ + x^- \geq 0$.

Let $\alpha = \operatorname{supp} \bar{q}$. As usual, let $\beta = \{1, \ldots, n\} \backslash \alpha$. Notice, $\alpha = \operatorname{supp} x$ and $\beta \cap \gamma = \emptyset$. Now, the orthant $\operatorname{pos} C_I(\xi)$ will contain $\bar{q}$ if and only if $\xi \subseteq \beta$. By Theorem **4.1.2**, if $\xi \subseteq \beta$, then $\det \bar{M}_{\xi\xi} = \det M_{\xi\cup\gamma,\xi\cup\gamma} / \det M_{\gamma\gamma}$. Yet, for all $\xi \subseteq \beta$, the orthant $\operatorname{pos} C_I(\xi \cup \gamma)$ contains x. Thus, by the hypothesis of the theorem, $\det \bar{M}_{\xi\xi}$ must be nonzero and must have the same sign for all ξ contained in β. Since $\det \bar{M}_{\emptyset\emptyset} = 1$, we see that $\bar{M}_{\beta\beta} \in \boldsymbol{P}$.

We may now apply the results of this section to $f_{\bar{M}}$. Corollary **6.5.6** and Theorem **6.5.7** show that $f_{\bar{M}}$ is injective on $S_{\bar{q}}$ and that, for all $\varepsilon > 0$, the set $f_{\bar{M}}(B(\bar{q}, \varepsilon))$ contains an open ball around $\bar{q}$. We must now examine what this implies about f_M.

By Lemma **6.5.8**, $f_{\bar{M}}(C_I(\gamma)y) = C_{\bar{M}}(\gamma) f_M(y)$ for all $y \in R^n$. Notice that both of the complementary matrices are nonsingular and also notice that $C_I(\gamma)$ is its own inverse. Thus, as $f_{\bar{M}}(B(\bar{q}, \varepsilon))$ contains an open ball around $\bar{q}$ for all $\varepsilon > 0$, then $f_M(B(C_I(\gamma)\bar{q}, \varepsilon))$ contains an open ball around $C_{\bar{M}}^{-1}(\gamma)\bar{q}$ for all $\varepsilon > 0$. In other words, $f_M(B(x, \varepsilon))$ contains an open ball around q. We have now proven part of the theorem.

To prove the rest of the theorem, we recall that $f_{\bar{M}}$ is injective on $S_{\bar{q}}$. Thus, f_M is injective on $\{C_I(\gamma)y : y \in S_{\bar{q}}\}$. However, it is easy to see that this latter set is just S_x. $\square$

6.5.11 Corollary. Let $M \in R^{n\times n}$, $q \in R^n$, and $x \in R^n$ be given. If $f_M(x) = q$ and if the indexes of all the orthants in S_x exist and are identical, then for all $\varepsilon > 0$ small enough f_M bijectively maps the open ball $B(x, \varepsilon)$ onto a neighborhood of q.

Proof. Pick $\varepsilon > 0$ small enough that $B(x, \varepsilon) \subseteq S_x$. By Theorem **6.5.10**, f_M is injective on $B(x, \varepsilon)$. Since $f_M(x) = q$, to finish proving the corollary we must show that $f_M(B(x, \varepsilon))$ is an open set.

Select $y \in B(x,\varepsilon)$. As $y \in \operatorname{int} S_x$, it is easy to see that $S_y \subseteq S_x$. Invoking Theorem **6.5.10** with y, we see that, for all $\delta > 0$, the set $f_M(B(y,\delta))$ contains an open ball around $f_M(y)$. Selecting $\delta > 0$ small enough so that $B(y,\delta) \subseteq B(x,\varepsilon)$, we find that $f_M(B(x,\varepsilon))$ contains an open ball around $f_M(y)$. Since this is true for all $y \in B(x,\varepsilon)$, it follows that $f_M(B(x,\varepsilon))$ is open. □

We finish this section with an important result which will be used later in this chapter. This result will follow easily from the above material.

6.5.12 Theorem. Let $M \in R^{n\times n}$ and $q \in R^n$ be given. Consider complementary cones relative to M and their facets. Suppose that any complementary cone containing q is full and suppose that any facet containing q is proper. It follows that there is an $\varepsilon > 0$ and an integer $k \geq 0$ such that for all $q' \in B(q,\varepsilon)$ we have $|\operatorname{SOL}(q',M)| = k$.

Proof. Suppose $x \in f_M^{-1}(q)$. Since every complementary cone containing q is full, and since f_M maps the orthants onto the complementary cones, we conclude that the complementary cones associated with the orthants of S_x are all full. Thus, all the orthants of S_x have a well-defined index.

Suppose $\operatorname{pos} C_I(\alpha)$ and $\operatorname{pos} C_I(\alpha \triangle \{i\})$ are two adjacent orthants in S_x. Thus,

$$x \in \operatorname{pos} C_I(\alpha) \cap \operatorname{pos} C_I(\alpha \triangle \{i\}) = \operatorname{pos} C_I(\alpha)_{\bullet\bar{i}}.$$

Therefore, the facet $f_M(\operatorname{pos} C_I(\alpha)_{\bullet\bar{i}}) = \operatorname{pos} C_M(\alpha)_{\bullet\bar{i}}$ contains q and, so, must be proper. Hence, $\operatorname{ind}(\operatorname{pos} C_I(\alpha)) = \operatorname{ind}(\operatorname{pos} C_I(\alpha \triangle \{i\}))$. We conclude that any two adjacent orthants in S_x have the same index.

Suppose $\operatorname{pos} C_I(\alpha)$ and $\operatorname{pos} C_I(\beta)$ are two orthants in S_x. By Theorem **6.2.7**, we may select a $q \in \operatorname{int}(\operatorname{pos} C_I(\alpha))$ and a $q' \in \operatorname{int}(\operatorname{pos} C_I(\beta))$ such that all intersections of the line segment $\ell[q,q']$ with $\mathcal{K}(I)$ are nondegenerate. Clearly, S_x is convex and, thus, $\ell[q,q'] \subseteq S_x$. The orthants which intersect $\ell[q,q']$, if considered in the sequence encountered as the path of $\ell[q,q']$ is followed from q to q', give a sequence of orthants in S_x from $\operatorname{pos} C_I(\alpha)$ to $\operatorname{pos} C_I(\beta)$ such that any two consective orthants in the sequence are adjacent. We conclude that $\operatorname{ind}(\operatorname{pos} C_I(\alpha)) = \operatorname{ind}(\operatorname{pos} C_I(\beta))$.

We may now invoke Theorem **6.5.10** and Corollary **6.5.11** to show that f_M is injective on S_x and that, for all $\delta > 0$ small enough, f_M bijectively maps $B(x,\delta)$ onto a neighborhood of q.

For each $x \in f_M^{-1}(q)$ select a $\delta_x > 0$ such that f_M bijectively maps $B(x, \delta_x)$ onto a neighborhood of q. We may assume each δ_x is small enough to ensure that $B(x, \delta_x) \subseteq S_x$. Since no degenerate cone contains q, we have $|\operatorname{SOL}(q, M)| < \infty$ and, thus, $|f_M^{-1}(q)| < \infty$. Therefore, we may select each δ_x small enough so that if $x \neq y$, then $B(x, \delta_x) \cap B(y, \delta_y) = \emptyset$. We may now select $\varepsilon > 0$ small enough to ensure that $B(q, \varepsilon) \subseteq f_M(B(x, \delta_x))$ for each $x \in f_M^{-1}(q)$. We may also select $\varepsilon > 0$ small enough to ensure that the only complementary cones intersecting $B(q, \varepsilon)$ are those which contain q. Thus, as f_M is injective on S_x for each $x \in f_M^{-1}(q)$, we conclude that if $f_M(y) \in B(q, \varepsilon)$, then $y \in B(x, \delta_x)$ for some $x \in f_M^{-1}(q)$. Hence, letting $k = |f_M^{-1}(q)|$, we have $|f_M^{-1}(q')| = k$ for all $q' \in B(q, \varepsilon)$. As $|f_M^{-1}(q')| = |\operatorname{SOL}(q', M)|$, the theorem follows. □

6.6 Matrix Classes Revisited

In Chapter 3 we defined several matrix classes and discussed their properties within the context of the LCP. Some properties were geometric in nature. For example: $M \in \boldsymbol{Q}$ if and only if $K(M)$ is all of space, $M \in \boldsymbol{Q_0}$ if and only if $K(M)$ is convex, and M is column sufficient if and only if $\operatorname{SOL}(q, M)$ is convex for each vector q. However, in Chapter 3 we did not emphasize the geometric aspect of the properties we established.

In earlier sections of this chapter, we uncovered some further geometric properties of different matrix classes. For example: $M \in \boldsymbol{R_0} \setminus \boldsymbol{Q}$ implies $\deg M = 0$, $M \in \boldsymbol{R}$ implies $\deg M = 1$, and $M \in \boldsymbol{L}$ implies the strongly degenerate complementary cones relative to M are in the boundary of $K(M)$. However, this time our emphasis was on the underlying LCP geometry and not on examining the properties of specific matrix classes.

In this section we will bring these lines of inquiry together. Specifically, a close examination of the geometry we have developed suggests that it would be fruitful to define and examine certain matrix classes. It is in this direction that we will now turn our attention.

From the standpoint of the linear complementarity problem, a particularly simple and familiar matrix class is the $\boldsymbol{P}$-matrices. We have not explicitly considered the geometry of this class and so we will do well to begin this section by examining it. Theorem **3.3.7** characterizes the class

$\boldsymbol{P}$ as those matrices $M \in R^{n\times n}$ for which $|\operatorname{SOL}(q,M)| = 1$ for all vectors $q \in R^n$. This implies that the interior of any full complementary cone intersects no other complementary cone. In addition, we see that every complementary cone is full and (from **6.2.10**) that every facet is proper. Obviously, we must have $K(M) = R^n$. Thus, geometrically, a matrix is in $\boldsymbol{P}$ if every complementary cone is nondegenerate and if the complementary cones partition R^n. Therefore, both algebraically and geometrically, the $\boldsymbol{P}$-matrices provide us with a simple and intuitive structure. Using the $\boldsymbol{P}$-matrices as a starting point, we will now investigate some possible generalizations to see what further patterns we can uncover.

In a first attempt at developing a generalization of the class $\boldsymbol{P}$, it would be natural to investigate those matrices $M \in R^{n\times n}$ for which there exists a fixed positive integer k such that $|\operatorname{SOL}(q,M)| = k$ for all $q \in R^n$. The first thing to notice is that we must have $k < \infty$. To see this, note that Corollary **6.1.9** and Theorem **6.1.12** imply that the set of $q \in R^n$ which are not contained in any degenerate complementary cone is dense in R^n. For any q in this set we have $|\operatorname{SOL}(q,M)| \leq 2^n$ as each full complementary cone can contribute at most one solution to the LCP (q,M). Therefore, in any nonempty open set in R^n, there is a q for which $|\operatorname{SOL}(q,M)| < \infty$.

Unfortunately, we now quickly deduce that if $|\operatorname{SOL}(q,M)| = k$ for all $q \in R^n$, then $k = 1$ and, so, we have not gotten away from the class $\boldsymbol{P}$. All we need do is observe that $\operatorname{SOL}(0,M)$ is a cone. Therefore, either $\operatorname{SOL}(0,M) = \{0\}$ or $|\operatorname{SOL}(0,M)| = \infty$. The latter case we have ruled out and the former case implies $k = 1$.

The reader may feel it is somewhat unfair to use the special case where $q = 0$ to dismiss what might otherwise prove to be a useful extension of the class $\boldsymbol{P}$. As it happens, this line of generalization still fails to provide more than the $\boldsymbol{P}$-matrices even when only nonzero q are considered.

6.6.1 Theorem. Let $M \in R^{n\times n}$ be given. If there is a positive integer k such that $|\operatorname{SOL}(q,M)| = k$ for all nonzero $q \in R^n$, then $k = 1$ and $M \in \boldsymbol{P}$.

Proof. The case for $n = 1$ is trivial, so we will assume $n > 1$.

Theorem **6.1.27** states that $|\operatorname{SOL}(q,M)| = \infty$ for any q in the relative interior of a degenerate complementary cone. If any degenerate cones exist, then degenerate cones with dimension $n-1$ exist (see Exercise **6.10.18**).

Such a degenerate cone would have nonzero points in its relative interior. Yet we must have $k < \infty$ since, as noted earlier, in any nonempty open set in R^n there exists a q for which $|\,\mathrm{SOL}(q, M)| < \infty$. Hence, there can be no degenerate complementary cones relative to M.

Since all complementary cones are full, all facets must be either proper or reflecting. Suppose $\mathrm{pos}\, C(\alpha)_{\bullet \bar{i}}$ is a reflecting facet. Since $\mathrm{pos}\, C(\alpha)_{\bullet \bar{i}}$ is $(n-1)$-dimensional and since $\mathcal{L}(M)$ is contained in the finite union of $(n-2)$-dimensional subspaces, then $\mathrm{pos}\, C(\alpha)_{\bullet \bar{i}} \setminus \mathcal{L}(M)$ is nonempty.

Select $q \in \mathrm{pos}\, C(\alpha)_{\bullet \bar{i}} \setminus \mathcal{L}(M)$. Clearly, $q \neq 0$. Let H and δ be as described in Theorem **6.2.4**, where $\mathrm{pos}\, C(\alpha)_{\bullet \bar{i}}$ takes the place of F. Let B^+ and B^- be the two open hemiballs which are the connected components of $B(q, \delta) \setminus H$. As $\mathrm{pos}\, C(\alpha)_{\bullet \bar{i}}$ is assumed to be reflecting, Theorem **6.2.12** implies that one of these hemiballs, say B^+, is contained in both $\mathrm{pos}\, C(\alpha)$ and $\mathrm{pos}\, C(\alpha \triangle \{i\})$.

Select $q' \in B^+$ with $q' \neq 0$. We claim that any complementary cone containing q must contain q'. To see this, suppose $\mathrm{pos}\, C(\beta)$ contains q but not q'. The boundary of $\mathrm{pos}\, C(\beta)$ must intersect $B(q, \delta)$ and, thus, Theorem **6.2.4** implies that H is a supporting hyperplane to $\mathrm{pos}\, C(\beta)$. In addition, Theorem **6.2.12** implies that $\mathrm{pos}\, C(\beta)$ contains exactly one of B^+ and B^-. As $q' \notin \mathrm{pos}\, C(\beta)$, we have $B^- \subseteq \mathrm{pos}\, C(\beta)$. Yet, as both $\mathrm{pos}\, C(\alpha)$ and $\mathrm{pos}\, C(\alpha \triangle \{i\})$ contain B^+, it follows that the vectors $I_{\bullet i}$ and $-M_{\bullet i}$ are contained in the open halfspace on the opposite side of H from $\mathrm{pos}\, C(\beta)$. This is not possible as one of these vectors is a generator for $\mathrm{pos}\, C(\beta)$. Therefore, $\mathrm{pos}\, C(\beta)$ cannot exist and our claim is shown.

As $q' \in B^+$, Theorem **6.2.4** implies that if it is contained in a complementary cone, then it is contained in the interior of the cone. Therefore, each complementary cone containing q' is associated with a distinct solution to (q', M). Also, we know that each solution to (q, M) is associated with some complementary cone containing q, and that each complementary cone is associated with at most one solution to (q, M). Thus, each complementary cone containing q must be associated with a distinct solution to (q, M) otherwise $|\,\mathrm{SOL}(q, M)| < |\,\mathrm{SOL}(q', M)|$ which would be a contradiction. However, $\mathrm{pos}\, C(\alpha)$ and $\mathrm{pos}\, C(\alpha \triangle \{i\})$ contain q in their common facet and, hence, they are associated with the same solution to (q, M). We conclude that $\mathrm{pos}\, C(\alpha)_{\bullet \bar{i}}$ cannot exist; there can be no reflecting facets.

We now know that all the facets of all the complementary cones relative to M are proper. By an argument similar to the one given in the proof of Theorem **6.5.12**, we conclude that every complementary cone of M has the same index. Since the nonnegative orthant always has an index of $+1$, it follows that $M \in \boldsymbol{P}$. □

The previous theorem shows that the LCP is a bit less flexible than one might have imagined. We have failed in our first attempt to uncover a new and interesting matrix class using the $\boldsymbol{P}$-matrices as a foundation on which to generalize. However, we have learned something about the geometry of the LCP, which is our primary goal.

In our next attempt to generalize beyond the class $\boldsymbol{P}$, we will work directly from the definition. In essence, a matrix is in $\boldsymbol{P}$ if it is nondegenerate and if every complementary cone has the same index. Suppose we allow exactly one complementary cone to have an index opposite to all the others. With this in mind, we are lead to the following.

6.6.2 Definition. A matrix $M \in R^{n\times n}$ is said to belong to the class $\boldsymbol{N}$ if all its principal minors (other than $\det M_{\emptyset\emptyset}$) are negative. Members of this class are called $\boldsymbol{N}$-matrices.

The class $\boldsymbol{N}$ seems reasonably similar to the class $\boldsymbol{P}$ in definition. One would hope that the geometry associated with the $\boldsymbol{N}$-matrices is not too complex and illuminates the structure of the LCP. As we will now demonstrate, this turns out to be the case. As it happens, there are two distinct types of $\boldsymbol{N}$-matrices. This, by itself, is interesting and provides insight into the geometry of the LCP.

6.6.3 Theorem. Given $M \in \boldsymbol{N} \cap R^{n\times n}$, if $M \leq 0$, then $\deg M = 0$ and

(a) $|\,\mathrm{SOL}(q, M)| = 0$, if $q \ngeq 0$,

(b) $|\,\mathrm{SOL}(q, M)| = 2$, if $q > 0$,

(c) $|\,\mathrm{SOL}(q, M)| = 1$, if $q \geq 0$ and $q \not> 0$.

Proof. If $q_i < 0$, then $(Mz + q)_i < 0$ for any $z \geq 0$. Thus, if $q \ngeq 0$, then the LCP (q, M) is not feasible. This proves part (a). In addition, if M has a well-defined degree, then $\deg M = 0$. However, it is clear that $\boldsymbol{N} \subseteq \boldsymbol{R_0}$, thus M has a well-defined degree.

Suppose $q > 0$ and $q \notin \mathcal{K}(M)$. We must have $\deg(q) = 0$. Yet, $\deg(q)$ is the sum of the indexes of the complementary cones containing q. The nonnegative orthant contains q and contributes an index of $+1$ to its degree. All the other complementary cones have an index of -1. Hence, exactly one other complementary cone contains q. Thus, $|\operatorname{SOL}(q, M)| = 2$.

Suppose $q > 0$. Clearly, any complementary cone containing q is full as there are no degenerate complementary cones. Also, any facet containing q is proper. This follows as the only complementary cone with a positive index is the nonnegative orthant and, thus, the only reflecting facets are the facets of the nonnegative orthant. All other facets are proper. We may now invoke Theorem **6.5.12** to conclude that there is some $\varepsilon > 0$ such that for all $q' \in B(q, \varepsilon)$ we have $|\operatorname{SOL}(q', M)| = |\operatorname{SOL}(q, M)|$. By Theorem **6.1.12**, for any $\varepsilon > 0$, there must exist some $q' \in B(q, \varepsilon)$ with $q' > 0$ and $q' \notin \mathcal{K}(M)$. Therefore, $|\operatorname{SOL}(q, M)| = 2$.

Suppose $q \geq 0$ and $q \not> 0$. Clearly, $(w, z) = (q, 0)$ is a solution to the LCP (q, M). If $M < 0$, then no other solution is possible. The reason is that $q_i = 0$ for some $i \in \{1, \ldots, n\}$. Given $M < 0$, if any element of $z \geq 0$ was positive, then $(Mz + q)_i < 0$. We will now show that $M < 0$ and this will complete the proof.

As $M \in \boldsymbol{N}$, no element of M can be zero. To see this, note that the diagonal elements must be negative. Given this, if $i \neq j$ and $m_{ij} = 0$, then $\det M_{\alpha\alpha} > 0$ where $\alpha = \{i, j\}$. Thus, M has no zero elements and, as $M \leq 0$, we conclude that $M < 0$. □

6.6.4 Theorem. Given $M \in \boldsymbol{N} \cap R^{n \times n}$, if $M \not\leq 0$, then $\deg M = -1$ and

(a) $|\operatorname{SOL}(q, M)| = 1$, if $q \not\geq 0$,

(b) $|\operatorname{SOL}(q, M)| = 3$, if $q > 0$,

(c) $|\operatorname{SOL}(q, M)| = 1$, if $q \geq 0$, $q \not> 0$, and $M_{\bar{\alpha}\bar{\alpha}} \not\leq 0$ where $\alpha = \operatorname{supp} q$,

(d) $|\operatorname{SOL}(q, M)| = 2$, if $q \geq 0$, $q \not> 0$, and $M_{\bar{\alpha}\bar{\alpha}} \leq 0$ where $\alpha = \operatorname{supp} q$.

Proof. Again, as $M \in \boldsymbol{N}$, all complementary cones are full and $\deg M$ exists. Since $M \not\leq 0$, some complementary cone contains a point outside of R^n_+. Therefore, part of the interior of that complementary cone is outside of R^n_+. By Theorem **6.1.12**, this complementary cone contains a point q which is in neither R^n_+ nor $\mathcal{K}(M)$. As $q \notin \mathcal{K}(M)$, $\deg(q)$ exists. As $q \notin R^n_+$, all complementary cones containing q have a negative index. Since at least one complementary cone does contain q, we conclude that $\deg(q) < 0$. Hence, $\deg M < 0$ and, thus, $M \in \boldsymbol{Q}$.

As $M \in \boldsymbol{Q}$, Proposition **3.1.5** implies the existence of an $x \in R^n$ where $x > 0$ and $Mx > 0$. Consider the LCP $(-Mx, M)$. Obviously, $(w, z) = (0, x)$ is a solution to this LCP. Suppose $(\bar{w}, \bar{z})$ is another solution to the LCP $(-Mx, M)$. As $-Mx < 0$, we cannot have $\bar{z} = 0$. Thus, for some $i \in \{1, \ldots, n\}$, we have $\bar{w}_i = 0$. Consider $S = \{x \in R^n : x_i \leq 0\}$. We see that S is a semiorthant which does not contain the nonnegative orthant. Thus, all orthants in S have the same index. We may now invoke Theorem **6.5.10** to show that f_M is injective on S. As $-x$ and $\bar{w} - \bar{z}$ are both in S, we cannot have that both $(0, x)$ and $(\bar{w}, \bar{z})$ are solutions of the LCP $(-Mx, M)$. We may conclude that $|\operatorname{SOL}(-Mx, M)| = 1$. By Theorem **6.1.12** and the fact that M is nonsingular, we may assume $-Mx \notin \mathcal{K}(M)$. Therefore, it must be that $\deg(-Mx) = -1$ and, hence, $\deg M = -1$.

Suppose $q \not\geq 0$ and $q \notin \mathcal{K}(M)$. We must have $\deg(q) = -1$. As $q \notin R^n_+$, any complementary cone containing q must have an index of -1. Thus, exactly one complementary cone contains q and, so, $|\operatorname{SOL}(q, M)| = 1$.

Suppose $q > 0$ and $q \notin \mathcal{K}(M)$. Again, we must have $\deg(q) = -1$. Notice, $q \in R^n_+$ and the index of the nonnegative orthant is $+1$. Also, any other complementary cone containing q will have an index of -1. Thus, exactly three complementary cones contain q and, so, $|\operatorname{SOL}(q, M)| = 3$.

Suppose $q \not\geq 0$. As mentioned in the proof of Theorem **6.6.3**, the only reflecting facets are the facets of the nonnegative orthant. All other facets are proper. Thus, all complementary cones containing q are nondegenerate and all facets containing q are proper. Hence, by Theorem **6.5.12**, there is some $\varepsilon > 0$ such that for all $q' \in B(q, \varepsilon)$ we have $|\operatorname{SOL}(q', M)| = |\operatorname{SOL}(q, M)|$. By **6.1.12**, there must exist some $q' \in B(q, \varepsilon)$ with $q' \not\geq 0$ and $q' \notin \mathcal{K}(M)$. Therefore, $|\operatorname{SOL}(q, M)| = 1$. An argument similar to the one given in this paragraph will show that $q > 0$ implies $|\operatorname{SOL}(q, M)| = 3$.

Suppose $q \geq 0$ and $q \not> 0$. We know $(w, z) = (q, 0)$ is one solution of the LCP (q, M). Suppose (w', z') is another solution. We must have $z' \neq 0$. Thus, using the notation of **6.5.2**, the semiorthant $S_{x'}$, with $x' = w' - z'$, does not contain the nonnegative orthant. Hence, all orthants within $S_{x'}$ have the same index. By Theorem **6.5.10**, f_M is injective on $S_{x'}$. Further, for any $\varepsilon > 0$, the set $f_M(B(x', \varepsilon))$ contains an open ball around q.

Suppose there is a third solution to (q, M), say (w'', z''). An argument similar to the above would show that, for any $\delta > 0$, the set $f_M(B(x'', \delta))$, with $x'' = w'' - z''$, contains an open ball around q. Since f_M is injective on $S_{x'}$, we have $x'' \notin S_{x'}$. Thus, we may assume $B(x', \varepsilon) \cap B(x'', \delta) = \emptyset$. However, all this implies that $|\,\mathrm{SOL}(\bar{q}, M)| \geq 2$ for some point $\bar{q} \not\geq 0$ close to q. This cannot happen, so the LCP (q, M) cannot have a third solution.

Let $\alpha = \operatorname{supp} q$ and suppose $M_{\bar{\alpha}\bar{\alpha}} \not\leq 0$. Clearly, $M_{\bar{\alpha}\bar{\alpha}} \in \boldsymbol{N}$ as $M \in \boldsymbol{N}$. From what we have so far shown, $M_{\bar{\alpha}\bar{\alpha}} \in \boldsymbol{Q}$. Thus, Corollary **6.5.6** implies that for every $\delta > 0$ the set $f_M(B(q, \delta))$ contains an open ball around q. Since f_M is injective on $S_{x'}$, we have $q \notin S_{x'}$ and, thus, we may assume $B(x', \varepsilon) \cap B(q, \delta) = \emptyset$. As before, this implies that $|\,\mathrm{SOL}(\bar{q}, M)| \geq 2$ for some point $\bar{q} \not\geq 0$ close to q. This cannot happen, so (w', z') cannot exist if $M_{\bar{\alpha}\bar{\alpha}} \not\leq 0$. Part (c) now follows.

The only thing left to prove is part (d). Suppose $M_{\bar{\alpha}\bar{\alpha}} \leq 0$. By Theorem **6.6.3**, $M_{\bar{\alpha}\bar{\alpha}} \notin \boldsymbol{Q}$. Hence, by Theorem **6.5.5**, $f_M(S_q)$ does *not* cover a neighborhood of q. Therefore, we can find a sequence of points $\{q^i\}$ such that $\lim_{i\to\infty} q^i = q$ and $q^i \notin f_M(S_q)$ for all i. As $M \in \boldsymbol{Q}$, there exists an x^i for each q^i such that $f_M(x^i) = q^i$. We may assume the sequence $\{q^i\}$ is bounded and, as M is nondegenerate, this implies the sequence $\{x^i\}$ is bounded. Therefore, let x be a limit point of $\{x^i\}$. By continuity, $f_M(x) = q$. Yet $x \neq q$ as $q \in \operatorname{int} S_q$ and $x^i \notin S_q$ for all i. This implies that (x^+, x^-) is a solution to the LCP (q, M) which is different from $(q, 0)$. Part (d) now follows. □

Just before we introduced Definition **6.6.2**, we had intended to study nondegenerate matrices in which every complementary cone had the same index except for exactly one such cone. (This was in an attempt to generalize beyond the class $\boldsymbol{P}$.) The $\boldsymbol{N}$-matrices certainly fall into this category, but it would seem that other matrices do as well. Suppose we had a matrix M which was nondegenerate and all the complementary cones relative to M had the same index except for $\operatorname{pos} C(\alpha)$. By Theorem **4.1.2**, if we pivot on the matrix $M_{\alpha\alpha}$, the resulting matrix we get would be an $\boldsymbol{N}$-matrix. Thus, in a real sense, the $\boldsymbol{N}$-matrices tell us all we need to know about nondegenerate matrices in which every complementary cone, but one, has the same index.

We should stop for a moment and consider the use we have just made of the concept of principal pivotal transforms. If $\bar{M}$ is the principal pivotal transform of M gotten by pivoting on $M_{\alpha\alpha}$, then the material in Sections 2.3 and 4.1 indicates that for each $\beta \subseteq \{1, \ldots, n\}$ the complementary cones $\text{pos}\, C_M(\beta)$ and $\text{pos}\, C_{\bar{M}}(\alpha \triangle \beta)$ have a strong correspondence. In some sense, we rearrange the complementary cones so that $\text{pos}\, C_M(\alpha)$ now acts as the nonnegative orthant, but the underlying geometry of the complementary cones is very much the same. We made heavy use of this idea in the previous section, particularly in Lemma **6.5.8** and Theorem **6.5.10**. We first did our local analysis for a point in the nonnegative orthant and used the correspondence between a matrix and its principal pivotal transforms to extend our results to a point in any full complementary cone. The basic procedure was to consider the full complementary cone $\text{pos}\, C_M(\alpha)$, then pivot on $M_{\alpha\alpha}$ to obtain the pivotal transform $\bar{M}$, and finally interpret the results we previously obtained for $\text{pos}\, C_{\bar{M}}(\emptyset)$ in the context of $\text{pos}\, C_M(\alpha)$.

At this point, one begins to wonder about the matrix classes we have so far encountered. Which of them are closed under the process of taking principal pivotal transformations? While considering this question, it will be convenient to give this distinction a name.

6.6.5 Definition. Let $\boldsymbol{Y}$ denote a (fixed) class of square matrices. Let $M \in R^{n \times n}$ be given. Suppose for all $\alpha \subseteq \{1, \ldots, n\}$ for which $\det M_{\alpha\alpha} \neq 0$, including $\alpha = \emptyset$, the pivotal transform of M with pivot $M_{\alpha\alpha}$ is a $\boldsymbol{Y}$-matrix. We then say that M is *fully-*$\boldsymbol{Y}$. The class of fully-$\boldsymbol{Y}$ matrices is denoted $\boldsymbol{Y^f}$. If $\boldsymbol{Y} = \boldsymbol{Y^f}$, then the matrix class $\boldsymbol{Y}$ is said to be *full.*

Using Theorem **4.1.2**, it is clear that the class of $\boldsymbol{P}$-matrices is full and the class of $\boldsymbol{N}$-matrices is not full. In fact, given this distinction, the way in which we have approached the study of these matrix classes seems typical. If a matrix class is full, we often use this fact when deriving properties of the class. If a matrix class is not full, we can often extend the results we prove about the class to matrices which are principal pivotal transforms of members of the class. For example, the matrix class $\boldsymbol{R}$ is not full as the matrix

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \tag{1}$$

is d-regular, for any $d > 0$, but $M^{-1} \notin \boldsymbol{R}$. The result given in Theorem

6.3.10 concerns $\boldsymbol{R}$-matrices. In essence, Corollary **6.3.11** extends this result to include a more general class of matrices which, as (1) shows, is also not full. Finally, Corollary **6.3.12** extends these results to include the principal pivotal transforms of the (generalized) $\boldsymbol{R}$-matrices.

The notion of a full matrix class gives us a direction in which to search for new matrix classes and geometric insight. What if we took a matrix class $\boldsymbol{Y}$, which is not full, and considered the class $\boldsymbol{Y}^{\boldsymbol{f}}$? Would the added restriction produce anything interesting? The answer depends, of course, on the matrix class $\boldsymbol{Y}$ which is used. However, of the classes we have studied which are not full, there is a particularly good candidate to use as the matrix class $\boldsymbol{Y}$. This is the class of semimonotone matrices which, as (1) shows, is not full. Part (b) of Theorem **3.9.3** characterizes the class $\boldsymbol{E_0}$ as those matrices for which no (other) complementary cone intersects the interior of the nonnegative orthant. Since, intuitively, a matrix class is full if anything true for one full complementary cone is true for all full complementary cones, it would seem that a fully-semimonotone matrix would be characterized by having no (other) complementary cone intersect the interior of any given full complementary cone. Geometrically, this would be an interesting property for a matrix class to have and, indeed, this characterization holds.

6.6.6 Theorem. Let $M \in R^{n \times n}$ be given. The matrix M is fully-semimonotone ($M \in \boldsymbol{E_0^f}$) if and only if for each full complementary cone $\operatorname{pos} C_M(\alpha)$ no other complementary cone intersects $\operatorname{int}(\operatorname{pos} C_M(\alpha))$.

Proof. The cone $\operatorname{pos} C_M(\alpha)$ is full if and only if $\det M_{\alpha\alpha} \neq 0$. Let $\bar{M}$ be the principal pivotal transform of M with pivot block $M_{\alpha\alpha}$. From Theorem **3.9.3(b)**, $\bar{M} \in \boldsymbol{E_0}$ if and only if no (other) complementary cone relative to $\bar{M}$ intersects $\operatorname{int}(\operatorname{pos} C_{\bar{M}}(\emptyset))$. In other words, $\bar{M} \in \boldsymbol{E_0}$ if and only if $|f_{\bar{M}}^{-1}(q)| = 1$ for all $q > 0$.

It is easy to check that $C_I(\alpha) = (C_I(\alpha))^{-1}$ and $C_{\bar{M}}(\alpha) = (C_M(\alpha))^{-1}$. Thus, by Lemma **6.5.8**, $f_{\bar{M}}(x) = (C_M(\alpha))^{-1} f_M(C_I(\alpha)x)$. Hence, $\bar{M} \in \boldsymbol{E_0}$ if and only if $|f_M^{-1}(C_M(\alpha)q)| = 1$ for all $q > 0$. In other words, $\bar{M} \in \boldsymbol{E_0}$ if and only if no (other) complementary cone relative to M intersects the interior of $\operatorname{pos} C_M(\alpha)$. The theorem now follows. □

6.6.7 Corollary. Let $M \in R^{n \times n}$ be given. If $M \in \boldsymbol{P_0}$, then $M \in \boldsymbol{E_0^f}$.

Proof. Part (b) of Theorem **3.4.2** shows that $\boldsymbol{P}_0 \subseteq \boldsymbol{E}_0$. Using Theorem **4.1.2** we deduce that $\boldsymbol{P}_0$ is a full class. The corollary now follows. □

Although we did not explicitly set out to do so, our characterization in Theorem **6.6.6** made the class $\boldsymbol{E}_0^f$ look like an extension of the class $\boldsymbol{P}$. A matrix M is in $\boldsymbol{P}$ if and only if the LCP (q, M) has a unique solution for every q. A matrix M is in $\boldsymbol{E}_0^f$ if and only if the LCP (q, M) has a unique solution for every q contained in the interior of a full complementary cone.

This way of thinking about $\boldsymbol{P}$ and $\boldsymbol{E}_0^f$ should bring to mind the matrix class $\boldsymbol{U}$. According to Definition **4.1.12**, a matrix M is in $\boldsymbol{U}$ if and only if the LCP (q, M) has a unique solution for every q contained in $\operatorname{int} K(M)$. Clearly, the interior of any full complementary cone is contained in $\operatorname{int} K(M)$. Thus, we have the inclusions $\boldsymbol{P} \subseteq \boldsymbol{U} \subseteq \boldsymbol{E}_0^f$. The example given in Remark **4.1.14** shows that the second inclusion is proper. The zero matrix shows that the first inclusion is proper.

One might wonder if there are any other interesting matrix classes within this line of inclusions. For example, consider the class of matrices M for which the LCP (q, M) has a unique solution for every q contained in $K(M)$. Clearly, this class contains the $\boldsymbol{P}$-matrices and is contained by the $\boldsymbol{U}$-matrices. Unfortunately, as the reader is essentially asked to show in Exercise **6.10.23**, this class is precisely the $\boldsymbol{P}$-matrices. Thus, we will not obtain a new matrix class with this approach. However, there does exist an interesting matrix class strictly between the $\boldsymbol{P}$-matrices and the $\boldsymbol{U}$-matrices.

6.6.8 Definition. A matrix $M \in R^{n \times n}$ is said to belong to the class $\boldsymbol{W}$ if, for any index set $\alpha \subseteq \{1, \ldots, n\}$, the complementary cones $\operatorname{pos} C_M(\alpha)$ and $\operatorname{pos} C_M(\bar{\alpha})$ intersect only at the origin. Members of this class are called $\boldsymbol{W}$-matrices.

It is not hard to show that $\boldsymbol{P} \subseteq \boldsymbol{W}$, and the reader is asked to do so in Exercise **6.10.24**. To show that $\boldsymbol{W} \subseteq \boldsymbol{U}$ is a bit more involved. We first need to prove that the matrix class $\boldsymbol{W}$ is both full and complete. The reader is asked to show that $\boldsymbol{W}$ is full in Exercise **6.10.25**. We will now show that $\boldsymbol{W}$ is complete.

6.6.9 Theorem. The matrix class $\boldsymbol{W}$ is complete.

Proof. Let $M \in \boldsymbol{W} \cap R^{n\times n}$. Clearly, all principal submatrices of M are in $\boldsymbol{W}$ if $n = 1$. Assume, by induction, that the theorem holds for all $(n-1) \times (n-1)$ real matrices. It is easy to see that the induction will be complete, and the theorem will follow, if we can show that $M_{\bar{n}\bar{n}} \in \boldsymbol{W}$ given that $M \in \boldsymbol{W}$.

Let $\alpha \subseteq \{1, \ldots, n-1\}$ and let $\bar{\alpha} = \{1, \ldots, n-1\} \setminus \alpha$. Suppose we have $q \in (\operatorname{pos} C_{M_{\bar{n}\bar{n}}}(\alpha)) \cap (\operatorname{pos} C_{M_{\bar{n}\bar{n}}}(\bar{\alpha}))$. This implies the existence of $w, z, \bar{w}, \bar{z} \in R^{n-1}_{+}$ such that

$$q = w - M_{\bar{n}\bar{n}} z = \bar{w} - M_{\bar{n}\bar{n}} \bar{z}, \tag{2}$$

where $w_\alpha = \bar{z}_\alpha = 0$ and $\bar{w}_{\bar{\alpha}} = z_{\bar{\alpha}} = 0$. We must show that $q = 0$.

Let $\lambda = M_{n,\bar{n}}(\bar{z} - z)$. Notice that λ is a scalar. Let $\lambda^+ = \max(0, \lambda)$ and let $\lambda^- = \max(0, -\lambda)$. Thus, $\lambda^- - M_{n,\bar{n}} z = \lambda^+ - M_{n,\bar{n}} \bar{z}$. Using this and (2) gives

$$(w, \lambda^-) - M(z, 0) = (\bar{w}, \lambda^+) - M(\bar{z}, 0), \tag{3}$$

where we have written the n-vectors out in a convenient partitioned form. Equation (3) shows that the n-vector $(w, \lambda^-) - M(z, 0)$ is contained in $(\operatorname{pos} C_M(\beta)) \cap (\operatorname{pos} C_M(\bar{\beta}))$, where $\beta = \alpha$ if $\lambda^- > 0$, and $\beta = \alpha \cup \{n\}$ if $\lambda^- = 0$. Since $M \in \boldsymbol{W}$, this n-vector is zero. This implies, using (2) and (3), that $q = 0$. The theorem now follows. □

It is now relatively easy to show $\boldsymbol{W} \subseteq \boldsymbol{E}_0^f$. This brings us closer to our goal of showing $\boldsymbol{W} \subseteq \boldsymbol{U}$.

6.6.10 Theorem. Let $M \in R^{n\times n}$ be given. If $M \in \boldsymbol{W}$, then $M \in \boldsymbol{E}_0^f$.

Proof. Since $\boldsymbol{W}$ is a full matrix class, we need only show that $M \in \boldsymbol{W}$ implies $M \in \boldsymbol{E}_0$. Suppose $M \notin \boldsymbol{E}_0$. By Theorem **3.9.3**, there is an index set $\alpha \subseteq \{1, \ldots, n\}$ and a vector $x_\alpha \geq 0$ such that $M_{\alpha\alpha} x_\alpha < 0$. This means $(\operatorname{pos} C_{M_{\alpha\alpha}}(\emptyset)) \cap (\operatorname{pos} C_{M_{\alpha\alpha}}(\alpha)) \neq \{0\}$. Thus, $M_{\alpha\alpha} \notin \boldsymbol{W}$. Theorem **6.6.9** now implies $M \notin \boldsymbol{W}$. This completes the proof. □

According to Theorem **6.1.27**, if q is in the relative interior of a degenerate complementary cone, then $|\operatorname{SOL}(q, M)| = \infty$. Thus, if $M \in \boldsymbol{U}$, then no degenerate complementary cone intersects $\operatorname{int} K(M)$. The next two theorems show that this is the case for $M \in \boldsymbol{W}$.

6.6.11 Theorem. Let $M \in R^{n\times n}$ be given. If $M \in \boldsymbol{W}$, then no complementary cone relative to M is weakly degenerate.

Proof. Suppose that $\operatorname{pos} C(\alpha)$ is weakly degenerate. There is then an $x \in R^n$ such that $C(\alpha)x = 0$ and $C(\alpha)x^+ \neq 0$. This implies $C(\alpha)x^+ = C(\alpha)x^-$.

Let $\beta = \alpha \triangle \operatorname{supp} x^+$. We then have

$$C(\beta)x^- = C(\alpha)x^- = C(\alpha)x^+ = C(\bar{\beta})x^+.$$

Hence, $\operatorname{pos} C(\beta)$ and $\operatorname{pos} C(\bar{\beta})$ intersect at a point other than the origin. This contradicts the fact that $M \in \boldsymbol{W}$. The theorem follows. □

6.6.12 Theorem. Let $M \in R^{n\times n}$ be given. If $M \in \boldsymbol{W}$, then no strongly degenerate complementary cone relative to M intersects $\operatorname{int} K(M)$.

Proof. Suppose $q \in \operatorname{int} K(M)$ and $q \in \operatorname{pos} C(\alpha)$ where $\operatorname{pos} C(\alpha)$ is strongly degenerate. Let $s = \dim(\operatorname{pos} C(\alpha))$. Since $\operatorname{pos} C(\alpha)$ is degenerate, Proposition **2.9.14** implies $s < n$.

Since $q \in \operatorname{int} K(M)$, we may select an $\varepsilon > 0$ small enough so that $B(q,\varepsilon) \subseteq \operatorname{int} K(M)$. As the complementary cones are closed, we may select $\varepsilon > 0$ small enough so that the only complementary cones intersecting $B(q,\varepsilon)$ are the ones which contain q. As the relative interior of $\operatorname{pos} C(\alpha)$ intersects $B(q,\varepsilon)$, Proposition **2.9.14** shows that $\operatorname{pos} C(\alpha) \cap B(q,\varepsilon)$ has dimension equal to s.

Consider the union of the nondegenerate complementary cones. It is easy to show this union contains $\operatorname{int} K(M)$ and, thus, contains $B(q,\varepsilon)$. Hence, the intersection of this union with $\operatorname{pos} C(\alpha)$ is a set of dimension s. Proposition **2.9.16** now implies that there is a nondegenerate complementary cone, say $\operatorname{pos} C(\beta)$, such that $q \in \operatorname{pos} C(\alpha) \cap \operatorname{pos} C(\beta)$ and $\dim(\operatorname{pos} C(\alpha) \cap \operatorname{pos} C(\beta)) = s$.

We know from Theorem **6.6.10** that $M \in \boldsymbol{E}_{\mathbf{0}}^{\boldsymbol{f}}$. Thus Theorem **6.6.6** implies that

$$\operatorname{pos} C(\alpha) \cap \operatorname{pos} C(\beta) \subseteq \operatorname{bd}(\operatorname{pos} C(\beta)).$$

As the intersection is convex, there must be an $i \in \{1, \ldots, n\}$ such that

$$\operatorname{pos} C(\alpha) \cap \operatorname{pos} C(\beta) \subseteq \operatorname{pos} C(\beta)_{\cdot \bar{i}}.$$

Let H equal the affine hull of the facet $\operatorname{pos} C(\beta)_{\bullet\bar{\imath}}$. Note, $\dim(H) = n-1$. As $\operatorname{pos} C(\alpha) \cap \operatorname{pos} C(\beta)$ has the same dimension as $\operatorname{pos} C(\alpha)$, we conclude that $\operatorname{pos} C(\alpha) \subseteq H$. Thus, $C(\alpha)_{\bullet i} \in H$. Therefore, $C(\alpha)_{\bullet i} \neq C(\beta)_{\bullet i}$. Hence, $\operatorname{pos} C(\beta \triangle \{i\})$ is a degenerate cone and is contained in H. By Theorem **6.6.11**, $\operatorname{pos} C(\beta \triangle \{i\})$ must be strongly degenerate. As the facet $\operatorname{pos} C(\beta)_{\bullet\bar{\imath}}$ must contain q, and as the facet is contained in $\operatorname{pos} C(\beta \triangle \{i\})$, we lose no generality in assuming $\alpha = \beta \triangle \{i\}$.

Since $\dim(\operatorname{pos} C(\beta)_{\bullet\bar{\imath}}) = n-1$ and since $\mathcal{L}(M)$ is contained in the finite union of subspaces with dimension not exceeding $n-2$, we may assume $q \in \operatorname{pos} C(\beta)_{\bullet\bar{\imath}} \backslash \mathcal{L}(M)$. We may invoke Theorem **6.2.4** with H as given here and with $\varepsilon > 0$ assumed small enough to be used for δ. Let B^+ and B^- be the two open hemiballs which are the connected components of $B(q,\varepsilon) \backslash H$. The cone $\operatorname{pos} C(\beta)$ will contain exactly one of the hemiballs, say B^+, and will be disjoint from the other. Since $B(q,\varepsilon)$ is in the union of the full complementary cones, some full complementary cone, say $\operatorname{pos} C(\gamma)$, must contain B^-. Yet, Theorem **6.6.6** implies that $\operatorname{pos} C(\gamma)$ cannot contain B^+. Thus, the boundary of $\operatorname{pos} C(\gamma)$ intersects $B(q,\varepsilon)$ and we deduce from Theorem **6.2.4** that some facet, say $\operatorname{pos} C(\gamma)_{\bullet\bar{\jmath}}$ is contained in H. Notice, if we consider the open halfspaces defined by H, then $C(\gamma)_{\bullet j}$ is in the halfspace containing B^-. Also, $C(\beta)_{\bullet i}$ is in the halfspace containing B^+. Since $C(\alpha)_{\bullet i}$ and $C(\alpha)_{\bullet j}$ are both in H, we conclude that $i \neq j$ and $\{i,j\} \subseteq \beta \triangle \gamma$.

As $q \notin \mathcal{L}(M)$, we have $q \in \operatorname{ri}(\operatorname{pos} C(\beta)_{\bullet\bar{\imath}}) \cap \operatorname{ri}(\operatorname{pos} C(\gamma)_{\bullet\bar{\jmath}})$. Thus, there exist positive vectors $u_{\bar{\imath}}$ and $v_{\bar{\jmath}}$ such that $q = C(\beta)_{\bullet\bar{\imath}} u_{\bar{\imath}} = C(\gamma)_{\bullet\bar{\jmath}} v_{\bar{\jmath}}$. As $\operatorname{pos} C(\alpha)$ is strongly degenerate there is an $x \in R^n$ such that $0 \neq x \geq 0$ and $C(\alpha)x = 0$. As $C(\alpha)_{\bullet\bar{\imath}} = C(\beta)_{\bullet\bar{\imath}}$, these columns are linearly independent and, thus, we may assume $x_i = 1$. Thus, $C(\alpha)_{\bullet i} = -C(\alpha)_{\bullet\bar{\imath}} x_{\bar{\imath}}$. Since $\{i,j\} \subseteq \beta \triangle \gamma$, we know $C(\alpha)_{\bullet i} = C(\gamma)_{\bullet i}$. Hence, letting $\xi = \{i,j\}$, we have

$$\begin{aligned} C(\alpha)_{\bullet\bar{\imath}} u_{\bar{\imath}} \;=\; C(\beta)_{\bullet\bar{\imath}} u_{\bar{\imath}} \;=\; C(\gamma)_{\bullet\bar{\jmath}} v_{\bar{\jmath}} \;&=\; C(\gamma)_{\bullet\bar{\xi}} v_{\bar{\xi}} + v_i C(\gamma)_{\bullet i} \\ =\; C(\gamma)_{\bullet\bar{\xi}} v_{\bar{\xi}} + v_i C(\alpha)_{\bullet i} \;&=\; C(\gamma)_{\bullet\bar{\xi}} v_{\bar{\xi}} - v_i C(\alpha)_{\bullet\bar{\imath}} x_{\bar{\imath}}\,. \end{aligned}$$

Therefore, $C(\alpha)_{\bullet\bar{\imath}}(u_{\bar{\imath}} + v_i x_{\bar{\imath}}) = C(\gamma)_{\bullet\bar{\xi}} v_{\bar{\xi}}$, which gives us

$$C(\alpha)_{\bullet\bar{\imath}}(u_{\bar{\imath}} + v_i x_{\bar{\imath}}) + C(\beta)_{\bullet i} \;=\; C(\gamma)_{\bullet\bar{\xi}} v_{\bar{\xi}} + C(\beta)_{\bullet i}\,. \tag{4}$$

Since $C(\alpha)_{\bullet\bar{\imath}} = C(\beta)_{\bullet\bar{\imath}}$, and since $u_{\bar{\imath}} + v_i x_{\bar{\imath}} > 0$, the left side of (4) is a point in the interior of $\operatorname{pos} C(\beta)$. As $i \in \beta \triangle \gamma$, the right side of (4) is a point in $\operatorname{pos} C(\gamma \triangle \{i\})$. Since $i \neq j$ and $j \in \beta \triangle \gamma$, we know $\beta \neq \gamma \triangle \{i\}$. Thus, a point in the interior of a full complementary cone is contained in another complementary cone. As $M \in \boldsymbol{E}_0^f$, this violates **6.6.6**. Thus, q and $\operatorname{pos} C(\alpha)$ cannot exist and the theorem holds. □

We can now finally prove our main result concerning $\boldsymbol{W}$-matrices.

6.6.13 Theorem. Let $M \in R^{n \times n}$ be given. If $M \in \boldsymbol{W}$, then $M \in \boldsymbol{U}$.

Proof. Consider any $q \in \operatorname{int} K(M)$. By Theorems **6.6.11** and **6.6.12**, we know that q is not contained in any degenerate complementary cones. Part (d) of Theorem **6.2.25** indicates that if there are any reflecting facets, then there exists a point contained in the interiors of two full complementary cones. However, Theorems **6.6.6** and **6.6.10** forbid this. Hence, all the cones containing q must be full and all the facets containing q must be proper. Thus, Theorem **6.5.12** implies that, for some $\varepsilon > 0$ small enough, if $q' \in B(q, \varepsilon)$, then $|\operatorname{SOL}(q', M)| = |\operatorname{SOL}(q, M)|$.

From Theorem **6.1.12** we deduce that there are points in $K(M) \setminus \mathcal{K}(M)$ within $B(q, \varepsilon)$. Since any such point, q', is in the interior of a full complementary cone, Theorems **6.6.6** and **6.6.10** imply that q' is in no other complementary cone and, hence, $|\operatorname{SOL}(q', M)| = 1$. Thus, $|\operatorname{SOL}(q, M)| = 1$. Therefore, $M \in \boldsymbol{U}$. □

After all the work we did to obtain the preliminary results (Theorems **6.6.9** through **6.6.12**), the proof of Theorem **6.6.13** seems rather short. Perhaps this is a consequence of our preliminary work. In any case, we should spend a moment considering exactly what the proof of Theorem **6.6.13** is doing. In essence, the proof points out that if the interior of $K(M)$ does not intersect any degenerate cone or any reflecting facet, then Theorem **6.5.12** can be used to show that the LCP (q, M) has the same number of solutions around any $q \in \operatorname{int} K(M)$. This property seems distinctive enough to be given a name.

6.6.14 Definition. Let $M \in R^{n \times n}$ be given. The cone $K(M)$ is said to be *regular* if no degenerate cone relative to M intersects $\operatorname{int} K(M)$ and if no reflecting facet relative to M intersects $\operatorname{int} K(M)$.

6.6.15 Remark. The reader should be aware that the notion of regular, as given in Definition **6.6.14**, should not be confused with the notion of a regular matrix, as given in Definition **3.9.20**. Whether or not M is a regular matrix does not imply or prevent $K(M)$ from being regular.

As we shall see in a moment, the preceding definition of regular inspires the definition of the following matrix class.

6.6.16 Definition. Let k be a positive integer. A matrix $M \in R^{n \times n}$ is said to belong to the class $\boldsymbol{INS_k}$ if $|\,\mathrm{SOL}(q, M)| = k$ for all $q \in \mathrm{int}\, K(M)$. We define the matrix class $\boldsymbol{INS}$ to be the union of the matrix classes $\boldsymbol{INS_k}$ over all positive integers k. Members of the class $\boldsymbol{INS_k}$ are called $\boldsymbol{INS_k}$-matrices. Members of the class $\boldsymbol{INS}$ are called $\boldsymbol{INS}$-matrices.

6.6.17 Remark. The notation $\boldsymbol{INS}$ is an acronym for the phrase: Invariant Number of Solutions.

6.6.18 Remark. It follows by definition that $\boldsymbol{INS_1} = \boldsymbol{U}$. Therefore, we may view the $\boldsymbol{INS}$-matrices as just a generalization of the $\boldsymbol{U}$-matrices.

We now have the definitions needed to exploit the ideas in the proof of Theorem **6.6.13**. The following results show that the class $\boldsymbol{INS}$ is closely connected with the geometric notion of $K(M)$ being regular.

6.6.19 Lemma. The matrix class $\boldsymbol{INS_\infty}$ is empty.

Proof. Let $M \in R^{n \times n}$ be given. Just before Theorem **6.6.1** we noted that every nonempty open set in R^n contains a q such that $|\,\mathrm{SOL}(q, M)| < \infty$. Thus, as $\mathrm{int}\, K(M)$ is open and nonempty, it contains such a q. Therefore, $M \notin \boldsymbol{INS_\infty}$ and the proof is complete. □

6.6.20 Theorem. Let $M \in R^{n \times n}$ be given. If $M \in \boldsymbol{INS}$, then $K(M)$ is regular.

Proof. Assume $M \in \boldsymbol{INS}$. Suppose $\mathrm{pos}\, C(\alpha)$ is a degenerate complementary cone. Since $\mathrm{int}\, K(M)$ is open, if $\mathrm{pos}\, C(\alpha) \cap \mathrm{int}\, K(M) \neq \emptyset$, then $\mathrm{ri}(\mathrm{pos}\, C(\alpha)) \cap \mathrm{int}\, K(M) \neq \emptyset$. Hence, by Theorem **6.1.27**, there is a $q \in \mathrm{int}\, K(M)$ for which $|\,\mathrm{SOL}(q, M)| = \infty$. Invoking Lemma **6.6.19**, we conclude that no degenerate cone can intersect $\mathrm{int}\, K(M)$.

Suppose $q \in \operatorname{int} K(M) \cap \operatorname{pos} C(\alpha)_{\bullet\bar{\imath}}$ where $\operatorname{pos} C(\alpha)_{\bullet\bar{\imath}}$ is a reflecting facet. Note that, as no complementary cone containing q is degenerate, we have $\dim(\operatorname{pos} C(\alpha)_{\bullet\bar{\imath}}) = n - 1$. Since $\mathcal{L}(M)$ is contained in the finite union of $(n-2)$-dimensional subspaces, we may use Proposition **2.9.17** to show $\operatorname{pos} C(\alpha)_{\bullet\bar{\imath}} \setminus \mathcal{L}(M)$ is dense in $\operatorname{pos} C(\alpha)_{\bullet\bar{\imath}}$. Thus, as $\operatorname{int} K(M)$ is open, we may assume that $q \notin \mathcal{L}(M)$.

We may now use the same argument given in the proof of Theorem **6.6.1** to show there is a point q' arbitrarily close to q for which we have $|\operatorname{SOL}(q, M)| < |\operatorname{SOL}(q', M)|$. Since we may take q' to be in the interior of $K(M)$, we have a contradiction. Thus, no reflecting facet can intersect $\operatorname{int} K(M)$. Therefore, $K(M)$ is regular. □

6.6.21 Theorem. *Let $M \in R^{n \times n}$ be given and suppose $K(M)$ is regular. If $\mathcal{S}$ is a connected component of $\operatorname{int} K(M)$, then there is a positive integer k such that $|\operatorname{SOL}(q, M)| = k$ for all $q \in \mathcal{S}$.*

Proof. Let q^0 and q^1 be two points in $\mathcal{S}$. We know there exists a path q^t between q^0 and q^1 that is contained in $\mathcal{S}$. Let

$$s = \sup\{ t \in [0,1] : |\operatorname{SOL}(q^t, M)| = |\operatorname{SOL}(q^0, M)| \}.$$

As $K(M)$ is regular, q^s is contained in no degenerate complementary cones and no reflecting facets. Thus, Theorem **6.5.12** implies that, for some $\varepsilon > 0$ small enough, if $q' \in B(q^s, \varepsilon)$, then $|\operatorname{SOL}(q', M)| = |\operatorname{SOL}(q^s, M)|$. We may conclude that $s = 1$ and $|\operatorname{SOL}(q^1, M)| = |\operatorname{SOL}(q^0, M)|$.

We have shown there is some integer k such that $|\operatorname{SOL}(q, M)| = k$ for all $q \in \mathcal{S}$. As $\mathcal{S} \subseteq K(M)$, we must have $k > 0$. As $\mathcal{S}$ is nonempty and open, an argument similar to the one given in the proof of Lemma **6.6.19** shows that $k < \infty$. □

6.6.22 Corollary. *Let $M \in R^{n \times n}$ be given and suppose $\operatorname{int} K(M)$ is connected. It follows that $M \in$ **INS** if and only if $K(M)$ is regular.* □

The ***INS***-matrices bring us back to the beginning of this section. In our first attempt to define a matrix class which is in some sense a natural extension of the ***P***-matrices we suggested using what we now realize to be the matrix class ***INS*** $\cap$ ***Q***. However, Theorem **6.6.1** showed us that ***INS*** $\cap$ ***Q*** $=$ ***P*** and, so, we continued our investigation in other directions.

Thus, we have returned to our starting point with the matrix class ***INS***, which is a natural extension of the ***P***-matrices in the direction which we initially tried investigating.

Before we end this section, there is a question concerning matrix classes and the geometry of the LCP which we should address. In fact, it is a question which should have arisen well before now. A key concept we introduced in this chapter is that of the degree of a matrix. Another key concept used throughout this book is that of a principal pivotal transform of a matrix. It seems obvious to ask how the process of taking a principal pivotal transform affects the degree of a matrix. We will now answer this question in the final result of this section.

6.6.23 Theorem. Let $M \in \boldsymbol{R_0} \cap R^{n\times n}$ be given. Suppose $\det M_{\alpha\alpha} \neq 0$, for some $\alpha \subseteq \{1,\ldots,n\}$, and let $\bar{M}$ be the principal pivotal transform of M with $M_{\alpha\alpha}$ as pivot block. It follows that $\bar{M} \in \boldsymbol{R_0}$. Furthermore, we have $\deg \bar{M} = \deg M \times \operatorname{sgn}(\det M_{\alpha\alpha})$. In particular, $|\deg M| = |\deg \bar{M}|$.

Proof. As $M \in \boldsymbol{R_0}$, we know that $f_M^{-1}(0) = \{0\}$. By Lemma **6.5.8**, we have $f_{\bar{M}}(C_I(\alpha)x) = C_{\bar{M}}(\alpha)f_M(x)$ for all $x \in R^n$. We know $C_I(\alpha)$ and $C_{\bar{M}}(\alpha)$ are nonsingular (see Remark **6.5.9**). From this we may conclude that $f_{\bar{M}}^{-1}(0) = \{0\}$. In other words, $\bar{M} \in \boldsymbol{R_0}$.

Select $q \notin \mathcal{K}(M)$. From (2.3.8), (2.3.9), (2.3.10), (2.3.11) and Proposition **2.3.3**, we see that (w, z) solves the LCP (q, M) if and only if $(\bar{w}, \bar{z})$ solves the LCP $(\bar{q}, \bar{M})$ where $(\bar{w}_\alpha, \bar{w}_{\bar{\alpha}}, \bar{z}_\alpha, \bar{z}_{\bar{\alpha}}) = (z_\alpha, w_{\bar{\alpha}}, w_\alpha, z_{\bar{\alpha}})$ and $\bar{q} = C_{\bar{M}}(\alpha)q$. Since $q \notin \mathcal{K}(M)$, if (w, z) solves the LCP (q, M), then $w + z > 0$. Furthermore, letting $\beta = \operatorname{supp} z$, we have $q \in \operatorname{pos} C_M(\beta)$ and, so, $\det M_{\beta\beta} \neq 0$. Therefore, using Theorem **4.1.2** and letting $\gamma = \alpha \triangle \beta$, we may deduce that $\bar{w} + \bar{z} > 0$ and that $\det \bar{M}_{\gamma\gamma} = \det M_{\beta\beta} / \det M_{\alpha\alpha} \neq 0$, where $\gamma = \operatorname{supp} \bar{z}$. We conclude that $\bar{q} \notin \mathcal{K}(\bar{M})$.

As $\det \bar{M}_{\gamma\gamma} = \det M_{\beta\beta} / \det M_{\alpha\alpha}$, it is apparent that

$$\operatorname{ind}_{\bar{M}}(\bar{w}, \bar{z}) = \operatorname{ind}_M(w, z) \times \operatorname{sgn}(\det M_{\alpha\alpha}).$$

Since this is true for each corresponding pair of solutions to (q, M) and $(\bar{q}, \bar{M})$, we finally derive that

$$\deg \bar{M} = \deg M \times \operatorname{sgn}(\det M_{\alpha\alpha}). \quad \square$$

6.7 Superfluous Matrices

In the previous section we discussed several matrix classes which have strong ties with the geometry of the linear complementarity problem. There is one particular matrix class which we failed to mention in the previous section but which is quite interesting from the standpoint of LCP geometry. This is the class of superfluous matrices. We briefly mentioned this class in Section 6.3. We also stated that superfluous matrices of arbitrarily large degree existed. The main goal of this section is to prove that if there exists a matrix with degree k, then there exists a superfluous matrix with degree k. In doing so we will gain more insight into the geometry of the LCP. First, we formally define the class of superfluous matrices.

6.7.1 Definition. A matrix $M \in \boldsymbol{R_0} \cap R^{n \times n}$ is said to be *superfluous* if for all $q \notin \mathcal{K}(M)$ we have $|\,\mathrm{SOL}(q, M)| > |\deg M|$.

By attempting to construct a superfluous matrix, the reader should become convinced that the existence of superfluous matrices is not at all obvious. They do exist, but constructing one takes a little work.

6.7.2 Theorem. If

$$M = \begin{bmatrix} -4 & 3 & 3 & 6 \\ 3 & -4 & 3 & 6 \\ 3 & 3 & -4 & 6 \\ 6 & 6 & 6 & -4 \end{bmatrix},$$

then $M \in \boldsymbol{Q} \cap \boldsymbol{R_0}$ and $\deg M = 0$. Therefore, M is superfluous.

Proof. It is easy to check that M is nondegenerate. Thus, $M \in \boldsymbol{R_0}$ and $\deg M$ is well-defined. If $q = (-28, 28, -28, 28)$, then the LCP (q, M) has exactly two solutions. One solution is $w = (0, 0, 0, 112)$ and $z = (2, 10, 2, 0)$. This solution is nondegenerate and has an index of $+1$. The other solution is $w = (14, 70, 14, 0)$ and $z = (0, 0, 0, 7)$. This solution is nondegenerate and has an index of -1. Thus, $\deg(q) = 0$ and, hence, $\deg M = 0$. We must now show $M \in \boldsymbol{Q}$.

Consider the matrix $M_{\bar{4}\bar{4}}$ which, since it is a principal submatrix of M, must be nondegenerate. If $q_{\bar{4}} = (2, -2, 2)$, then the LCP $(q_{\bar{4}}, M_{\bar{4}\bar{4}})$ has exactly one solution. This solution is $w_{\bar{4}} = (0, 10, 0)$ and $z_{\bar{4}} = (2, 0, 2)$. This solution is nondegenerate and has an index of +1. Thus, $\deg M_{\bar{4}\bar{4}} = 1$ and, hence, $M_{\bar{4}\bar{4}} \in \boldsymbol{Q}$.

Suppose $q \in R^4$ and $q_4 \geq 0$. Let $(w_{\bar{4}}, z_{\bar{4}})$ be a solution to the LCP $(q_{\bar{4}}, M_{\bar{4}\bar{4}})$, which must exist as $M_{\bar{4}\bar{4}} \in \boldsymbol{Q}$. Set $z_4 = 0$. As $M_{4,\bar{4}} \geq 0$, we conclude that z provides us with a solution to the LCP (q, M).

Consider the matrix $M_{\bar{1}\bar{1}}$ which, since it is a principal submatrix of M, must be nondegenerate. If $q_{\bar{1}} = (4, -4, 4)$, then the LCP $(q_{\bar{1}}, M_{\bar{1}\bar{1}})$ has exactly one solution. This solution is $w_{\bar{1}} = (10, 2, 0)$ and $z_{\bar{1}} = (0, 0, 1)$. This solution is nondegenerate and has an index of -1. Thus, $\deg M_{\bar{1}\bar{1}} = -1$ and, hence, $M_{\bar{1}\bar{1}} \in \boldsymbol{Q}$.

Suppose $q \in R^4$ and $q_1 \geq 0$. Let $(w_{\bar{1}}, z_{\bar{1}})$ be a solution to the LCP $(q_{\bar{1}}, M_{\bar{1}\bar{1}})$, which must exist as $M_{\bar{1}\bar{1}} \in \boldsymbol{Q}$. Set $z_1 = 0$. As $M_{1,\bar{1}} \geq 0$, we conclude that z provides us with a solution to the LCP (q, M). As $M_{\bar{1}\bar{1}} = M_{\bar{2}\bar{2}} = M_{\bar{3}\bar{3}}$, we may also conclude that the LCP (q, M) will have a solution if $q_2 \geq 0$ or $q_3 \geq 0$.

To complete the proof, we must show that the LCP (q, M) has a solution if $q < 0$. Let $\alpha = \{1, 2, 3\}$ and let $\beta = \{3, 4\}$. Some calculation gives us

$$(C_M(\alpha))^{-1} = \frac{1}{14}\begin{bmatrix} -1 & -3 & -3 & 0 \\ -3 & -1 & -3 & 0 \\ -3 & -3 & -1 & 0 \\ -42 & -42 & -42 & 14 \end{bmatrix}$$

and

$$(C_M(\beta))^{-1} = \frac{1}{10}\begin{bmatrix} 10 & 0 & -24 & -21 \\ 0 & 10 & -24 & -21 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & -3 & -2 \end{bmatrix}.$$

Suppose $q < 0$ and $q \notin \operatorname{pos} C(\alpha)$. This implies $q_4 < 3(q_1 + q_2 + q_3)$. If $q \notin \operatorname{pos} C(\beta)$, then $\min\{10q_1, 10q_2\} < 24q_3 + 21q_4$. Thus, $\min\{10q_1, 10q_2\} < 87q_3 + 63(q_1 + q_2)$, which is impossible as $q < 0$. Therefore, if $q < 0$, then $q \in \operatorname{pos} C(\alpha) \cup \operatorname{pos} C(\beta)$ and, so, the LCP (q, M) has a solution. □

Our original interest in superfluous matrices had to do with Lemke's method. Given a matrix M, Corollary **6.3.12** suggests that d would be a good covering vector to use with Lemke's method if $d \in K(M) \setminus \mathcal{K}(M)$ and if all the complementary cones containing d have the same index. For such a d, the artificial variable z_0 can never increase without bound. Thus, this is one less way in which Lemke's method could terminate without providing a solution to the LCP. As our discussion after Corollary **6.3.12** indicated, no such d can exist for a superfluous matrix. However, our discussion also indicated that no such d can exist for a matrix $M \in \boldsymbol{R_0}$ if $\deg M = 0$. Therefore, Theorem **6.7.2** does not really answer the question we raised in Section 6.3. We were really interested in whether there were any superfluous matrices with a nonzero degree. In Section 6.3 we indicated that such matrices do exist. We will now prove this is the case.

6.7.3 Theorem. If

$$M = \begin{bmatrix} -4 & 3 & 3 & 6 & 6 \\ 3 & -4 & 3 & 6 & 6 \\ 3 & 3 & -4 & 6 & 6 \\ 6 & 6 & 6 & -4 & 6 \\ 6 & 6 & 6 & 6 & -4 \end{bmatrix},$$

then M is superfluous and $\deg M = -1$.

Proof. It is easy to check that M is nondegenerate. Thus, $M \in \boldsymbol{R_0}$ and $\deg M$ is well-defined. If $q = (4, -4, 4, -4, 4)$, then the LCP (q, M) has exactly three solutions. One solution is $w = (0, 20, 0, 44, 52)$ and $z = (4, 0, 4, 0, 0)$. This solution is nondegenerate and has an index of $+1$. Another solution is $w = (22, 0, 22, 0, 28)$ and $z = (0, 2, 0, 2, 0)$. This solution is nondegenerate and has an index of -1. The remaining solution is $w = (10, 2, 10, 2, 0)$ and $z = (0, 0, 0, 0, 1)$. This solution is nondegenerate and has an index of -1. Thus, $\deg(q) = -1$ and, hence, $\deg M = -1$.

To show that M is superfluous we must show that $|\,\mathrm{SOL}(q, M)| > 1$ for every $q \notin \mathcal{K}(M)$. Since $\deg(q) = -1$ for every $q \notin \mathcal{K}(M)$, if q is in a complementary cone with index equal to $+1$, then q must be contained in at least two additional complementary cones each with index -1. Therefore, one way to prove $|\,\mathrm{SOL}(q, M)| > 1$ for a given $q \notin \mathcal{K}(M)$ is to show that q

is contained in a complementary cone with index equal to +1. We will use this approach later.

Consider the matrix $M_{\bar{5}\bar{5}}$. This is the matrix given in Theorem **6.7.2**. Thus, $\deg M_{\bar{5}\bar{5}} = 0$ and $M_{\bar{5}\bar{5}} \in \boldsymbol{Q}$. This implies that if $q_{\bar{5}} \notin \mathcal{K}(M_{\bar{5}\bar{5}})$, then $q_{\bar{5}}$ must be contained in at least two complementary cones relative to $M_{\bar{5}\bar{5}}$. Using Theorem **6.1.12** and the fact that complementary cones are closed, we may conclude that every $q_{\bar{5}} \in R^4$ is contained in at least two complementary cones.

Suppose $q \in R^5$ and $q_5 \geq 0$. If $(w_{\bar{5}}, z_{\bar{5}})$ is a solution to the LCP $(q_{\bar{5}}, M_{\bar{5}\bar{5}})$, then as $M_{5,\bar{5}} \geq 0$ we may obtain a solution to the LCP (q, M) by setting $z_5 = 0$ and $w = q + Mz$. From this we may deduce that as $q_{\bar{5}}$ is contained in at least two complementary cones relative to $M_{\bar{5}\bar{5}}$, then q is contained in at least two complementary cones relative to M. In addition, as $M_{\bar{4}\bar{4}} = M_{\bar{5}\bar{5}}$, we may conclude that if $q \in R^5$ and $q_4 \geq 0$, then q is contained in at least two complementary cones relative to M.

Consider the matrix $M_{\bar{1}\bar{1}}$ which, since it is a principal submatrix of M, must be nondegenerate. If $q_{\bar{1}} = (4, -4, 4, -4)$, then the LCP $(q_{\bar{1}}, M_{\bar{1}\bar{1}})$ has exactly two solutions. One solution is $w_{\bar{1}} = (10, 2, 0, 2)$ and $z_{\bar{1}} = (0, 0, 1, 0)$. This solution is nondegenerate and has an index of -1. The other solution is $w_{\bar{1}} = (22, 0, 28, 0)$ and $z_{\bar{1}} = (0, 2, 0, 2)$. This solution is nondegenerate and has an index of -1. Thus, $\deg M_{\bar{1}\bar{1}} = -2$. It follows that if $q_{\bar{1}} \notin \mathcal{K}(M_{\bar{1}\bar{1}})$, then $q_{\bar{1}}$ must be contained in at least two complementary cones relative to $M_{\bar{1}\bar{1}}$. Using Theorem **6.1.12** and the fact that complementary cones are closed, we may conclude that every $q_{\bar{1}} \in R^4$ is contained in at least two complementary cones.

Suppose $q \in R^5$ and $q_1 \geq 0$. If $(w_{\bar{1}}, z_{\bar{1}})$ is a solution to the LCP $(q_{\bar{1}}, M_{\bar{1}\bar{1}})$, then as $M_{1,\bar{1}} \geq 0$ we may obtain a solution to the LCP (q, M) by setting $z_1 = 0$ and $w = q + Mz$. From this we may deduce that as $q_{\bar{1}}$ is contained in at least two complementary cones relative to $M_{\bar{1}\bar{1}}$, then q is contained in at least two complementary cones relative to M. In addition, as $M_{\bar{1}\bar{1}} = M_{\bar{2}\bar{2}} = M_{\bar{3}\bar{3}}$, we may conclude that if $q \in R^5$ and if either $q_2 \geq 0$ or $q_3 \geq 0$, then q is contained in at least two complementary cones relative to M.

To complete the proof, we must show that the LCP (q, M) has at least two solutions if $q \notin \mathcal{K}(M)$ and $q < 0$. Let $\alpha = \{1, 2, 3\}$ and let $\beta = \{3, 4\}$.

Some calculation gives us

$$(C_M(\alpha))^{-1} = \frac{1}{14}\begin{bmatrix} -1 & -3 & -3 & 0 & 0 \\ -3 & -1 & -3 & 0 & 0 \\ -3 & -3 & -1 & 0 & 0 \\ -42 & -42 & -42 & 14 & 0 \\ -42 & -42 & -42 & 0 & 14 \end{bmatrix}$$

and

$$(C_M(\beta))^{-1} = \frac{1}{10}\begin{bmatrix} 10 & 0 & -24 & -21 & 0 \\ 0 & 10 & -24 & -21 & 0 \\ 0 & 0 & -2 & -3 & 0 \\ 0 & 0 & -3 & -2 & 0 \\ 0 & 0 & -30 & -30 & 10 \end{bmatrix}.$$

Suppose $q \notin \mathcal{K}(M)$ and $q < 0$. If $q \in \operatorname{pos} C(\alpha)$, then as $\operatorname{ind}(\operatorname{pos} C(\alpha)) = +1$ we may conclude that $|\operatorname{SOL}(q, M)| > 1$. Therefore, let us now make the additional assumption that $q \notin \operatorname{pos} C(\alpha)$. As $q < 0$, this assumption is equivalent to the requirement that $\min\{q_4, q_5\} < 3(q_1 + q_2 + q_3)$.

Suppose $q_4 \leq q_5$. If $q \notin \operatorname{pos} C(\beta)$, then we must have $\min\{10q_1, 10q_2\} < 24q_3+21q_4$. Thus, $\min\{10q_1, 10q_2\} < 87q_3+63(q_1+q_2)$, which is impossible as $q < 0$. Therefore, $q \in \operatorname{pos} C(\beta)$. It now follows from the symmetry in the matrix M that q will also be contained in the cones $\operatorname{pos} C(\{1,4\})$ and $\operatorname{pos} C(\{2,4\})$. Hence, $|\operatorname{SOL}(q, M)| > 1$. If at the start of this paragraph we had supposed that $q_4 \geq q_5$, instead of $q_4 \leq q_5$, then from the symmetry in the matrix M we see that the arguments given here would lead us to conclude that q was contained in the cones $\operatorname{pos} C(\{1,5\})$, $\operatorname{pos} C(\{2,5\})$, and $\operatorname{pos} C(\{3,5\})$. Hence, $|\operatorname{SOL}(q, M)| > 1$. This completes the proof. □

At the beginning of this section we stated that our goal was to prove that if matrices of degree k exist, then superfluous matrices of degree k exist. This will follow as a corollary after we prove a few simple results.

6.7.4 Theorem. Let $M \in \boldsymbol{R_0} \cap R^{n\times n}$, $q \in R^n$ and $\alpha \subseteq \{1, \ldots, n\}$ be given. Let $\beta = \bar{\alpha} = \{1, \ldots, n\} \setminus \alpha$. Suppose $M_{\alpha\beta} = 0$ and $M_{\beta\alpha} = 0$. That is, after an appropriate principal rearrangement,

$$M = \begin{bmatrix} M_{\alpha\alpha} & 0 \\ 0 & M_{\beta\beta} \end{bmatrix}.$$

We then have $|\,\mathrm{SOL}(q, M)| = |\,\mathrm{SOL}(q_\alpha, M_{\alpha\alpha})| \times |\,\mathrm{SOL}(q_\beta, M_{\beta\beta})|$. (Note, $\infty \times 0 = 0$ and $\infty \times k = \infty$ if $k > 0$.) In addition, $M_{\alpha\alpha}$, $M_{\beta\beta} \in \boldsymbol{R_0}$ and $\deg M = (\deg M_{\alpha\alpha})(\deg M_{\beta\beta})$.

Proof. If $M_{\alpha\alpha} \notin \boldsymbol{R_0}$, then there is a $z_\alpha \neq 0$ which solves the LCP $(0, M_{\alpha\alpha})$. Thus, if we set $z_\beta = 0$, we obtain a $z \neq 0$ which solves the LCP $(0, M)$. This contradicts that $M \in \boldsymbol{R_0}$, thus $M_{\alpha\alpha} \in \boldsymbol{R_0}$. Similarly, $M_{\beta\beta} \in \boldsymbol{R_0}$.

We now make some simple observations. First, (w, z) solves (q, M) if and only if (w_α, z_α) solves $(q_\alpha, M_{\alpha\alpha})$ and (w_β, z_β) solves $(q_\beta, M_{\beta\beta})$. Clearly, (w, z) is nondegenerate if and only if both (w_α, z_α) and (w_β, z_β) are nondegenerate. In addition, it is easy to see that $\mathrm{ind}(w, z)$ is well-defined if and only if both $\mathrm{ind}(w_\alpha, z_\alpha)$ and $\mathrm{ind}(w_\beta, z_\beta)$ are well-defined. Further, if the indexes are well-defined, then $\mathrm{ind}(w, z) = \mathrm{ind}(w_\alpha, z_\alpha) \times \mathrm{ind}(w_\beta, z_\beta)$. The conclusions of the theorem are now straightforward consequences of these observations.

6.7.5 Lemma. For any integer k, if there exists an $\boldsymbol{R_0}$-matrix with degree equal to k, then there exists an $\boldsymbol{R_0}$-matrix with degree equal to $-k$.

Proof. We have already seen examples of matrices with degrees $+1$ and -1. Thus, suppose $M \in \boldsymbol{R_0}$ and $|\deg M| \neq 1$. If M has a principal submatrix with negative determinant, then we could pivot on this submatrix to obtain a matrix with degree equal to $-\deg M$ (see Theorem **6.6.23**). Otherwise, we have $M \in \boldsymbol{P_0}$. Corollary **6.6.7** implies that $M \in \boldsymbol{E_0^f}$. Theorem **6.6.6** states that the interior of each full complementary cone intersects no other complementary cone. Therefore, we must have $\deg M = 1$. This is not allowed, and the lemma follows. □

6.7.6 Remark. Notice, if $k \neq 1$, then the proof of Lemma **6.7.5** shows that if there is an $n \times n$ $\boldsymbol{R_0}$-matrix of degree k, then there is an $n \times n$ $\boldsymbol{R_0}$-matrix of degree $-k$.

We can now move on to the main result of this section.

6.7.7 Corollary. For any integer k, if there exists an $\boldsymbol{R_0}$-matrix with degree equal to k, then there exists a superfluous matrix with degree equal to k.

Proof. Suppose there exists an $\boldsymbol{R_0}$-matrix with degree equal to k. By Lemma **6.7.5**, we may assume the existence of $M \in \boldsymbol{R_0} \cap R^{n\times n}$ where $\deg M = -k$. We will define a matrix $\bar{M} \in R^{(n+5)\times(n+5)}$ as follows. Let $\alpha = \{1, \ldots, 5\}$ and $\beta = \{6, \ldots, n+5\}$. Set $\bar{M}_{\alpha\beta} = 0$ and $\bar{M}_{\beta\alpha} = 0$. Take $\bar{M}_{\beta\beta}$ to be M and take $\bar{M}_{\alpha\alpha}$ to be the matrix given in Theorem **6.7.3**. We conclude that $\bar{M} \in \boldsymbol{R_0}$ and, from Theorem **6.7.4**, that $\deg(\bar{M}) = k$.

Let $q \in R^{n+5} \setminus \mathcal{K}(\bar{M})$ be given. We may assume $k \neq 0$ as the case of $k = 0$ is covered by Theorem **6.7.2**. From the proof of Theorem **6.7.4** we see that $q \notin \mathcal{K}(\bar{M})$ if and only if $q_\alpha \notin \mathcal{K}(\bar{M}_{\alpha\alpha})$ and $q_\beta \notin \mathcal{K}(\bar{M}_{\beta\beta})$. From Theorem **6.7.3** we have $|\deg \bar{M}_{\alpha\alpha}| < |\,\mathrm{SOL}(q_\alpha, \bar{M}_{\alpha\alpha})|$. Clearly, $|\deg \bar{M}_{\beta\beta}| \leq |\,\mathrm{SOL}(q_\beta, \bar{M}_{\beta\beta})|$. Thus, using Theorem **6.7.4**, we have

$$\begin{aligned} |\deg \bar{M}| &= |\deg \bar{M}_{\alpha\alpha}| \times |\deg \bar{M}_{\beta\beta}| \\ &< |\,\mathrm{SOL}(q_\alpha, \bar{M}_{\alpha\alpha})| \times |\,\mathrm{SOL}(q_\beta, \bar{M}_{\beta\beta})| \;=\; |\,\mathrm{SOL}(q, \bar{M})|\,. \end{aligned}$$

We conclude that $\bar{M}$ is superfluous. The corollary follows. □

Before ending this section, there is a very interesting superfluous matrix we wish to mention. This matrix partially answers a question which we have not yet addressed. This is the question of whether or not the set $\boldsymbol{Q} \cap R^{n\times n}$ is open (or closed) in $R^{n\times n}$. In Exercise **6.10.32**, the reader is asked to verify that, for any $n \geq 1$, the set $\boldsymbol{Q} \cap R^{n\times n}$ is not closed in $R^{n\times n}$. The question of whether or not the set $\boldsymbol{Q} \cap R^{n\times n}$ is open in $R^{n\times n}$ is more difficult. However, it turns out that $\boldsymbol{Q} \cap R^{n\times n}$ is not open for $n = 4$. To see this, consider the matrix

$$M = \begin{bmatrix} 21 & 25 & -27 & -36 \\ 7 & 3 & -9 & 36 \\ 12 & 12 & -20 & 0 \\ 4 & 4 & -4 & -8 \end{bmatrix}.$$

In Kelly and Watson (1979) it is shown that $M \in \boldsymbol{Q}$. One can check that M is nondegenerate and, hence, $M \in \boldsymbol{R_0}$. Some calculation shows that if $q = (-48, 48, -48, 48)$, then the LCP (q, M) has exactly two solutions. One solution is $w = (272, 0, 0, 104)$ and $z = (0, 29, 15, 0)$. This solution is nondegenerate and has an index of $+1$. The other solution is $w = (0, 2464, 1008, 0)$ and $z = (88, 0, 0, 50)$. This solution is nondegenerate and has an index of -1. Thus, $\deg M = 0$ and M is superfluous.

Let $M^\varepsilon \in R^{4\times 4}$ be equal to M except that

$$m^\varepsilon_{14} = -36 - \varepsilon \quad \text{and} \quad m^\varepsilon_{24} = 36 + \varepsilon.$$

Let

$$q^\varepsilon = (26\varepsilon,\ -2\varepsilon,\ 3200 - 120\varepsilon,\ -8\varepsilon).$$

Kelly and Watson (1979) show that if $0 < \varepsilon < 1$, then $\text{SOL}(q^\varepsilon, M^\varepsilon) = \emptyset$. Thus, M is on the boundary of the set $\boldsymbol{Q} \cap R^{4\times 4}$ and, so, the set $\boldsymbol{Q} \cap R^{4\times 4}$ is not open in $R^{4\times 4}$.

6.8 Bounds on Degree

In the previous section we showed that if a matrix exists with degree equal to k, then a superfluous matrix exists with degree equal to k. This does not quite prove the statement we made in Section 6.3 that superfluous matrices with arbitrarily large degree exist. We still need to show that, in fact, there are matrices with arbitrarily large degree.

Actually, it is quite easy to show that matrices with arbitrarily large degree exist. Let M be the matrix given in Theorem **6.7.3**. Recall that in the proof of **6.7.3** we showed $\deg M_{\bar{1}\bar{1}} = -2$. Using $M_{\bar{1}\bar{1}}$, along with Theorem **6.7.4** and Lemma **6.7.5**, it is not hard to construct a matrix with an arbitrarily large degree. However, the larger the degree we wish the matrix to have, then the larger the matrix we must construct. One begins to wonder how large a degree an $n \times n$ $\boldsymbol{R_0}$-matrix can have. We will now turn our attention to this question.

Lower bounds on degree

The goal of this subsection is to construct an $n \times n$ $\boldsymbol{R_0}$-matrix which has a relatively large degree. As one might expect, it will be convenient to first prove some technical lemmas. The reader is reminded that e denotes the appropriately-dimensioned vector containing all ones.

6.8.1 Lemma. Let n be a positive integer and let a and b be real numbers. If M is the $n \times n$ matrix

$$a(ee^{\mathrm{T}}) + bI = \begin{bmatrix} a+b & a & \cdots & a \\ a & a+b & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a+b \end{bmatrix}, \tag{1}$$

then $\det M = b^{n-1}(na+b)$. (By convention, $0^0 = 1$.)

Proof. The lemma is obvious if $n = 1$ or if $b = 0$. Thus, assume $n > 1$ and $b \neq 0$. Let M' be the matrix obtained from M by subtracting the first row from each of the other rows. Let M'' be the matrix obtained from M' by subtracting from the first row a/b times the sum of the other rows. Clearly, $\det M = \det M' = \det M''$. In addition, we find that M'' is a lower triangular matrix with $m''_{11} = na + b$ and $m''_{ii} = b$ for $i = 2, \ldots, n$. Hence, $\det M'' = b^{n-1}(na + b)$, and the lemma follows. □

6.8.2 Lemma. Let $M \in R^{n \times n}$ be the matrix given in (1) and suppose $\det M \neq 0$. If $na + b > 0$, then $e \in \operatorname{int}(\operatorname{pos} M)$. If $na + b < 0$, then $e \notin \operatorname{pos} M$.

Proof. If $na + b > 0$, then letting $x = \lambda e$, with $\lambda = 1/(na + b)$, we have $Mx = e$. Since $\det M \neq 0$, this shows that $e \in \operatorname{int}(\operatorname{pos} M)$.

If $na + b < 0$, then letting $x = -\lambda e$, with $\lambda = 1/(na + b)$, we have $Mx = -e$. This shows that $-e \in \operatorname{pos} M$. If $My = e$ for some $y \geq 0$, then $x + y > 0$ and $M(x + y) = 0$. This contradicts the hypothesis that $\det M \neq 0$. Hence, $e \notin \operatorname{pos} M$. □

We now proceed to construct an $n \times n$ $\boldsymbol{R_0}$-matrix with a relatively large degree. It is interesting to note that there are two distinct cases depending on whether n is even or odd.

6.8.3 Theorem. Let n be an odd positive integer. If M is the $n \times n$ matrix

$$2(ee^{\mathrm{T}}) - nI = \begin{bmatrix} 2-n & 2 & \cdots & 2 \\ 2 & 2-n & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 2-n \end{bmatrix},$$

then $M \in \boldsymbol{R_0}$ and

$$|\deg M| = \binom{n-1}{(n-1)/2}.$$

Proof. Consider the principal submatrix $M_{\alpha\alpha}$ of M. Lemma **6.8.1** gives us $\det M_{\alpha\alpha} = (-n)^{|\alpha|-1}(2|\alpha| - n)$. As n is odd we have $\det M_{\alpha\alpha} \neq 0$. Thus, M is nondegenerate and, hence, $M \in \boldsymbol{R_0}$.

We will now calculate the degree of M by finding all the solutions to the LCP (e, M). If (w, z) solves (e, M) with $\alpha = \operatorname{supp} z$, then we must have $-M_{\alpha\alpha} z_\alpha = e$. Conversely, as $M_{\bar{\alpha}\alpha} \geq 0$, if $z_\alpha \geq 0$ and $-M_{\alpha\alpha} z_\alpha = e$, then letting $z_{\bar{\alpha}} = 0$ gives us a $z \in R^n$ which solves the LCP (e, M). Thus, to find the solutions of (e, M) we must find those index sets α for which $e \in \operatorname{pos}(-M_{\alpha\alpha})$. Using Lemma **6.8.2**, we see that $e \in \operatorname{pos}(-M_{\alpha\alpha})$ if and only if $|\alpha| < n/2$. Since n is odd, this happens if and only if $|\alpha| \leq (n-1)/2$. Further, using Lemma **6.8.2**, if $|\alpha| \leq (n-1)/2$, then $e \in \operatorname{int}(\operatorname{pos}(-M_{\alpha\alpha}))$. Thus, for each index set α with $|\alpha| \leq (n-1)/2$, we obtain a distinct nondegenerate solution of the LCP (e, M) and, moreover, these are the only solutions of (e, M). The degree of M may now be calculated as the sum of the indexes of the solutions to (e, M). The index of the solution associated with α is $\operatorname{sgn}(\det M_{\alpha\alpha})$. Since $|\alpha| \leq (n-1)/2$ and $\det M_{\alpha\alpha} = (-n)^{|\alpha|-1}(2|\alpha| - n)$, the index of the solution associated with α is $(-1)^{|\alpha|}$. Therefore

$$\begin{aligned} \deg M &= \sum_{k=0}^{(n-1)/2} (-1)^k \binom{n}{k} \\ &= \binom{n-1}{0} + \sum_{k=1}^{(n-1)/2} (-1)^k \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right) \end{aligned}$$

$$= (-1)^{(n-1)/2}\binom{n-1}{(n-1)/2}.$$

The theorem now follows. □

6.8.4 Theorem. Let n be an even positive integer. If M is the $n \times n$ matrix

$$2(ee^{\mathrm{T}}) - (n+1)I = \begin{bmatrix} 1-n & 2 & \cdots & 2 \\ 2 & 1-n & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 1-n \end{bmatrix},$$

then $M \in \boldsymbol{R_0}$ and

$$|\deg M| = \binom{n-1}{n/2}.$$

Proof. This is Exercise **6.10.35**. □

Theorems **6.8.3** and **6.8.4**, along with Lemma **6.7.5** and Remark **6.7.6**, give a lower bound on the largest degree that an $n \times n$ matrix can have. It has been conjectured that this lower bound is, in fact, the largest degree attained by any of the matrices in $\boldsymbol{R_0} \cap R^{n\times n}$ (see Morris (1990a)). While it is not known whether this conjecture is true or false, it should be possible for us to obtain some upper bound, albeit crude, on the degree of an $n \times n$ $\boldsymbol{R_0}$-matrix. This will be our goal for the next part of this section.

Upper bounds on degree

We will first examine a geometrical way of viewing the linear complementarity problem which is different from the ways we have so far discussed. Given $M \in R^{n\times n}$, consider the polytope

$$\mathcal{P}_M \equiv \{x \in R^n : x \geq 0,\, Mx \geq 0,\, e^{\mathrm{T}}x = 1\}. \tag{2}$$

Notice that $\mathcal{P}_M$ is a subset of the $(n-1)$-dimensional regular simplex $\{x \in R^n : x \geq 0,\, e^{\mathrm{T}}x = 1\}$. A vertex x of $\mathcal{P}_M$ is said to be $\bar{\imath}$*-complementary* if $x_j(Mx)_j = 0$ for all $j \in \{1,\dots,n\} \setminus \{i\}$. We may refer to a vertex of $\mathcal{P}_M$ as being *complementary* without specifically mentioning to which index set it is complementary. The following is a key result concerning $\mathcal{P}_M$.

6.8.5 Theorem. Let $M \in \boldsymbol{R_0} \cap R^{n \times n}$ be given. If, for some $i \in \{1, \ldots, n\}$, we have $M_{\bullet i} \notin \mathcal{K}(M)$, then there is a bijective correspondence between solutions to the LCP $(M_{\bullet i}, M)$ and $\bar{\imath}$-complementary vertices of $\mathcal{P}_M$.

Proof. Suppose $M_{\bullet i} \notin \mathcal{K}(M)$. There is then a bijective correspondence between complementary cones containing $M_{\bullet i}$ and solutions to the LCP $(M_{\bullet i}, M)$. Suppose $M_{\bullet i} \in \operatorname{pos} C(\alpha)$ and let (w, z) be the corresponding solution to $(M_{\bullet i}, M)$. If we let $x = (z + I_{\bullet i})/(e^{\mathrm{T}}z + 1)$, then it is easy to check that x is an $\bar{\imath}$-complementary vertex of $\mathcal{P}_M$. Further, $\operatorname{supp} x = \alpha \cup \{i\}$. Notice, as $\operatorname{pos} C(\alpha)$ is nondegenerate, since $M_{\bullet i} \notin \mathcal{K}(M)$, we must have $i \notin \alpha$. Thus, for each element in $\mathrm{SOL}(M_{\bullet i}, M)$ we have a distinct $\bar{\imath}$-complementary vertex of $\mathcal{P}_M$.

Conversely, suppose x is an $\bar{\imath}$-complementary vertex of $\mathcal{P}_M$. If $x_i = 0$, then $x^{\mathrm{T}}Mx = 0$ and it follows that $x \in \mathrm{SOL}(0, M)$. This is impossible as $M \in \boldsymbol{R_0}$. Thus, $x_i > 0$. It is now easy to check that if $z = (x - x_i I_{\bullet i})/x_i$ and $w = Mx/x_i$, then (z, w) solves $(M_{\bullet i}, M)$ and, if $\alpha = \operatorname{supp} x$, then $\operatorname{supp} z = \alpha \setminus \{i\}$. Therefore, the correspondence we found in the previous paragraph between elements in $\mathrm{SOL}(M_{\bullet i}, M)$ and $\bar{\imath}$-complementary vertices of $\mathcal{P}_M$ is invertible and, hence, a bijection. □

The condition given in Theorem **6.8.5**, that $M_{\bullet i} \notin \mathcal{K}(M)$, is not true for general $\boldsymbol{R_0}$-matrices. (For example, consider the identity matrix in $R^{n \times n}$ with $n \geq 2$.) To circumvent this problem we will restrict our attention to a certain matrix class contained within the class $\boldsymbol{R_0}$. It will turn out that $M_{\bullet i} \notin \mathcal{K}(M)$ is always true for matrices in this class and, yet, the degree bounds we obtain for this matrix class will hold for general $\boldsymbol{R_0}$-matrices.

6.8.6 Definition. A matrix $M \in R^{n \times n}$ is said to be *totally nondegenerate* if every square submatrix of M is nonsingular.

6.8.7 Proposition. A matrix $M \in R^{n \times n}$ is totally nondegenerate if and only if every $n \times n$ submatrix of the $n \times 2n$ matrix $[\,I,\ M\,]$ is nonsingular. Further, the class of totally nondegenerate $n \times n$ matrices is dense in $R^{n \times n}$. In addition, if $M \in R^{n \times n}$ is totally nondegenerate, then:

(a) $M \in \boldsymbol{R_0}$;

(b) $M_{\bullet i} \notin \mathcal{K}(M)$ for every $i \in \{1, \ldots, n\}$;

(c) the class of totally nondegenerate matrices is both full and complete;

(d) $|\{i : x_i = 0\}| + |\{i : (Mx)_i = 0\}| = n - 1$ for every vertex x of $\mathcal{P}_M$;

(e) if $\mathcal{P}_M \neq \emptyset$, then $\dim \mathcal{P}_M = n - 1$.

Proof. This is Exercise **6.10.36**. □

We now use the polytope $\mathcal{P}_M$ to derive an upper bound on the degree of totally nondegenerate matrices.

6.8.8 Lemma. For $M \in R^{n \times n}$, let $\text{cv}(M)$ denote the number of complementary vertices of $\mathcal{P}_M$. Let

$$g(n) \;=\; \max\{\text{cv}(M) : M \in R^{n\times n} \text{ is totally nondegenerate}\}.$$

For $n \geq 2$, we have $(n-1)\, g(n) \leq 2n\, g(n-1)$.

Proof. Suppose M is totally nondegenerate and $\mathcal{P}_M \neq \emptyset$. Part (e) of **6.8.7** implies $\dim \mathcal{P}_M = n - 1$. Part (d) of **6.8.7** implies every vertex of $\mathcal{P}_M$ is contained in exactly $n-1$ facets and, further, these facets determine the vertex. Since $\mathcal{P}_M$ has at most $2n$ facets, we have $\text{cv}(M) \leq \binom{2n}{n-1}$. Thus, $g(n)$ is bounded from infinity and, hence, it is truly a maximum (as opposed to a supremum). We may assume $g(n) = \text{cv}(M)$.

For some fixed $i \in \{1, \ldots, n\}$, consider the facet $F = \{x \in \mathcal{P}_M : x_i = 0\}$. If x is a complementary vertex of $\mathcal{P}_M$ contained in F, then **6.8.7(d)** implies that $x_{\bar{\imath}}$ is a complementary vertex of $\mathcal{P}_{M_{\bar{\imath}\bar{\imath}}}$. Thus, **6.8.7(c)** implies that F contains no more than $g(n-1)$ complementary vertices of $\mathcal{P}_M$.

Now suppose $F = \{x \in \mathcal{P}_M : (Mx)_i = 0\}$. Let $\bar{M}$ be the principal pivotal transform of M with pivot m_{ii}. If x is a complementary vertex of $\mathcal{P}_M$ contained in F, then **6.8.7(d)** along with (2.3.8), (2.3.9), and (2.3.10) implies that $x_{\bar{\imath}}/e^{\mathrm{T}}x_{\bar{\imath}}$ is a complementary vertex of $\mathcal{P}_{\bar{M}_{\bar{\imath}\bar{\imath}}}$. Thus, **6.8.7(c)** implies that F contains no more than $g(n-1)$ complementary vertices of $\mathcal{P}_M$.

We now know that every facet of $\mathcal{P}_M$ contains no more than $g(n-1)$ complementary vertices. As $\mathcal{P}_M$ has at most $2n$ facets, if we were to count the number of distinct ordered pairs (F, x) where F is a facet of $\mathcal{P}_M$ and x is a complementary vertex of $\mathcal{P}_M$ contained in F, then this number would be no larger than $2n\, g(n-1)$. However, we know that each complementary vertex of $\mathcal{P}_M$ is contained in exactly $n-1$ facets, thus the number of ordered pairs (F, x) is exactly equal to $(n-1)\, g(n)$. The lemma now follows. □

6.8.9 Theorem. Let $M \in R^{n \times n}$ be given. If $n \geq 4$, and if M is totally nondegenerate, then $|\deg M| \leq 3 \times 2^{n-4}$.

Proof. Part (d) of **6.8.7** implies that no vertex of $\mathcal{P}_M$ can be both $\bar{\imath}$-complementary and $\bar{\jmath}$-complementary with $i \neq j$. From Theorem **6.8.5** and part (b) of **6.8.7**, there must be at least $|\deg M|$ $\bar{\imath}$-complementary vertices of $\mathcal{P}_M$, for each $i \in \{1, \ldots, n\}$. Hence, using the notation of Lemma **6.8.8**, we have $|\deg M| \leq g(n)/n$. Using Lemma **6.8.8**, we have

$$\frac{g(n)}{n} \leq \frac{2g(n-1)}{n-1} \leq \frac{2^2 g(n-2)}{n-2} \leq \cdots \leq \frac{2^{n-4} g(4)}{4}.$$

According to Theorem **2.6.33**, the number of edges of a 3-dimensional polytope is two less than the number of vertices plus the number of facets. If $M' \in R^{4 \times 4}$ is totally nondegenerate and if $\mathcal{P}_{M'} \neq \emptyset$, then **6.8.7** implies that $\dim \mathcal{P}_{M'} = 3$ and that every vertex is contained in exactly three edges. Clearly, every edge contains exactly two vertices and $\mathcal{P}_M$ has no more than eight facets. Putting all these facts together, we deduce that $\mathcal{P}_{M'}$ has at most 12 vertices. Therefore, $g(4) \leq 12$ and, so, $|\deg M| \leq 3 \times 2^{n-4}$. □

As we mentioned earlier, Theorem **6.8.9** can be extended to cover all $\boldsymbol{R}_0$-matrices.

6.8.10 Corollary. Let $M \in R^{n \times n}$ be given. If $n \geq 4$ and if $M \in \boldsymbol{R}_0$, then $|\deg M| \leq 3 \times 2^{n-4}$.

Proof. As the reader is asked to show in Exercise **6.10.7**, a matrix is pseudo-regular if and only if none of its complementary cones are strongly degenerate. Thus, by Theorem **6.1.25**, the set $\boldsymbol{R}_0 \cap R^{n \times n}$ is open in $R^{n \times n}$. Hence, there is an $\varepsilon > 0$ such that if $\|M - M'\| \leq \varepsilon$ then $M' \in \boldsymbol{R}_0$. Proposition **6.8.7** implies there is a totally nondegenerate M' such that $\|M - M'\| \leq \varepsilon$. Clearly, every matrix of the form $tM + (1-t)M'$, for $t \in [0, 1]$, is pseudo-regular. We now conclude from Theorem **6.1.22** that $\deg M = \deg M'$. The corollary now follows from Theorem **6.8.9**. □

At first glance Corollary **6.8.10** is quite disappointing. After all, it is immediate that $|\deg M| \leq 2^n$, for $M \in \boldsymbol{R}_0 \cap R^{n \times n}$, as there are only 2^n complementary cones. Thus, Corollary **6.8.10** simply reduces this by a constant factor. On the other hand, according to the well-known Stirling

Approximation (see Knuth (1973)),

$$\lim_{n \to \infty} \frac{n!\, e^n}{n^n \sqrt{2\pi n}} = 1 .$$

Therefore, $\binom{n-1}{(n-1)/2} = O(2^n/\sqrt{n})$ as $n \to \infty$. Thus, by Theorem **6.8.3**, we cannot hope to bound $|\deg M|$ for $M \in \boldsymbol{R_0} \cap R^{n \times n}$ by anything smaller than $O(2^n/\sqrt{n})$. Hence, the true bound lies somewhere between $O(2^n/\sqrt{n})$ and $O(2^n)$. There is evidence to suggest that the true bound is $O(2^n/\sqrt{n})$ but a complete proof of this does not yet exist (see Morris (1990a)).

6.9 Q_0-matrices and Pseudomanifolds

The main goal of this section is to prove a most interesting theorem characterizing a certain subclass of the $\boldsymbol{Q_0}$-matrices. However, in developing the insights needed for the proof, we will study the geometry of the LCP from a new viewpoint. To explore the underlying geometric and combinatorial structure exhibited by this new viewpoint we will introduce the concept of a pseudomanifold. Thus, at the end of this section we will not only have obtained a result concerning $\boldsymbol{Q_0}$-matrices but, also, we will have gained additional understanding concerning the geometry of the LCP.

The classes $\boldsymbol{Q}$ and $\boldsymbol{Q_0}$ do not, as of yet, have complete characterizations that are computationally practical. Thus, since we wish to obtain a relatively simple characterization for $\boldsymbol{Q_0}$-matrices, we will be forced to place some restrictions on the matrices we will consider. First, we will work only with totally nondegenerate matrices (see **6.8.6**). This is a reasonably strong assumption. However, as with many other areas of mathematical programming, if one works under a nondegeneracy assumption, then cleaner results may be obtained, with the key ideas behind those results more clearly displayed, without a lengthy digression to deal with the degenerate cases. We mention, as the standard consolation, that the class of totally nondegenerate $n \times n$ matrices is dense in $R^{n \times n}$ (see **6.8.7**).

In addition to total nondegeneracy, we will assume that $\operatorname{pos}(I, -M)$ is pointed (see **2.6.25**). This is required by the geometric approach we will take in studying the LCP. Note, if $\operatorname{pos}(I, -M)$ equals all of space, then $M \in \boldsymbol{Q_0}$ if and only if $M \in \boldsymbol{Q}$. Thus, if we wish to specifically study matrices which are in $\boldsymbol{Q_0}$ but not $\boldsymbol{Q}$, which would be particularly interesting considering how much attention we have given to $\boldsymbol{Q}$-matrices compared with $\boldsymbol{Q}_0$-matrices, then this pointedness assumption is quite appropriate. Of course, this assumption avoids those cases in which pos $(I, -M)$ is not pointed but is still not all of space. However, the assumption of total nondegeneracy would also eliminate these cases.

Therefore, throughout the rest of this section we will assume that the matrix $M \in R^{n\times n}$ is totally nondegenerate and that the cone pos $(I, -M)$ is pointed.

Complementary simplices

In Exercise **2.10.29** the reader is asked to show that if a finite cone is not all of space, then there is a hyperplane that intersects the cone precisely on the cone's lineality space. Thus, as pos $(I, -M)$ is pointed, there is an $(n-1)$-dimensional hyperplane which intersects pos $(I, -M)$ at the origin and at no other point. We may select some hyperplane $\tilde{H}$ that intersects int(pos $(I, -M)$) and that is parallel to the hyperplane through the origin. Since M can have no zero columns, it is easy to see that if x is a column of the matrix $(I, -M)$, then $\tilde{H}$ intersects the open ray $\{\lambda x : \lambda > 0\}$ in exactly one point.

6.9.1 Notation. Within this section, we will let $\tilde{H}$ be as described above. If x is a column of the matrix $(I, -M)$, then we define the point $[\,x\,]$ by letting

$$[\,x\,] \;=\; \{\lambda x : \lambda > 0\} \cap \tilde{H}\,.$$

If $A \in R^{n\times k}$ is a submatrix of $(I, -M)$, then $[\,A\,]$ will denote the convex hull of $\{[\,A_{\bullet i}\,]\}_{i=1}^{k}$. If A and B are both submatrices of $(I, -M)$, each with n rows, then $[\,A, B\,]$ is the convex hull of $[\,A\,]$ and $[\,B\,]$. Notice, all these convex hulls are contained in $\tilde{H}$. Further, within this section, we will let

$$\mathcal{S} \;=\; \{[\,x\,] : x \text{ is a column of } (I, -M)\}\,.$$

Clearly, if $M \in R^{n\times n}$, then $|\mathcal{S}| = 2n$. Also, $[\,I, -M\,]$ is the convex hull of the points in $\mathcal{S}$ and $\dim[\,I, -M\,] = n-1$.

As M is totally nondegenerate, we deduce that if A is any $n \times n$ submatrix of $(I, -M)$, then $\dim[\,A\,] = n-1$. It is clear that the polytope $[\,I, -M\,]$ equals $\tilde{H} \cap \text{pos}(I, -M)$ and that, for any index set α, the simplex $[\,C_M(\alpha)\,]$ equals $\tilde{H} \cap \text{pos}\, C_M(\alpha)$.

6.9.2 Definition. If $A \in R^{n\times k}$ is a submatrix of $(I, -M)$, then $[A]$ is said to be *distinctly labelled* if, for each $i \in \{1, \dots, n\}$, the matrix A does not contain both the column $I_{\cdot i}$ and the column $-M_{\cdot i}$. If $A \in R^{n\times n}$ and if $[A]$ is distinctly labelled, then $[A]$ is said to be a *complementary simplex* relative to M.

Clearly, $[A]$ is a complementary simplex relative to M if and only if $A = C_M(\alpha)$ for some index set α. Using Proposition **3.2.1**, we deduce the following characterization for when M is a $\boldsymbol{Q_0}$-matrix. It is this characterization which we will use in the sequel.

6.9.3 Proposition. If $M \in R^{n\times n}$ is totally nondegenerate and if the cone $\text{pos}(I, -M)$ is pointed, then $M \in \boldsymbol{Q_0}$ if and only if the union of all the complementary simplices relative to M equals the polytope $[I, -M]$. □

For the rest of this section we will study the LCP by examining the geometric and combinatorial structure of the polytope $[I, -M]$ and the complementary simplices. Since $\tilde{H} = \text{affn}[I, -M]$, it will be convenient to have the following definition. Since this definition is only for this section, we list it as a notation.

6.9.4 Notation. Let S be an affine space in R^n. If $C \subseteq R^n$, we will let C_S denote the orthogonal projection of C into S. Let $\bar{S}$ be another affine space in R^n. If $x \in S \subseteq \bar{S}$, then the *orthogonal complement* of S in $\bar{S}$ around x refers to the affine space of all vectors $y \in \bar{S}$ such that $(y - x)^{\mathrm{T}}(z - x) = 0$ for all $z \in S$.

A characterization of $\boldsymbol{Q_0}$

The main goal of this section is to prove the following result.

6.9.5 Theorem. Let $M \in R^{n\times n}$ be given. If M is totally nondegenerate and $\text{pos}(I, -M)$ is pointed, then $M \in \boldsymbol{Q_0}$ if and only if each facet of $[I, -M]$ is distinctly labelled.

Proof of necessity. Suppose F is a facet of $[I, -M]$ which is not distinctly labelled. Let $H = \text{affn}\,F$. Select $q \in \text{ri}\,F$. If $[C(\alpha)]$ is a complementary simplex containing q, then $H \cap [C(\alpha)]$ must be a face of $[C(\alpha)]$.

Thus, $H \cap [C(\alpha)] = [C(\alpha)_{\bullet\beta}]$ for some index set β. However, total nondegeneracy implies that F has exactly $n-1$ vertices and these are the only points of $\mathcal{S}$ contained in H. Since $q \in \operatorname{ri} F$, we conclude that $[C(\alpha)_{\bullet\beta}]$ must equal F. This is impossible as F is not distinctly labelled. Therefore, q is not contained in any complementary simplex and, so, $M \notin \boldsymbol{Q_0}$. □

The proof of sufficiency is more involved and will require some additional tools which we will develop in the rest of this section. Unfortunately, the arguments which establish these tools require $n \geq 3$. (It should come as no surprise that lower dimensions are somewhat anomalous.) Fortunately, we can prove Theorem **6.9.5** directly for $n \leq 2$ by considering all possible cases.

Proof for n = 1. If $n = 1$, then M satisfies the hypothesis of the theorem if and only if $M < 0$. In this case $[I, -M]$ is a single point with no facets and $M \in \boldsymbol{Q_0}$. Thus, the theorem is vacuously true. □

Proof for n = 2. If $n = 2$ and if M satisfies the hypothesis of the theorem, then $[I, -M]$ is a line segment. Clearly, the facets of the line segment (the endpoints) must be distinctly labelled as they each have one label. Thus, for the theorem to hold, M must be a $\boldsymbol{Q_0}$-matrix. To see that $M \in \boldsymbol{Q_0}$ we can argue as follows. Without loss of generality, we may assume $[I_{\bullet 1}]$ is one of the two endpoints of $[I, -M]$. If $[I_{\bullet 2}]$ is the other endpoint, then the complementary simplex $[C(\emptyset)]$ will contain $[I, -M]$. Similarly, $[C(\{2\})]$ will contain $[I, -M]$ if $[-M_{\bullet 2}]$ is the other endpoint. If $[-M_{\bullet 1}]$ is the other endpoint, then $[I_{\bullet 2}]$ is in the relative interior of $[I, -M]$ and so $[I, -M]$ is contained in the union of the simplices $[C(\emptyset)]$ and $[C(\{1\})]$. In all cases, **6.9.3** implies $M \in \boldsymbol{Q_0}$. □

As previously mentioned, we cannot present the proof of sufficiency for Theorem **6.9.5** in the case $n \geq 3$ until we introduce some additional concepts. Therefore, we now turn to the next subsection to begin our discussion of the needed background material. For the rest of this section, we will assume $n \geq 3$.

Pseudomanifolds

In this subsection we will begin our study of the combinatorial structure of the polytope $[I, -M]$. We start with the following basic results.

6.9.6 Proposition. Let $M \in R^{n\times n}$ be given with $n \geq 3$. Suppose M is totally nondegenerate and $\text{pos}(I, -M)$ is pointed. If F is an m-face of $[I, -M]$, where $0 \leq m \leq n-2$, then F is the convex hull of $m+1$ points in $\mathcal{S}$.

Proof. By **2.7.5**, $[I, -M]$ has a supporting hyperplane H such that $F = [I, -M] \cap H$. Thus, F is the convex hull of those points of $\mathcal{S}$ which lie in H. As $\dim F = m$ and as M is totally nondegenerate, we conclude that F is the convex hull of $m+1$ points in $\mathcal{S}$. □

6.9.7 Proposition. Let $M \in R^{n\times n}$ be given with $n \geq 3$. Suppose M is totally nondegenerate and $\text{pos}(I, -M)$ is pointed. If F is an m-face of $[I, -M]$ and if F' is a k-face of F, where $0 \leq k < m \leq n-2$, then F' is a k-face of $[I, -M]$.

Proof. By **2.7.5**, there exist b, $d \in R$ and nonzero vectors a, $c \in R^n$ such that

$$\begin{aligned} a^{\mathrm{T}}x &= b && \text{for all } x \in F, \\ a^{\mathrm{T}}x &> b && \text{for all } x \in [I, -M] \setminus F, \\ c^{\mathrm{T}}x &= d && \text{for all } x \in F', \\ c^{\mathrm{T}}x &> d && \text{for all } x \in F \setminus F'. \end{aligned}$$

It is not hard to see that F is the convex hull of those points $x \in \mathcal{S}$ for which $a^{\mathrm{T}}x = b$. Similarly, F' is the convex hull of those points $x \in \mathcal{S} \cap F$ for which $c^{\mathrm{T}}x = d$. Let

$$\bar{\lambda} = \min\{(a^{\mathrm{T}}x - b)/(d - c^{\mathrm{T}}x) : x \in \mathcal{S} \setminus F \text{ and } d > c^{\mathrm{T}}x\}$$

with $\bar{\lambda} = 1$ if the minimum is taken over the empty set. If $x \in \mathcal{S} \setminus F$, then $a^{\mathrm{T}}x > b$ and, so, $\bar{\lambda} > 0$. Select $\lambda \in R$ such that $0 < \lambda < \bar{\lambda}$. It follows that $H = \{x \in R^n : (a + \lambda c)^{\mathrm{T}}x = (b + \lambda d)\}$ is a supporting hyperplane to $[I, -M]$ with $F' = H \cap [I, -M]$. Thus, F' is a k-face of $[I, -M]$. □

We now introduce the key combinatorial structure that will be used in our study of $[I, -M]$.

6.9.8 Definition. Let V be a finite and nonempty set of elements (called *vertices*). We say that a collection P of subsets of V is an *n-dimensional pseudomanifold* (on V) if

(a) $S \in P$ implies that $|S| = n+1$,

(b) $F \subseteq V$ and $|F| = n$ implies that F is a subset of at most two elements in P,

(c) for every pair S, $\bar{S} \in P$, there is a sequence $S = S_0, S_1, \ldots, S_m = \bar{S}$ of elements in P such that $|S_{i-1} \cap S_i| = n$ for $1 \leq i \leq m$.

If P is an n-dimensional pseudomanifold, we refer to the elements in P as n-simplices. In addition, if $F \subseteq S \in P$ with $|F| = n$, then we say that F is a *facet* of S. The collection of all facets which are contained in exactly one element of P is called the *boundary* of the pseudomanifold P and is denoted by $\operatorname{bd} P$. We define $P = \{\emptyset\}$ to be the unique nonempty (-1)-dimensional pseudomanifold, and, we note, P has no facets.

Several of the ideas used in Sections 6.2 and 6.3 reappear in the above definition. Indeed, the sequence of cones and facets encountered in Lemke's method, or the cones and facets associated with a family of facets, can be expressed as pseudomanifolds. We will now show that important parts of the structure of $[\,I, -M\,]$ are pseudomanifolds.

6.9.9 Theorem. Let $M \in R^{n\times n}$ be given with $n \geq 3$. Suppose M is totally nondegenerate and $\operatorname{pos}(I, -M)$ is pointed. If we consider a polytope as representing the set of its extreme points, then the collection of complementary simplices is an $(n-1)$-dimensional pseudomanifold on $\mathcal{S}$ whose boundary is empty.

Proof. Clearly, **6.9.8(a)** holds. Suppose $F \subseteq \mathcal{S}$ with $|F| = n-1$. If both $[\,I_{\bullet i}\,]$ and $[\,-M_{\bullet i}\,]$ are contained in F, for some $i \in \{1, \ldots, n\}$, then F is not contained by any complementary simplex. Therefore, if F is contained by some complementary simplex, we must have $F = [\,C(\alpha)_{\bullet \bar{i}}\,]$ for some index set α and some $i \in \{1, \ldots, n\}$. Thus, F is contained by exactly two complementary simplices and these are $[\,C(\alpha)\,]$ and $[\,C(\alpha \triangle \{i\})\,]$. Hence, **6.9.8(b)** holds and, further, the pseudomanifold will not have a boundary.

If $\alpha \triangle \beta = \{i_1, \ldots, i_m\}$, then the sequence $S_0 = [\,C(\alpha)\,]$ and $S_k = [\,C(\alpha \triangle \{i_1, \ldots, i_k\})\,]$, for $k = 1, \ldots, m$, shows that $[\,C(\alpha)\,]$ and $[\,C(\beta)\,]$ can be connected as described in (and required by) **6.9.8(c)**. $\square$

6.9.10 Theorem. Let $M \in R^{n \times n}$ be given with $n \geq 3$. Suppose M is totally nondegenerate and pos $(I, -M)$ is pointed. If we consider a polytope as representing the set of its extreme points, then the facets of $[I, -M]$ form an $(n-2)$-dimensional pseudomanifold on $\mathcal{S}$ whose boundary is empty.

Proof. Proposition **6.9.6** implies **6.9.8(a)** holds. Suppose F is a facet of $[I, -M]$ and F' is a facet of F. (Note, $\dim F' = n - 3$ which requires the assumption that $n \geq 3$.) Select some point $x \in \operatorname{ri} F'$ and let H be the orthogonal complement of aff F' in $\tilde{H}$ around x. We will now show **6.9.8(b)** holds by examining the geometry of $[I, -M]_H$.

Clearly, $\dim H = 2$ and $F'_H = x$. Since orthogonal projection is an affine transformation, we can show using convexity that $[I, -M]_H$ is just the convex hull of the points $\{y_H : y \in \mathcal{S}\}$. Thus, $[I, -M]_H$ is a polygon. Since $\dim[I, -M] = n - 1$, we must have $\dim[I, -M]_H = 2$. By **6.9.7**, there is an $(n-2)$-dimensional hyperplane $H' \subseteq \tilde{H}$ such that $F' = [I, -M] \cap H'$. It is not hard to show that $\dim H'_H = 1$ and that, for any $y \in \tilde{H}$, if $y_H \in H'_H$, then $y \in H'$. It follows that $H'_H \cap [I, -M]_H = \{x\}$. Hence, x is a vertex of $[I, -M]_H$. Using convexity and total nondegeneracy, we conclude that there is a bijection between facets of $[I, -M]$ containing F' and edges of $[I, -M]_H$ containing x. Therefore, as $[I, -M]_H$ is a 2-dimensional polygon, $[I, -M]$ has exactly two distinct facets which contain F'. This not only proves that **6.9.8(b)** holds but also shows that the pseudomanifold of facets of $[I, -M]$ will not have a boundary.

Suppose F and $\bar{F}$ are two facets of $[I, -M]$. Select $y \in \operatorname{ri}[I, -M]$. Let F' be a k-face of $[I, -M]$ where $k \leq n - 4$. (If $n = 3$, then no such F' exists and this part of the argument is trivial.) The affine hull of $\{y\} \cup F'$ will have dimension $n - 3$ or less. The (finite) union of all such affine hulls will have dimension $n - 3$ or less (see **2.9.16**). The complement of this union in $\tilde{H}$ is path connected (see **2.9.15(c)**). This complement will contain some $x \in \operatorname{ri} F$ and $\bar{x} \in \operatorname{ri} \bar{F}$ along with a path between them (see **2.9.17**). If z is a point in this path, then $z \neq y$ and, by convexity, the ray $\{y + \lambda(z - y) : \lambda \geq 0\}$ intersects the relative boundary of $[I, -M]$ in exactly one point. By associating this point of intersection with z, we obtain a second path between x and $\bar{x}$ which is contained in the relative boundary of $[I, -M]$ and does not intersect any k-face of $[I, -M]$ with $k \leq n - 4$. It follows that every point along this second path is contained in no more than two facets of $[I, -M]$. Further, if a point is contained in two facets, then the two facets share a common $(n-3)$-face. Therefore, by following the path from x to $\bar{x}$ we will obtain from the facets we encounter a sequence from F to $\bar{F}$ as required by **6.9.8(c)**. □

We now know that both the complementary simplices and the facets of $[\, I, -M \,]$ form boundaryless pseudomanifolds. Neither of these is precisely what we will want. The pseudomanifolds which we will find most valuable turn out to be certain subsets of the complementary simplices having the facets of $[\, I, -M \,]$ as boundary. These pseudomanifolds will be defined via the following equivalence relation.

6.9.11 Definition. Let $M \in R^{n\times n}$ be given. Suppose M is totally nondegenerate and pos $(I, -M)$ is pointed. We will say that two complementary simplices $[\, C(\alpha) \,]$ and $[\, C(\beta) \,]$ are *related*, denoted $[\, C(\alpha) \,] \sim [\, C(\beta) \,]$, if $\alpha = \beta$ or if there exists a sequence of index sets $\alpha = \gamma_0, \gamma_1, \ldots, \gamma_m = \beta$ such that for each $i \in \{1, \ldots, m\}$ the simplices $[\, C(\gamma_{i-1}) \,]$ and $[\, C(\gamma_i) \,]$ have a facet in common and, further, this common facet is *not* a facet of $[\, I, -M \,]$.

The reader is reminded that the simplices $[\, C(\gamma_{i-1}) \,]$ and $[\, C(\gamma_i) \,]$, in Definition **6.9.11**, have a common facet if and only if $|\gamma_{i-1} \bigtriangleup \gamma_i| \leq 1$.

Just before **6.9.11**, we mentioned that we were going to define an equivalence relation. For $\sim$ to be an equivalence relation we must have

(a) $[\, C(\alpha) \,] \sim [\, C(\alpha) \,]$ for all α,

(b) $[\, C(\alpha) \,] \sim [\, C(\beta) \,]$ if $[\, C(\beta) \,] \sim [\, C(\alpha) \,]$,

(c) $[\, C(\alpha) \,] \sim [\, C(\beta) \,]$ if $[\, C(\alpha) \,] \sim [\, C(\gamma) \,]$ and $[\, C(\gamma) \,] \sim [\, C(\beta) \,]$.

The reader can easily verify that all these conditions hold and, thus, the relation $\sim$ is indeed an equivalence relation. We may therefore consider an equivalence class of complementary simplices, that is, a set of complementary simplices each related by $\sim$ to the others and none related to any complementary simplex outside the set. The collection of all complementary simplices relative to M may be partitioned into equivalence classes. We will now show that these equivalence classes are pseudomanifolds. These particular pseudomanifolds will end up being quite important to us.

6.9.12 Theorem. Let $M \in R^{n\times n}$ be given with $n \geq 3$. Suppose M is totally nondegenerate and pos $(I, -M)$ is pointed. Let $\mathcal{C}$ be the set of complementary simplices in some $\sim$ equivalence class. If we consider a polytope as representing the set of its extreme points, then $\mathcal{C}$ is an $(n-1)$-dimensional pseudomanifold. Further, if all the facets of $[\,I, -M\,]$ are distinctly labelled, then if F is a facet of $[\,I, -M\,]$, there is some complementary simplex in $\mathcal{C}$ which has F as a facet.

Proof. Since the collection of all the complementary simplices is a $(n-1)$-dimensional pseudomanifold (see **6.9.9**), then $\mathcal{C}$ satisfies conditions (a) and (b) of **6.9.8**. It follows from the definition of $\sim$ that a $\sim$ equivalence class must satisfy **6.9.8(c)**. Therefore, $\mathcal{C}$ is a pseudomanifold.

Now assume that all the facets of $[\,I, -M\,]$ are distinctly labelled. We must show that if F is a facet of $[\,I, -M\,]$, then some complementary simplex in $\mathcal{C}$ has F as a facet. This will follow from the fact that the facets of $[\,I, -M\,]$ form a pseudomanifold (see **6.9.10**) if we can show

(a) there is some facet of $[\,I, -M\,]$ which is also a facet of some complementary simplex in $\mathcal{C}$,

(b) if F and $\bar{F}$ are facets of $[\,I, -M\,]$ which share a common $(n-3)$-face (a facet of the facets), and if F is a facet of some complementary simplex in $\mathcal{C}$, then $\bar{F}$ is a facet of some complementary simplex in $\mathcal{C}$.

We will first prove (a). Let F be a facet of $[\,I, -M\,]$. Since F is distinctly labelled, we must have $F = [\,C(\beta)_{\bullet\bar{j}}\,]$ for some $j \in \{1, \dots, n\}$ and some index set β. Let $[\,C(\alpha)\,]$ be an element of $\mathcal{C}$. If $\alpha = \beta$, we are done. If $\alpha \neq \beta$, let $\alpha \triangle \beta = \{i_1, \dots, i_m\}$ and let $\gamma_k = \alpha \triangle \{i_1, \dots, i_k\}$ for $1 \leq k \leq m$. If none of the facets $[\,C(\gamma_k)_{\bullet\overline{i_k}}\,]$, for $1 \leq k \leq m$, is a facet of $[\,I, -M\,]$, then we may deduce that $[\,C(\alpha)\,] \sim [\,C(\beta)\,]$. Thus, $[\,C(\beta)\,] \in \mathcal{C}$ and, so, (a) would follow. Otherwise, let k' be the minimum value of k for which $[\,C(\gamma_k)_{\bullet\overline{i_k}}\,]$ is a facet of $[\,I, -M\,]$. If $k' = 1$, then a facet of $[\,C(\alpha)\,]$ is a facet of $[\,I, -M\,]$, so (a) would follow. If $k' > 1$, then we see $[\,C(\alpha)\,] \sim [\,C(\gamma_{k'-1})\,]$, so $[\,C(\gamma_{k'-1})\,] \in \mathcal{C}$. Further, a facet of $[\,C(\gamma_{k'-1})\,]$ is a facet of $[\,I, -M\,]$. In all cases, (a) holds.

We will now prove (b). Suppose F is a facet of $[\,I, -M\,]$ and suppose F is also a facet of some element of $\mathcal{C}$. Again, as F is distinctly labelled, we may assume $F = [\,C(\beta)_{\bullet\bar{j}}\,]$. In addition, we may assume $[\,C(\beta)\,] \in \mathcal{C}$.

Pick some $i \neq j$ and let $\gamma = \{i, j\}$. Without loss of generality, we may assume $\beta \cap \gamma = \emptyset$ and, thus, $F = [\,C(\beta)_{\bullet\bar{\gamma}}, I_{\bullet i}\,]$. Now, consider $[\,C(\beta)_{\bullet\bar{\gamma}}\,]$ which is a facet of F. Let F' be the unique other facet of $[\,I, -M\,]$ which has $[\,C(\beta)_{\bullet\bar{\gamma}}\,]$ as a facet. We must show that F' is a facet of some element of $\mathcal{C}$. Since F' is distinctly labelled, there are three cases.

(1) $F' = [\,C(\beta)_{\bullet\bar{\gamma}}, I_{\bullet j}\,]$. In this case, F' is a facet of $[\,C(\beta)\,] \in \mathcal{C}$.

(2) $F' = [\,C(\beta)_{\bullet\bar{\gamma}}, -M_{\bullet i}\,]$. Thus, F' is a facet of $[\,C(\beta \triangle \{i\})\,]$. The common facet between $[\,C(\beta)\,]$ and $[\,C(\beta \triangle \{i\})\,]$ is $[\,C(\beta)_{\bullet\bar{\gamma}}, I_{\bullet j}\,]$. Since only two facets of $[\,I, -M\,]$ can have $[\,C(\beta)_{\bullet\bar{\gamma}}\,]$ as a facet, $[\,C(\beta)_{\bullet\bar{\gamma}}, I_{\bullet j}\,]$ is not a facet of $[\,I, -M\,]$. Thus, $[\,C(\beta)\,] \sim [\,C(\beta \triangle \{i\})\,]$, therefore $[\,C(\beta \triangle \{i\})\,] \in \mathcal{C}$ and (b) would follow.

(3) $F' = [\,C(\beta)_{\bullet\bar{\gamma}}, -M_{\bullet j}\,]$. Thus, F' is a facet of $[\,C(\beta \triangle \gamma)\,]$. The common facet between $[\,C(\beta \triangle \gamma)\,]$ and $[\,C(\beta \triangle \{i\})\,]$ is $[\,C(\beta)_{\bullet\bar{\gamma}}, -M_{\bullet i}\,]$. The common facet between $[\,C(\beta \triangle \{i\})\,]$ and $[\,C(\beta)\,]$ is $[\,C(\beta)_{\bullet\bar{\gamma}}, I_{\bullet j}\,]$. Both $[\,C(\beta)_{\bullet\bar{\gamma}}, -M_{\bullet i}\,]$ and $[\,C(\beta)_{\bullet\bar{\gamma}}, I_{\bullet j}\,]$ are not facets of $[\,I, -M\,]$ as only two facets of $[\,I, -M\,]$ can have $[\,C(\beta)_{\bullet\bar{\gamma}}\,]$ as a facet. Therefore, $[\,C(\beta \triangle \gamma)\,] \sim [\,C(\beta)\,]$ so $[\,C(\beta \triangle \gamma)\,] \in \mathcal{C}$ and (b) holds. □

6.9.13 Corollary. Let $M \in R^{n \times n}$ be given with $n \geq 3$. Suppose M is totally nondegenerate and $\operatorname{pos}(I, -M)$ is pointed. If all the facets of $[\,I, -M\,]$ are distinctly labelled, then there are at most two $\sim$ equivalence classes.

Proof. Let F be a facet of $[\,I, -M\,]$. Theorem **6.9.12** implies that each equivalence class must contain a complementary simplex with F as a facet. However, F cannot be the facet of more than two complementary simplices (see **6.9.9**). Thus, there are at most two equivalence classes. □

We will eventually show that, under the hypothesis of Corollary **6.9.13**, there are always two $\sim$ equivalence classes. Given this, the next result supplies us with the proof of sufficiency for Theorem **6.9.5** in the case $n \geq 3$.

6.9.14 Theorem. Let $M \in R^{n \times n}$ be given with $n \geq 3$. Suppose M is totally nondegenerate and $\operatorname{pos}(I, -M)$ is pointed. Suppose all the facets of $[\,I, -M\,]$ are distinctly labelled. If there are exactly two $\sim$ equivalence classes, then $M \in \boldsymbol{Q_0}$.

Proof. Let $\mathcal{C}$ be one of the equivalence classes. We will show that if $q \in [I, -M]$, then q is contained in some element of $\mathcal{C}$. The theorem will then follow from **6.9.3**.

First, we note that if $q \in K(M)$, then there is a unique $\lambda > 0$ such that $\lambda q \in \tilde{H}$. We will let $\tilde{H}(q)$ denote λq. If S is a set contained in $K(M)$, then $\tilde{H}(S)$ will denote the set $\{\tilde{H}(q) : q \in S\} \subseteq \tilde{H}$.

If q is in the relative boundary of $[I, -M]$, then **6.9.12** implies that q is contained in some element of $\mathcal{C}$. Thus, we will assume $q \in \text{ri}[I, -M]$.

As noted in previous proofs, we may select some index set α and $i \in \{1, \ldots, n\}$ such that $[C(\alpha)_{\bullet\bar{i}}]$ is a facet of $[I, -M]$. Theorem **6.9.12** allows us to assume $[C(\alpha)] \in \mathcal{C}$. Using Theorem **6.2.7**, one may show the existence of a point $\bar{q} \in \text{ri}(\text{pos}\, C(\alpha)_{\bullet\bar{i}})$ such that all the intersections of $\mathcal{K}(M)$ with the path of $\ell[q, \bar{q}]$ are nondegenerate.

We may now use Algorithm **6.3.1** (Lemke's method) in the extended manner described between Corollaries **6.3.11** and **6.3.12**. Let $d = \bar{q} - q$ and $z_0 = 1$. The initial distinguished complementary cone will be $\text{pos}\, C(\alpha)$, and we begin by having z_0 decrease. (Note, $q + dz_0 \notin K(M)$ if $z_0 > 1$.) The following are some observations concerning what will happen as the algorithm processes this particular problem.

Suppose the algorithm changes distinguished cones at the point $q + dz_0$. Let these two cones be $\text{pos}\, C(\beta)$ and $\text{pos}\, C(\beta \triangle \{j\})$. Hence, we have $q + dz_0 \in \text{ri}(\text{pos}\, C(\beta)_{\bullet\bar{j}})$. If $0 \leq z_0 < 1$, then $\tilde{H}(q + dz_0) \in \text{ri}[I, -M]$. It follows that $[C(\beta)] \sim [C(\beta \triangle \{j\})]$. Hence, as long as we have $0 \leq z_0 < 1$, the complementary simplices associated with the distinguished cones will all be elements of $\mathcal{C}$.

As M is totally nondegenerate, Algorithm **6.3.1** will not terminate with $z_0 \in (0, 1)$. Suppose, after we start the algorithm, that z_0 first leaves the interval $(0, 1)$ by taking the value zero. If $\text{pos}\, C(\beta)$ is the distinguished complementary cone at that point, then $q \in [C(\beta)] \in \mathcal{C}$ and the theorem would follow. Therefore, suppose z_0 first leaves the interval $(0, 1)$ by taking the value one.

As $\bar{q} \in \text{ri}(\text{pos}\, C(\alpha)_{\bullet\bar{i}})$, one can show $\tilde{H}(\bar{q}) \in \text{ri}[C(\alpha)_{\bullet\bar{i}}]$. Thus, any complementary simplex containing $\tilde{H}(\bar{q})$ must have $[C(\alpha)_{\bullet\bar{i}}]$ as a facet. (See the proof of necessity for Theorem **6.9.5**.) Therefore, if $\text{pos}\, C(\beta)$ is the distinguished cone at the time z_0 returns to the value one, we must have $\beta = \alpha$ or $\beta = \alpha \triangle \{i\}$. Since z_0 must return to the value one by increasing, Theorem **6.3.7(a)** implies $\alpha \neq \beta$. (Also, see the discussion just before Corollary **6.3.12**.) Thus, $\beta = \alpha \triangle \{i\}$, and we conclude $[\,C(\alpha \triangle \{i\})\,] \in \mathcal{C}$. Yet, Theorem **6.9.12** implies that every equivalence class has an element with $[\,C(\alpha)_{\bullet\bar{\imath}}\,]$ as a facet. There are only two complementary simplices with $[\,C(\alpha)_{\bullet\bar{\imath}}\,]$ as a facet, and we have shown that $\mathcal{C}$ contains both of them. Therefore, there can only be one equivalence class. This contradicts the theorem's hypothesis; thus, z_0 must always reach the value zero. □

Restricted pseudomanifolds

In the final part of the proof of Theorem **6.9.5**, we will need to use induction on the dimension of a family of pseudomanifolds on $\mathcal{S}$. The lower-dimensional pseudomanifolds in this family will be restrictions of the higher-dimensional pseudomanifolds. To explain what we mean by the word *restriction*, we introduce the following definition.

6.9.15 Definition. Let P be an n-dimensional pseudomanifold on the set V. If $S \subseteq V$, we define

$$P(S) \;=\; \{S' \subseteq (V \setminus S) : S \cup S' \in P\}.$$

If $P(S)$ is a, possibly empty, $(n-|S|)$-dimensional pseudomanifold for every set $S \subseteq V$, then we will say that P has the *restricted property* and we will refer to the pseudomanifolds $P(S)$ as *restricted* pseudomanifolds of P.

6.9.16 Remark. If P has the restricted property and has no boundary, then $P(S)$ has no boundary. If P has the restricted property, then so does $P(S)$. The reader is asked to show these assertions in Exercise **6.10.38**.

6.9.17 Theorem. Let $M \in R^{n \times n}$ be given with $n \geq 3$. Suppose M is totally nondegenerate and $\operatorname{pos}(I, -M)$ is pointed. If we consider a polytope as representing the set of its extreme points, then the collection of complementary simplices is an $(n-1)$-dimensional pseudomanifold on $\mathcal{S}$ with the restricted property.

Proof. This is Exercise **6.10.39**. □

The following result contains the essential geometric property of the polytope $[\,I, -M\,]$ which we will need.

6.9.18 Theorem. Let $M \in R^{n \times n}$ be given with $n \geq 3$. Suppose M is totally nondegenerate and pos $(I, -M)$ is pointed. Suppose all the facets of $[\, I, -M\,]$ are distinctly labelled. If we consider a polytope as representing the set of its extreme points, then the facets of $[\, I, -M\,]$ form an $(n-2)$-dimensional pseudomanifold on $\mathcal{S}$ with the restricted property.

Proof. Let P be the collection of facets of $[\, I, -M\,]$. Let A be an $n \times k$ submatrix of $(I, -M)$. For ease of notation we will denote $P([\, A\,])$ as simply $P(A)$. Theorem **6.9.10** states that P is an $(n-2)$-dimensional pseudomanifold. We must now show that $P(A)$ is also a pseudomanifold.

If $[\, A\,]$ is not distinctly labelled, or if $k \geq n$, then $P(A) = \emptyset$. If $k = n-1$, and if $P(A) \neq \emptyset$, then we must have $[A] \in P$ and so $P(A) = \{\emptyset\}$. If $k = n-2$ and if $P(A) \neq \emptyset$, then we must have $A = C(\alpha)_{\bullet\beta}$ where $|\beta| = n-2$ and where $[\, A\,]$ is an $(n-3)$-face of $[\, I, -M\,]$. There are precisely two elements in P containing $[\, A\,]$. Thus, there are precisely two distinct elements in $P(A)$, each of which is a one-point set. Hence, $P(A)$ satisfies all the conditions in Definition **6.9.8** to be a 0-dimensional pseudomanifold.

We will now prove the theorem by induction on k. We know $P(A)$ is a pseudomanifold if $k \geq n-2$. We will now assume this is true for $k+1$ and show it is true for k, where $0 \leq k < n-2$. As before, if $P(A) \neq \emptyset$, then we must have $A = C(\alpha)_{\bullet\beta}$ where $|\beta| = k$ and where $[\, A\,]$ is a $(k-1)$-face of $[\, I, -M\,]$. It is clear that $P(A)$ satisfies **6.9.8(a)**. Let A' be an $n \times (n-k-2)$ submatrix of $(I, -M)$. As P is a pseudomanifold, $[\, A, A'\,]$ can be a facet of at most two elements of P. Thus, $[\, A'\,]$ can be a facet of at most two elements in $P(A)$. Therefore, $P(A)$ satisfies **6.9.8(b)**.

We now turn our attention to showing that $P(A)$ satisfies **6.9.8(c)**. Notice that **6.9.8(c)** defines an equivalence relation in that we may partition the elements of $P(A)$ into disjoint equivalence classes so that for any two elements in $P(A)$, there is a sequence as given in **6.9.8(c)** if and only if the two elements are in the same equivalence class.

Let d be some column of the matrix $(I_{\bullet\bar{\beta}}, -M_{\bullet\bar{\beta}})$. Let B and B' be $n \times (n-k-2)$ submatrices of $(I, -M)$ such that $[\, d, B\,]$ and $[\, d, B'\,]$ are in $P(A)$. Clearly, $[\, B\,]$ and $[\, B'\,]$ are elements of $P(d, A)$ and, by induction, there is a sequence in $P(d, A)$, as given in **6.9.8(c)**, for $[\, B\,]$ and $[\, B'\,]$. If we adjoin d to each element in this sequence, we obtain a sequence in $P(A)$, as given in **6.9.8(c)**, for $[\, d, B\,]$ and $[\, d, B'\,]$. From this we conclude no column in $(I_{\bullet\bar{\beta}}, -M_{\bullet\bar{\beta}})$ can appear in elements of more than one equivalence class.

Let B be an $n \times (n-k-1)$ submatrix of $(I, -M)$ such that $[B] \in P(A)$. Clearly, B must be a submatrix of $(I_{\bullet\bar{\beta}}, -M_{\bullet\bar{\beta}})$. Since $k < n-2$, there is a facet of $[A, B]$, containing the face $[A]$. There must be another element of P also containing this facet. (P has no boundary, see **6.9.10**.) This other element of P gives us an element of $P(A)$ distinct from $[B]$ but sharing a facet with $[B]$. We conclude that each equivalence class must contain at least two elements. Thus, at least $n-k$ columns of the matrix $(I_{\bullet\bar{\beta}}, -M_{\bullet\bar{\beta}})$ must appear in each equivalence class. Since $(I_{\bullet\bar{\beta}}, -M_{\bullet\bar{\beta}})$ has $2(n-k)$ columns, we conclude there can be at most two equivalence classes. If there is only one equivalence class, then **6.9.8(c)** is satisfied, and the theorem follows. Therefore, we will now assume there are two equivalence classes, and we will show that this leads to a contradiction.

Suppose there are two equivalence classes. Recall that $A = C(\alpha)_{\bullet\beta}$. Given our observations above and the fact that P has no boundary, we may assume the two equivalence classes are as follows:

$$\{\, [B] : B \text{ is an } n \times (n-k-1) \text{ submatrix of } C(\alpha)_{\bullet\bar{\beta}} \,\},$$

$$\{\, [B] : B \text{ is an } n \times (n-k-1) \text{ submatrix of } C(\bar{\alpha})_{\bullet\bar{\beta}} \,\}.$$

If we select $i \in \bar{\beta}$, then $[C(\alpha)_{\bullet\bar{\imath}}]$ is a facet of $[I, -M]$. Thus, there is a vector $x \in R^n$ such that

$$x^{\mathrm{T}}C(\alpha)_{\bullet\bar{\imath}} = 0, \quad x^{\mathrm{T}}C(\alpha)_{\bullet i} > 0, \quad \text{and} \quad x^{\mathrm{T}}C(\bar{\alpha}) > 0.$$

We may assume x is scaled such that $x^{\mathrm{T}}C(\alpha)_{\bullet i} = 1$. This means we may assume $x = (C(\alpha))^{-1}_{i\bullet}$. Hence, $(C(\alpha))^{-1}_{\bar{\beta}\bullet}C(\bar{\alpha}) > 0$.

If we select $j \in \bar{\beta}$, then $[C(\alpha)_{\bullet\beta}, C(\bar{\alpha})_{\bullet\bar{\beta}\setminus\{j\}}]$ is a facet of $[I, -M]$. Thus, there is a vector $y \in R^n$ such that $y^{\mathrm{T}}C(\alpha)_{\bullet\beta} = 0$, $y^{\mathrm{T}}C(\bar{\alpha})_{\bullet\bar{\beta}\setminus\{j\}} = 0$, and $y^{\mathrm{T}}C(\alpha)_{\bullet\bar{\beta}} > 0$. Since $k < n-2$, we have $|\bar{\beta}| > 2$. Thus, select some $i \in \bar{\beta} \setminus \{j\}$. We have

$$\begin{aligned}
0 &= y^{\mathrm{T}}C(\bar{\alpha})_{\bullet i} \\
&= y^{\mathrm{T}}C(\alpha)(C(\alpha))^{-1}C(\bar{\alpha})_{\bullet i} \\
&= y^{\mathrm{T}}C(\alpha)_{\bullet\beta}(C(\alpha))^{-1}_{\beta\bullet}C(\bar{\alpha})_{\bullet i} + y^{\mathrm{T}}C(\alpha)_{\bullet\bar{\beta}}(C(\alpha))^{-1}_{\bar{\beta}\bullet}C(\bar{\alpha})_{\bullet i} \\
&= y^{\mathrm{T}}C(\alpha)_{\bullet\bar{\beta}}(C(\alpha))^{-1}_{\bar{\beta}\bullet}C(\bar{\alpha})_{\bullet i} > 0.
\end{aligned}$$

Thus, we arrive at a contradiction, and the theorem follows. □

6.9.19 Notation. For the rest of this section we will use the following notation. Let P denote the collection of facets of $[I, -M]$. Let $\bar{P}$ denote the collection of complementary simplices. If A is a submatrix of $(I, -M)$, let $P(A)$ and $\bar{P}(A)$ denote $P([A])$ and $\bar{P}([A])$, respectively.

We can now extend Definition **6.9.11** so that it applies to the restricted pseudomanifolds of $\bar{P}$. The reader should find the following extension to be quite natural.

6.9.20 Definition. Let $M \in R^{n\times n}$ be given with $n \geq 3$. Suppose M is totally nondegenerate and $\operatorname{pos}(I, -M)$ is pointed. Suppose all the facets of $[I, -M]$ are distinctly labelled. Given $A = C(\alpha)_{\bullet\beta}$, we say the simplices $[C(\gamma)_{\bullet\bar{\beta}}]$ and $[C(\gamma')_{\bullet\bar{\beta}}]$ contained in $\bar{P}(A)$ are *related*, which we denote by $[C(\gamma)_{\bullet\bar{\beta}}] \overset{A}{\sim} [C(\gamma')_{\bullet\bar{\beta}}]$, if $\bar{\beta} \cap \gamma = \bar{\beta} \cap \gamma'$ or if there exists a sequence of index sets $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_m = \gamma'$ such that for each $i \in \{1, \ldots, m\}$ the simplices $[C(\gamma_{i-1})_{\bullet\bar{\beta}}]$ and $[C(\gamma_i)_{\bullet\bar{\beta}}]$ have a facet in common and, further, this common facet is *not* an element of $P(A)$.

Note that $\overset{A}{\sim}$ is the same as $\sim$ if $\beta = \emptyset$. It should be clear that $\overset{A}{\sim}$ is an equivalence relation. As for the equivalence classes determined by $\overset{A}{\sim}$, we can quickly deduce the following.

6.9.21 Theorem. Let $M \in R^{n\times n}$ be given with $n \geq 3$. Suppose M is totally nondegenerate and $\operatorname{pos}(I, -M)$ is pointed. Suppose all the facets of $[I, -M]$ are distinctly labelled. Given $A = C(\alpha)_{\bullet\beta}$, let $\mathcal{C}$ be the set of elements of $\bar{P}(A)$ in some $\overset{A}{\sim}$ equivalence class. If we consider a polytope as representing the set of its extreme points, then $\mathcal{C}$ is an $(n - 1 - |\beta|)$-dimensional pseudomanifold. Further, if $F \in P(A)$, then there is some element of $\mathcal{C}$ which has F as a facet and, therefore, there are at most two $\overset{A}{\sim}$ equivalence classes.

Proof. The proof is similar to the proofs of Theorem **6.9.12** and Corollary **6.9.13**. We leave it to the reader as Exercise **6.10.40**. □

We are now in a position to show that there are exactly two $\sim$ equivalence classes. We will first show this in the case where not all the points in $\mathcal{S}$ are vertices of $[I, -M]$. This is the easy case as it can be shown directly, i.e., without the use of induction.

6.9.22 Theorem. Let $M \in R^{n\times n}$ be given with $n \geq 3$. Suppose M is totally nondegenerate and pos$(I, -M)$ is pointed. Suppose all the facets of $[\,I, -M\,]$ are distinctly labelled. Let $A = C(\alpha)_{\bullet\beta}$ be given and let x be a column of $(I_{\bullet\bar\beta}, -M_{\bullet\bar\beta})$. If $P(A) \neq \emptyset$ and if $[\,x\,]$ is not a vertex of any element in $P(A)$, then there are exactly two $\overset{A}{\sim}$ equivalence classes.

Proof. The case $|\bar\beta| = 1$ is trivial, thus we will assume $|\bar\beta| \geq 2$. Without loss of generality, we may assume $x = I_{\bullet i}$ where $i \in \bar\beta$. There are two cases depending on whether or not $[\,-M_{\bullet i}\,]$ is a vertex of any element in $P(A)$.

Suppose $[\,-M_{\bullet i}\,]$ is not a vertex of any element in $P(A)$. In what follows, B represents an appropriately sized submatrix of $(I, -M)$. Let

$$\mathcal{C} \;=\; \{\,[\,B,\, I_{\bullet i}\,] \in \bar P(A) : [\,B\,] \in P(A)\,\}\,.$$

As $P(A) \neq \emptyset$, then $\mathcal{C} \neq \emptyset$. We will show that if $S \in \mathcal{C}$ and $S' \in \bar P(A)$ share a common facet, then either $S' \in \mathcal{C}$ or the common facet is in $P(A)$. It follows from this that $\mathcal{C}$ contains an $\overset{A}{\sim}$ equivalence class. Yet, $\mathcal{C}$ is not all of $\bar P(A)$ as it contains no simplex with $[\,-M_{\bullet i}\,]$ as vertex. Thus, in this case, the theorem would follow from **6.9.21**.

As $|\bar\beta| \geq 2$, we may take $[\,B,\, I_{\bullet j},\, I_{\bullet i}\,]$, with $[\,B\,] \in P(A, I_{\bullet j})$, as an arbitrary element in $\mathcal{C}$. Now $P(A)$ is a pseudomanifold (see **6.9.18**), and it has no boundary (see **6.9.10** and **6.9.16**). Thus, there is exactly one other element of $P(A)$, aside from $[\,B,\, I_{\bullet j}\,]$, which contains $[\,B\,]$. Since no element of $P(A)$ has $[\,I_{\bullet i}\,]$ or $[\,-M_{\bullet i}\,]$ as a vertex, and as P is distinctly labelled, then we must have $[\,B,\, -M_{\bullet j}\,] \in P(A)$. Hence, $[\,B,\, -M_{\bullet j},\, I_{\bullet i}\,] \in \mathcal{C}$. We deduce that every element of $\bar P(A)$ adjacent to $[\,B,\, I_{\bullet j},\, I_{\bullet i}\,]$ is in $\mathcal{C}$ except for possibly $[\,B,\, I_{\bullet j},\, -M_{\bullet i}\,]$. However, the common facet between $[\,B,\, I_{\bullet j},\, I_{\bullet i}\,]$ and $[\,B,\, I_{\bullet j},\, -M_{\bullet i}\,]$ is in $P(A)$. The theorem, in this case, now follows.

Suppose $[\,-M_{\bullet i}\,]$ is a vertex of some element in $P(A)$. Let

$$\mathcal{C} \;=\; \{\,[\,B,\, -M_{\bullet i}\,] \in \bar P(A) : [\,B\,] \in P(A)\,\}\,.$$

Notice, if $F \in P(A)$ has $[\,-M_{\bullet i}\,]$ as a vertex, then there is a unique other element of $P(A)$ which shares a facet with F but doesn't have $[\,-M_{\bullet i}\,]$ as a vertex. Hence, there are elements of $P(A)$ which do not have $[\,-M_{\bullet i}\,]$ as a vertex and, so, $\mathcal{C} \neq \emptyset$. We will show that if $S \in \mathcal{C}$ and $S' \in \bar P(A)$ share a common facet, then either $S' \in \mathcal{C}$ or the common facet is in $P(A)$. As before, the theorem will follow from this.

Again, as $|\bar{\beta}| \geq 2$, we may take $[B, I_{\bullet j}, -M_{\bullet i}]$, with $[B] \in P(A, I_{\bullet j})$, as an arbitrary element in $\mathcal{C}$. There is exactly one other element of $P(A)$, aside from $[B, I_{\bullet j}]$, which contains $[B]$. Since no element of $P(A)$ has $[I_{\bullet i}]$ as a vertex, and as P is distinctly labelled, then we must have either $[B, -M_{\bullet i}] \in P(A)$ or $[B, -M_{\bullet j}] \in P(A)$, but not both. In the case where $[B, -M_{\bullet j}] \in P(A)$, we have $[B, -M_{\bullet j}, -M_{\bullet i}] \in \mathcal{C}$. Thus, as before, if a facet of $[B, I_{\bullet j}, -M_{\bullet i}]$ is not in $P(A)$, then the unique other element of $\bar{P}(A)$ containing that facet is in $\mathcal{C}$. The theorem follows. □

We finally move on to the main result needed to finish the proof of Theorem **6.9.5**.

6.9.23 Theorem. Let $M \in R^{n \times n}$ be given with $n \geq 3$. Suppose M is totally nondegenerate and pos$(I, -M)$ is pointed. Suppose all the facets of $[I, -M]$ are distinctly labelled. Let $A = C(\alpha)_{\bullet \beta}$ be given. If $P(A) \neq \emptyset$, then there are exactly two $\overset{A}{\sim}$ equivalence classes.

Proof. The case $|\bar{\beta}| = 1$ is trivial. Suppose $|\bar{\beta}| = 2$ and assume $\bar{\beta} = \{i, j\}$. Since $P(A)$ is a pseudomanifold (see **6.9.18**) and it has no boundary (see **6.9.10** and **6.9.16**), then $P(A)$ consists of exactly two vertices. As the reader can check, there are essentially two cases. An example of one of the cases would be $P(A) = \{[I_{\bullet i}], [I_{\bullet j}]\}$. There are two $\overset{A}{\sim}$ equivalence classes, namely,

$$\{[I_{\bullet i}, I_{\bullet j}]\} \quad \text{and} \quad \{[-M_{\bullet i}, I_{\bullet j}], [I_{\bullet i}, -M_{\bullet j}], [-M_{\bullet i}, -M_{\bullet j}]\}.$$

An example of the other case would be $P(A) = \{[I_{\bullet i}], [-M_{\bullet i}]\}$. There are two $\overset{A}{\sim}$ equivalence classes, namely,

$$\{[I_{\bullet i}, I_{\bullet j}], [-M_{\bullet i}, I_{\bullet j}]\} \quad \text{and} \quad \{[I_{\bullet i}, -M_{\bullet j}], [-M_{\bullet i}, -M_{\bullet j}]\}.$$

Thus, in all cases, the theorem holds if $|\bar{\beta}| = 2$.

We now proceed by induction. Select $k \in \{3, \ldots, n\}$. We will assume that if $|\bar{\beta}| < k$, then

(a) the theorem holds,

(b) if $[x] \in \mathcal{S}$ and if $[B], [B'] \in \bar{P}(A, x)$, then $[B, x] \overset{A}{\sim} [B', x]$ implies $[B] \overset{A,x}{\sim} [B']$.

Notice that (a) and (b) hold if $|\bar{\beta}| \leq 2$. We will now show that (a) and (b) hold if $|\bar{\beta}| = k$.

We will first show that (a) holds if $|\bar{\beta}| = k$. We may assume that, for every column x of the matrix $(I_{\bullet\bar{\beta}}, -M_{\bullet\bar{\beta}})$, there is some element of $P(A)$ which has $[\,x\,]$ as a vertex. Otherwise, (a) follows directly from Theorem **6.9.22**. Thus, if we select $i \in \bar{\beta}$, then $P(A, I_{\bullet i})$ and $P(A, -M_{\bullet i})$ are nonempty. By induction the elements of $\bar{P}(A, I_{\bullet i})$ are partitioned into two equivalence classes by $\overset{A,I_{\bullet i}}{\sim}$. Denote these two classes by $\mathcal{C}_I$ and $\mathcal{C}'_I$. Similarly, the elements of $\bar{P}(A, -M_{\bullet i})$ are partitioned into two equivalence classes by $\overset{A,-M_{\bullet i}}{\sim}$. Denote these two classes by $\mathcal{C}_M$ and $\mathcal{C}'_M$. Let $\bar{\mathcal{C}}_I = \{\,[\,B,\, I_{\bullet i}\,] : [\,B\,] \in \mathcal{C}_I\}$ and $\bar{\mathcal{C}}_M = \{\,[\,B,\, -M_{\bullet i}\,] : [\,B\,] \in \mathcal{C}_M\}$, with similar definitions for $\bar{\mathcal{C}}'_I$ and $\bar{\mathcal{C}}'_M$. Notice, $\bar{\mathcal{C}}_I$, $\bar{\mathcal{C}}_M$, $\bar{\mathcal{C}}'_I$, and $\bar{\mathcal{C}}'_M$, are mutually disjoint and their union is $\bar{P}(A)$. In addition, note that $\bar{\mathcal{C}}_I$ is entirely contained in some $\overset{A}{\sim}$ equivalence class and, further, the same can be said of $\bar{\mathcal{C}}_M$, $\bar{\mathcal{C}}'_I$, and $\bar{\mathcal{C}}'_M$.

Theorem **6.9.21** implies that the only way in which (a) can fail is for all of $\bar{P}(A)$ to be one equivalence class. We will now show that this cannot happen.

We may take $[\,B,\, I_{\bullet j},\, I_{\bullet i}\,]$ to be an arbitrary element of $\bar{\mathcal{C}}_I$, where $[\,B\,]$ is in $\bar{P}(A, I_{\bullet j}, I_{\bullet i})$. Suppose $[\,B,\, I_{\bullet i}\,] \notin P(A)$. This implies $[\,B\,] \notin P(A, I_{\bullet i})$. Thus, $[\,B,\, -M_{\bullet j}\,] \in \mathcal{C}_I$ and, hence, $[\,B,\, -M_{\bullet j},\, I_{\bullet i}\,] \in \bar{\mathcal{C}}_I$. We have now shown that if $S \in \bar{\mathcal{C}}_I$ and $S' \in \bar{P}(A)$ share a common facet which is not in $\mathcal{C}_I$, then either $S' \in \bar{\mathcal{C}}_I$ or the common facet is in $P(A)$. From this we conclude that if $\mathcal{C}_I \subseteq P(A)$, then $\bar{\mathcal{C}}_I$ is a $\overset{A}{\sim}$ equivalence class and, hence, (a) would follow. Therefore, assume $[\,B,\, I_{\bullet j}\,] \notin P(A)$ and, hence, $[\,B,\, I_{\bullet j},\, I_{\bullet i}\,] \overset{A}{\sim} [\,B,\, I_{\bullet j},\, -M_{\bullet i}\,]$. We may assume $[\,B,\, I_{\bullet j},\, -M_{\bullet i}\,] \in \bar{\mathcal{C}}_M$. Thus, $\bar{\mathcal{C}}_I \cup \bar{\mathcal{C}}_M$ is contained in a $\overset{A}{\sim}$ equivalence class.

Since we now know that an element in $\bar{\mathcal{C}}_I$ and an element in $\bar{\mathcal{C}}_M$ share the common facet $[\,B,\, I_{\bullet j}\,] \notin P(A)$, the question arises as to whether it is possible for an element in $\bar{\mathcal{C}}_I$ and an element in $\bar{\mathcal{C}}'_M$ to share a common facet $F \notin P(A)$. There are two cases to consider depending on whether or not F and $[\,B,\, I_{\bullet j}\,]$ have any common vertices.

Suppose F and $[\,B,\, I_{\bullet j}\,]$ share a common vertex. We may take F to be $[\,B',\, I_{\bullet j}\,]$ where $[\,B'\,] \in \bar{P}(A, I_{\bullet j}, I_{\bullet i})$. Thus, we have $[\,B',\, I_{\bullet j}\,] \in \mathcal{C}_I \cap \mathcal{C}'_M$. We know $[\,B,\, I_{\bullet j}\,] \overset{A,I_{\bullet i}}{\sim} [\,B',\, I_{\bullet j}\,]$ because both are elements in $\mathcal{C}_I$. By part (b) of our induction assumption, we have $[\,B\,] \overset{A,I_{\bullet j},I_{\bullet i}}{\sim} [\,B'\,]$. This implies $[B, I_{\bullet i}] \overset{A,I_{\bullet j}}{\sim} [B', I_{\bullet i}]$. As $[B, I_{\bullet j}]$ and $[B', I_{\bullet j}]$ are not in $P(A)$, we have $[B, I_{\bullet i}] \overset{A,I_{\bullet j}}{\sim} [B, -M_{\bullet i}]$ and $[B', I_{\bullet i}] \overset{A,I_{\bullet j}}{\sim} [B', -M_{\bullet i}]$. Thus, $[B, -M_{\bullet i}] \overset{A,I_{\bullet j}}{\sim} [B', -M_{\bullet i}]$. Again, using part (b) of our induction assumption, we have $[B] \overset{A,I_{\bullet j},-M_{\bullet i}}{\sim} [B']$. Hence, $[B, I_{\bullet j}] \overset{A,-M_{\bullet i}}{\sim} [B', I_{\bullet j}]$. However, as know $[B, I_{\bullet j}] \in \mathcal{C}_M$ and $[B', I_{\bullet j}] \in \mathcal{C}'_M$, we have a contradiction. Therefore, F and $[B, I_{\bullet j}]$ cannot share a common vertex.

The only possibility left is that $F = [B', -M_{\bullet j}]$, where there is a index set γ for which $B = C(\gamma)_{\bullet\bar{\beta}\setminus\{i,j\}}$ and $B' = C(\bar{\gamma})_{\bullet\bar{\beta}\setminus\{i,j\}}$. Now consider $\bar{\mathcal{C}}'_I$. Using a previous argument in this proof, we know that if $\mathcal{C}'_I \subseteq P(A)$, then $\bar{\mathcal{C}}'_I$ is a $\overset{A}{\sim}$ equivalence class and, hence, (a) would follow. Therefore, there is some $F' \in \mathcal{C}'_I$ which is not in $P(A)$. As $[B, I_{\bullet j}]$ and $[B', -M_{\bullet j}]$ are elements of $\mathcal{C}_I$, then F' is neither of these. From this we see that F' must have a vertex in common with both $[B, I_{\bullet j}]$ and $[B', -M_{\bullet j}]$. Thus, using F' and $[B, I_{\bullet j}]$ if $F' \in \mathcal{C}_M$, or using F' and $[B', -M_{\bullet j}]$ if $F' \in \mathcal{C}'_M$, an argument similar to the one given in the previous paragraph will lead us to a contradiction. Therefore, F cannot exist.

We have now shown that if F is the common facet between an element in $\bar{\mathcal{C}}_I$ and an element in $\bar{\mathcal{C}}'_M$, then $F \in P(A)$. We may similarly show that if F is the common facet between an element in $\bar{\mathcal{C}}_M$ and an element in $\bar{\mathcal{C}}'_I$, then $F \in P(A)$. We conclude from this that $\bar{\mathcal{C}}_I \cup \bar{\mathcal{C}}_M$ is a $\overset{A}{\sim}$ equivalence class. Therefore, (a) is true in all cases.

We will now show that (b) holds if $|\bar{\beta}| = k$. Suppose $[B, I_{\bullet i}] \overset{A}{\sim} [B', I_{\bullet i}]$ where $[B]$ and $[B']$ are elements of $\bar{P}(A, I_{\bullet i})$. To prove (b), we must show $[B] \overset{A,I_{\bullet i}}{\sim} [B']$.

First, we will assume that $[I_{\bullet i}]$ is not a vertex of any element in $P(A)$. This is the easy case because the assumption we just made implies that $P(A, I_{\bullet i}) = \emptyset$. Consequently, all of $\bar{P}(A, I_{\bullet i})$ forms one $\overset{A,I_{\bullet i}}{\sim}$ equivalence class and, hence, $[B] \overset{A,I_{\bullet i}}{\sim} [B']$.

Now we will assume that $[I_{\bullet i}]$ is a vertex of some element in $P(A)$. Thus, $P(A, I_{\bullet i}) \neq \emptyset$. Define $\mathcal{C}_I$, $\mathcal{C}'_I$, $\bar{\mathcal{C}}_I$, and $\bar{\mathcal{C}}'_I$ as before. It is not hard to see that there must be some element in $\mathcal{C}_I$ which has a common facet with some element in $\mathcal{C}'_I$. We may assume these elements are $[D, I_{\bullet j}] \in \mathcal{C}_I$ and $[D, -M_{\bullet j}] \in \mathcal{C}'_I$, where $j \in \bar{\beta}\setminus\{i\}$ and $[D] \in \bar{P}(A, I_{\bullet i}, I_{\bullet j})$. Of course, this means $[D] \in P(A, I_{\bullet i})$ and, hence, $[D, I_{\bullet i}] \in P(A)$.

Our goal is to show $[B] \overset{A,I_{\bullet i}}{\sim} [B']$. That is, we must show $[B]$ and $[B']$ are both in $\mathcal{C}_I$ or are both in $\mathcal{C}'_I$. If not, by considering the assumption that $[B, I_{\bullet i}] \overset{A}{\sim} [B', I_{\bullet i}]$, we would have to conclude that $\bar{C}_I$ and $\bar{C}'_I$ are in the same $\overset{A}{\sim}$ equivalence class. Yet, we have already proven part (a) of the induction, so we know there are two $\overset{A}{\sim}$ equivalence classes. By Theorem **6.9.21**, both of these $\overset{A}{\sim}$ equivalence classes must contain an element having $[D, I_{\bullet i}]$ as a facet. However, if $\bar{C}_I$ and $\bar{C}'_I$ are in the same $\overset{A}{\sim}$ equivalence class, then that equivalence class contains both $[D, I_{\bullet j}, I_{\bullet i}]$ and $[D, -M_{\bullet j}, I_{\bullet i}]$. Thus, the other equivalence class would not contain an element having $[D, I_{\bullet i}]$ as a facet. Therefore, $\bar{C}_I$ and $\bar{C}'_I$ are in different $\overset{A}{\sim}$ equivalence classes and, so, $[B] \overset{A, I_\bullet}{\sim} [B']$. We have now shown that (b) is true in all cases. This completes the induction and, hence, the theorem holds. □

Proof of sufficiency for Theorem 6.9.5. We have already shown the theorem is true for $n \leq 2$. Thus, we assume $n \geq 3$. Since $P = P(\emptyset)$, and since $P \neq \emptyset$, Theorem **6.9.23** implies there are two $\sim$ equivalence classes. Theorem **6.9.5** now follows from Theorem **6.9.14**. □

6.10 Exercises

6.10.1 Let $A \in R^{n \times p}$ be given. Show that the affine hull of $\operatorname{pos} A$ is the subspace of R^n spanned by the columns of A. In addition, show that $\operatorname{pos} A$ has a nonempty relative interior.

6.10.2 Prove that Proposition **6.1.2** is true.

6.10.3 In **6.1.16** we determined the degree of the matrix whose complementary cones are shown in Figure 1.2. We also determined that the matrix depicted in Figure 1.4 does not have a well-defined degree. Determine, for Figures 1.3, 1.5, 1.6, and 1.7, whether or not the matrices depicted have a well-defined degree and, if so, the value of the degree. (Note: It is not necessary to determine actual numerical values for the entries of the matrices in order to do this exercise.)

6.10.4 Show that if $M \in \boldsymbol{P}$, then M has a well-defined degree and $\deg M = 1$. Give a proof or a counterexample to the converse of the previous statement.

6.10.5 Let $M \in R^{n\times n}$ be given. Remember that pos $(I, -M)$, the convex hull of $K(M)$, is the set of vectors q for which the LCP (q, M) is feasible. Show that $M \in \boldsymbol{S_0}$ if and only if $\text{pos}(I, -M)$ is not strictly pointed.

6.10.6 Let $\{M_i\}$ be a sequence of matrices in $R^{n\times n}$, and $\{x^i\}$ a sequence of points in R^n, such that $\lim_{i\to\infty} M_i = M$ and $\lim_{i\to\infty} x_i = x$. Show that $\lim_{i\to\infty} f_{M_i}(x^i) = f_M(x)$.

6.10.7 Let $M \in R^{n\times n}$ be given. Show that M is an $\boldsymbol{R_0}$-matrix if and only if no complementary cone of M is strongly degenerate.

6.10.8 For fixed $n \geq 2$, let $\mathcal{S} \subseteq R^n \times R^{n\times n}$ be the set of all vector-matrix pairs (q, M) for which $\deg_M(q)$ is well-defined by **6.1.4**. Is $\mathcal{S}$ a closed set in $R^n \times R^{n\times n}$? Is $\mathcal{S}$ an open set in $R^n \times R^{n\times n}$?

6.10.9 Given $M \in R^{n\times n}$, show that $K(M) = \text{cl}\big(\text{int}\, K(M)\big)$ if M is nondegenerate.

6.10.10 Given $M \in R^{n\times n}$, show that $M \in \boldsymbol{Q}$ if and only if the union of the nondegenerate complementary cones relative to M is equal to R^n.

6.10.11 Show that in the defintion of $\mathcal{L}(M)$, given in **6.2.3**, both conditions (a) and (b) are needed. (Give an example of a matrix M for which $\mathcal{L}(M)$ would change if condition (a) were dropped. Likewise, give an example of a matrix M for which $\mathcal{L}(M)$ would change if condition (b) were dropped.)

6.10.12 Let $M \in R^{n\times n}$, $\alpha \subseteq \{1, \ldots, n\}$, and $i \in \{1, \ldots, n\}$ be given. Suppose that neither $\text{pos}\, C_M(\alpha)$ nor $\text{pos}\, C_M(\alpha \triangle \{i\})$ is weakly degenerate. Show that for any $q \in \text{pos}\, C_M(\alpha)_{\cdot\bar{i}} \setminus \mathcal{L}(M)$, the family of facets around q containing $\text{pos}\, C_M(\alpha)_{\cdot\bar{i}}$ consists solely of $\text{pos}\, C_M(\alpha)_{\cdot\bar{i}}$. In addition, show that the class of the family does not depend on q.

6.10.13 Give an example of a matrix M and a vector $q \in K(M) \setminus \mathcal{L}(M)$ such that q is not contained in any full or weakly degenerate complementary cone relative to M. Is it possible to construct such an example so that q is contained in an odd number of complementary cones?

6.10.14 Supply the proof to Theorem **6.2.24**.

6.10.15 Give an example to show that Theorem **6.2.28** is stronger than Theorem **6.1.17**. That is, in the notation of the two theorems, give an example of a matrix M where the connected components of $R^n \setminus \mathcal{C}$ are not the same as the connected components of $R^n \setminus \mathrm{cl}\,\mathcal{S}$.

6.10.16 For the LCP given in Example **6.3.3**, the complementary cones and the ray $\{q + dz_0 : z_0 \geq 0\}$ are given in Figure 6.1 while the path of solutions which Lemke's method follows is shown in Figure 6.2. What would these two figures look like if

$$M = \begin{bmatrix} 0 & 1 \\ -3 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad \text{and} \quad d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}?$$

Using your figures, describe how Algorithms **4.4.5**, **4.5.4**, and **6.3.1** would process this LCP.

6.10.17 Given $M \in R^{n \times n}$ and $q \in R^n$, we know that if q is not contained in any degenerate complementary cone, then $|\,\mathrm{SOL}(q, M)| < \infty$. (For example, see the discussion just before Theorem **6.6.1**.) Is the converse true, that is, if $|\,\mathrm{SOL}(q, M)| < \infty$, must it be the case that all the cones containing q are nondegenerate? Give a proof or a counterexample.

6.10.18 Let $M \in R^{n \times n}$ be given. Suppose there is a complementary cone relative to M with dimension m. Show that if $m \leq k \leq n$, then there is a complementary cone relative to M with dimension k.

6.10.19 Let $M \in R^{n \times n}$ be given. Suppose there are n or fewer full complementary cones relative to M. Show that $M \notin \boldsymbol{Q}$.

6.10.20 Let $M \in \boldsymbol{R_0} \cap R^{3 \times 3}$ be given. Suppose $\deg M = 2$.

(a) Show that at least six of the eight complementary cones relative to M will have a positive index.

(b) Show that exactly six of the eight complementary cones relative to M must have a positive index and, further, that at least one of the two remaining complementary cones must have a negative index.

(c) Show that there is no continuous function $M_t : [\,0, 1\,] \to \boldsymbol{R_0} \cap R^{3 \times 3}$ with M_0 and M_1 as in (6.1.1).

6.10.21 Show that $\boldsymbol{E}^f = \boldsymbol{P}$.

6.10.22 With reference to Definition **6.6.5**, which of the following matrix classes are full? Column adequate; column sufficient; copositive-plus; $\boldsymbol{Z}$; $\boldsymbol{Q}$; $\boldsymbol{Q}_0$. For each matrix class, either give a proof that it is full or a counterexample showing that it is not full.

6.10.23 Let $M \in \boldsymbol{U} \cap R^{n \times n}$ be given.

(a) If $M \in \boldsymbol{Q}$, show that $M \in \boldsymbol{P}$.

(b) If $M \notin \boldsymbol{Q}$, show that $|\,\mathrm{SOL}(q, M)| = \infty$ for any $q \in \mathrm{bd}(\mathrm{int}\, K(M))$.

6.10.24 Given $M \in R^{n \times n}$, show that $M \in \boldsymbol{P}$ implies $M \in \boldsymbol{W}$.

6.10.25 Show that $\boldsymbol{W}$ is a full matrix class.

6.10.26 Show that the inclusions $\boldsymbol{P} \subseteq \boldsymbol{W} \subseteq \boldsymbol{U}$ are all proper.

6.10.27 Let $M \in R^{n \times n}$ be given. Suppose there is a positive integer k such that $|\,\mathrm{SOL}(q, M)| = k$ for all $q \in K(M)$. Show that $M \in \boldsymbol{P}$.

6.10.28 A matrix $M \in R^{n \times n}$ is said to be *weakly separating* if for all index sets $\alpha \subseteq \{1, \dots, n\}$ and all $i \in \{1, \dots, n\}$ there exists a hyperplane $H = \{x \in R^n : a^{\mathrm{T}}x = 0\}$, with $0 \neq a \in R^n$, such that $a_i \geq 0$, $a^{\mathrm{T}}M_{\bullet i} \geq 0$, and $\mathrm{pos}\, C_M(\alpha)_{\bullet \bar{\imath}} \subseteq H$. If we require that $a_i > 0$ and $a^{\mathrm{T}}M_{\bullet i} > 0$ (instead of $a_i \geq 0$ and $a^{\mathrm{T}}M_{\bullet i} \geq 0$), then M is said to be *strictly separating*. (The hyperplane H separates the vectors $I_{\bullet i}$ and $-M_{\bullet i}$.)

(a) Show that M is strictly separating if and only if each principal pivotal transform of M has a positive diagonal.

(b) Show that M is strictly separating if and only if $M \in \boldsymbol{P}$.

(c) Show that M is weakly separating if and only if each principal pivotal transform of M has a nonnegative diagonal.

(d) Show that the class of weakly separating matrices contains, but is not equal to, the class $\boldsymbol{P}_0$.

6.10.29 Let $M \in R^{n \times n}$ be given and suppose M is nondegenerate. Show that $M \in \boldsymbol{INS}$ if and only if $K(M)$ is regular.

6.10.30 Does there exist an $M \in \boldsymbol{U} \setminus \boldsymbol{W}$ such that no complementary cone relative to M is weakly degenerate?

6.10.31 A matrix $M \in R^{n \times n}$ is said to be *almost* $\boldsymbol{N}$ if all its proper principal minors (other than $\det M_{\emptyset\emptyset}$) are negative and if $\det M > 0$. A matrix $M \in R^{n \times n}$ is said to be almost $\boldsymbol{N}$ of *first category* if it is almost $\boldsymbol{N}$ and if both M and M^{-1} have at least one positive element.

(a) Let $M \in R^{n \times n}$ be given. Suppose all the principal minors of M of orders 1, 2, and 3 are negative. Show there exists an index set $\alpha \subseteq \{1, \ldots, n\}$ such that both $M_{\alpha\alpha}$ and $M_{\bar{\alpha}\bar{\alpha}}$ have only negative elements and both $M_{\alpha\bar{\alpha}}$ and $M_{\bar{\alpha}\alpha}$ have only positive elements.

(b) Let $M \in R^{n \times n}$ be given with $n \geq 4$. Suppose M is almost $\boldsymbol{N}$ and suppose $M \not\leq 0$. Let α be the index set described in part (a) of this exercise. Show that for each $q > 0$ there exists two complementary cones $\operatorname{pos} C_M(\beta)$ and $\operatorname{pos} C_M(\gamma)$ such that $q \in \operatorname{pos} C_M(\beta) \cap \operatorname{pos} C_M(\gamma)$, $\emptyset \neq \beta \subseteq \alpha$, and $\emptyset \neq \gamma \subseteq \bar{\alpha}$.

(c) Let $M \in R^{n \times n}$ be given with $n \geq 4$. Suppose M is almost $\boldsymbol{N}$ of first category. Show there exists a $q > 0$ such that $|\operatorname{SOL}(q, M)| = 3$ and $\deg_M(q) = -1$.

6.10.32 Consider the set $\boldsymbol{Q} \cap R^{n \times n}$ as a subset of $R^{n \times n}$.

(a) If $n \geq 1$, show that $\boldsymbol{Q} \cap R^{n \times n}$ is not closed in $R^{n \times n}$.

(b) If $n \geq 4$, show that $\boldsymbol{Q} \cap R^{n \times n}$ is not open in $R^{n \times n}$.

6.10.33 For $n \geq 3$, let $M \in R^{n \times n}$ be the matrix in which each diagonal element equals -1 and all other elements equal 2, i.e., $M = 2ee^{\mathrm{T}} - 3I$. Consider the matrix

$$\bar{M} = \begin{bmatrix} M & e \\ e^{\mathrm{T}} & 0 \end{bmatrix} \in R^{(n+1) \times (n+1)}.$$

Show that $\bar{M} \in \boldsymbol{Q}$.

6.10.34 Show that

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 1 & 1 & 1 & -3 & 0 \\ 1 & 1 & 1 & 0 & -3 \end{bmatrix}$$

is a $\boldsymbol{Q}$-matrix.

6.10.35 Supply a proof to Theorem **6.8.4**.

6.10.36 Supply a proof to Proposition **6.8.7**.

6.10.37 Let n be a positive integer. If n is odd, then let M be the matrix given in Theorem **6.8.3**. If n is even, then let M be the matrix given in Theorem **6.8.4**. Show there is a real number λ such that $\mathcal{P}_M$ is similar to the polytope $\{x \in R^n : 0 \leq x \leq e,\ e^{\mathrm{T}}x = \lambda\}$. Thus, $\mathcal{P}_M$ is a section of a cube. (Note, two polytopes in R^n are said to be *similar* if there is a function $f : R^n \to R^n$, which transforms one polytope into the other, such that for some $t > 0$ we have $\|f(x) - f(y)\| = t\|x - y\|$ for all $x, y \in R^n$.)

6.10.38 Let P be a pseudomanifold on the set V. Show that if P has the restricted property, as given in Definition **6.9.15**, then $P(S)$ has the restricted property for every $S \subseteq V$. Show that if P has the restricted property and has no boundary, then $P(S)$ has no boundary for every $S \subseteq V$.

6.10.39 Supply a proof to Theorem **6.9.17**.

6.10.40 Supply a proof to Theorem **6.9.21**.

6.11 Notes and References

6.11.1 There are many works which have applied degree theory to the study of the LCP. Some examples would include Garcia and Zangwill (1981), Garcia, Gould, and Turnbull (1983, 1984), Howe (1983), Howe and Stone (1983), Stone (1986), and Morris (1990a). Many of the basic definitions and results found in Section 6.1 can be seen throughout this literature. The reader may also wish to consult such works as Saigal and Simon (1973), Kojima and Saigal (1981), and Ha (1987), which use degree theory to study the general complementarity problem.

It is possible to extend the degree-theoretic results we have presented so that $\deg_M(q)$ is well-defined for points $q \in \mathcal{K}(M)$ which are not contained in any strongly degenerate complementary cones. For such a q there will exist an $\varepsilon > 0$ such that $\deg_M(q')$ is well-defined and takes on the same value for all $q' \in B(q, \varepsilon) \setminus \mathcal{K}(M)$. One may properly define $\deg_M(q)$ to be this common value. Furthermore, the set $\mathrm{SOL}(q, M)$ is compact and one may properly define an index for each of the connected components of $\mathrm{SOL}(q, M)$ such that the sum of the indexes is $\deg_M(q)$. For an example of how this might be accomplished in the case where all the connected components are individual points, and for examples of how such extended definitions of degree and index might be applied, the reader should consult Stewart (1993) and Gowda (1991b).

For a different topological approach to the LCP, see Naiman and Stone (1998) where homology theory is used to obtain a characterization of the class $\boldsymbol{Q}$ which leads to a test for membership. This test has better time complexity, but worse space complexity, than the one attributed to Gale (see Note **3.13.4**). Unfortunately, both tests are inefficient.

6.11.2 Theorem **6.1.12** can be viewed as an analogue of Sard's theorem for f_M. See Theorem 3.1.3 in Hirsch (1976) or Theorem 22.1.1 in Garcia and Zangwill (1981).

6.11.3 The term *strictly pointed* is taken from Doverspike (1982). The term *totally nondegenerate* is taken from Doverspike and Lemke (1982).

6.11.4 The example given in (6.1.1) is from Morris (1990b).

6.11.5 Theorem **6.1.23** is basically the contrapositive of Proposition 2.2 in Doverspike (1982).

6.11.6 From Theorem **6.1.27** one can deduce that $\mathrm{SOL}(q, M)$ is unbounded if and only if q is contained in a strongly degenerate cone relative to M. Moreover, from the proof of the theorem, one sees that q is contained in a strongly degenerate cone if and only if the LCP (q, M) has a solution ray. See Definition **7.5.4** and Note **7.7.12**.

6.11.7 As mentioned in Note **1.7.12**, the use of complementary cones to study the LCP can be traced back to Samelson, Thrall, and Wesler (1958) and Murty (1972), particularly the latter. Subsequently, complementary cones, their facets, and path-following arguments have become standard tools in the study of the LCP. Rather than give what (would have to be) an extremely abbreviated list of some of the works dealing with these topics, we invite the reader to select any of the material referenced in Notes **6.11.3**, **6.11.8**, **6.11.11**, **6.11.13**, or **6.11.25** as examples taken from this literature.

6.11.8 When path-following arguments are employed to study the LCP, it is common to require that certain "degenerate" points not be on the path which is used. At these points it is difficult to keep track of which complementary cones the path is entering and/or leaving. In this chapter, we have used $\mathcal{L}(M)$ as the set of degenerate points. By using Theorems **6.2.7** and **6.2.8** we insure that the paths we use never contain any points in $\mathcal{L}(M)$.

The set $\mathcal{L}(M)$ appears in Saigal (1972a) as the union of two sets. The two sets are denoted by $\bar{D}$ and E, and they correspond, respectively, to conditions (a) and (b) of Definition **6.2.3**. In addition, Saigal (1972a) contains a proof of Theorem **6.2.8**.

A smaller set of degenerate points is used in Doverspike (1982) where only the set $\bar{D}$, using Saigal's notation, is to be excluded from paths. We could have used this smaller set here, rather than $\mathcal{L}(M)$, but very few of our results would have gained anything by this (mainly Definition **6.2.20** would be slightly more general) while the proofs of many of them would have had additional complexity introduced.

6.11.9 Lemmas **6.2.15** and **6.2.17** are special cases of Lemma 3.1 in Saigal (1972a).

6.11.10 As pointed out in Remark **6.2.19**, the underlying idea behind the proof of Proposition **6.2.18** is a graph-theoretic argument. This is also the reasoning which underlies the proof of convergence of Lemke's algorithm (Theorem **4.4.4**). In fact, this argument can be found in the original papers describing Lemke's algorithm (Lemke and Howson (1964) and Lemke (1965)). One sometimes encounters this reasoning referred to as a "Lemke's ghost" argument alluding to a memorable interpretation due to B.C. Eaves (see Section 2.2.6 of Murty (1988)). Cottle and Dantzig (1970) explicitly describe the graph theory behind this argument. They then extend the argument to cover the vertical generalization of the LCP (see (1.5.3)). For a treatment of Lemke's algorithm in terms of graph theory and combinatorial topology see Shapley (1974).

6.11.11 The concept, in the nondegenerate case, of proper and reflecting facets (Definition **6.2.10**) can be seen implicitly in several works. With regard to facets, our terminology and definitions are derived from Saigal (1972b), Stone (1981, 1986), and Saigal and Stone (1985). Much of Section 6.2 can be traced to these references. It should be noted that Saigal (1972b) considered any facet contained in the boundary of $K(M)$ to be proper. By doing this the definition of regular (Definition **6.6.14**) can be simply stated, in the nondegenerate case, by saying that a matrix M is regular if all the facets of all the complementary cones relative to M are proper. Stone (1981, 1986) defines facets to be proper and reflecting only in the nondegenerate case. By the definitions given in Saigal (1972b), the common facet of a full cone and a degenerate cone would be considered a proper facet. This means that some results given in Saigal (1972b) require certain corrections, as is documented in Stone (1981) and Saigal and Stone (1985). Our Definition **6.2.20** is taken from Saigal and Stone (1985). However, we have split the class of absorbing facets, as defined in that paper, into the classes of absorbing and isolated facets. Also, we have added the class of cyclic facets.

6.11.12 The LCP parity theorem (Theorem **6.2.27**) is a stronger version of Theorem 3.1 in Saigal (1972a). The same idea underlies the proofs of Theorems **6.2.27**, **6.2.28**, and 3.1 (in Saigal (1972a)). Earlier (and less general) LCP parity theorems can be found in Saigal (1970b) and Murty (1972).

6.11.13 Many works describe and use Lemke's method from a geometric viewpoint. For some examples of this literature, and further references, the reader is directed to Saigal (1972b), Lemke (1980), Garcia and Zangwill (1981), Doverspike (1982), and Garcia, Gould, and Turnbull (1984).

6.11.14 The basic results concerning the behavior of Lemke's method as it relates to complementary cones and their facets is first taken up in Saigal (1972b). Theorem **6.3.7** and Corollary **6.3.9** are derived from the material in Saigal (1972b) and Saigal and Stone (1985).

6.11.15 An essential ingredient driving the proofs of Theorem **6.3.10**, Corollaries **6.3.11** and **6.3.12**, and Theorem **6.3.13** is the fact that, under nondegeneracy, the almost complementary path followed by Lemke's method must reach a solution if the only ray the path contains is the initial ray. This fact is stated explicitly as the corollary to Theorem 1 in Cottle and Dantzig (1968), however, it appears explicitly or implicitly in many of the works dealing with Lemke's method including Lemke (1965) itself.

6.11.16 Lemke's method with an arbitrary covering vector, as discussed in the text between Corollaries **6.3.11** and **6.3.12**, is essentially the same as the method proposed in Garcia, Gould, and Turnbull (1984). That paper uses a homotopy approach in describing and analyzing the method. The contents of Corollary **6.3.12** and Theorem 4.8 in Garcia, Gould, and Turnbull (1984) are basically the same. The reader should note that the matrix class $\boldsymbol{Q}_0$, as given in that paper, refers to the set of $\boldsymbol{R}_0$-matrices which have nonzero degree and are not superfluous. Why such a matrix class would be of interest can be seen in the discussion following Corollary **6.3.12**.

6.11.17 For the special case of copositive-plus matrices, Theorem **6.3.14** is part of Theorem 3.1 of Cottle (1974b). The proof of our theorem is based on the proof of Cottle's theorem.

6.11.18 Corollary **6.3.15** shows that Lemke's method will process an $\boldsymbol{L}$-matrix. As mentioned in Note **4.12.15**, this result is implied by Theorem **4.4.15**. We point out that Eaves (1971a), which first defined the class $\boldsymbol{L}$, was the first to show that Lemke's method would process an $\boldsymbol{L}$-matrix.

6.11.19 The augmented LCP addressed in Theorem **6.4.1** was discussed in Note **3.13.13**.

Bimatrix games, addressed in Theorem **6.4.2**, were discussed in Note **1.7.4**. The reader is also directed to Shapley (1974) which proves Theorem **6.4.2** via a graph-theoretic approach to index theory.

Theorem **6.4.3** is a generalization of Theorem 2.7 in Doverspike (1982). This latter theorem is similar to Theorem **6.4.3** but its hypotheses include the additional requirement that $\deg_M(q)$ be odd for all $q \in \mathcal{S}$.

Theorem **6.4.4** was first shown as part of Theorem 3.3 in Doverspike (1982).

6.11.20 Section 6.5 here is based on Section 4 of Howe and Stone (1983).

6.11.21 The notion given in Section 6.6, that $M \in R^{n\times n} \cap \boldsymbol{P}$ if and only if the complementary cones relative to M are nondegenerate and partition R^n, dates back to Samelson, Thrall, and Wesler (1958). Theorem **6.6.1** was first shown as Theorem 7.1 in Murty (1972).

6.11.22 The class of $\boldsymbol{N}$-matrices is defined and discussed in Nikaido (1968) which cites an earlier work by Inada (see Inada (1971)). Theorems **6.6.3** and **6.6.4** are derived from results in Kojima and Saigal (1979). (The reader should be aware that Theorems 3.3 and 3.4 of Kojima and Saigal (1979) are slightly inaccurate.) The complete version of Theorem **6.6.4**, as we have it here, appears as Theorem 4 in Gowda (1991b).

6.11.23 The class of fully-semimonotone matrices and the class of $\boldsymbol{U}$-matrices are defined and studied in Stone (1981) and Cottle and Stone (1983). The class of $\boldsymbol{W}$-matrices is defined and studied in Jeter and Pye (1987). The results we present in Section 6.6 concerning these matrix classes are derived from material taken from these sources.

We note that Stone (1981) shows that $\boldsymbol{U} \cap \boldsymbol{Q}_0 \subseteq \boldsymbol{P}_0$ and conjectures that $\boldsymbol{E}_0^f \cap \boldsymbol{Q}_0 \subseteq \boldsymbol{P}_0$. As of this writing, the conjecture remains open. For related material, see Aganagić and Cottle (1987) and Note **3.13.22**.

6.11.24 The concept of regular (Definition **6.6.14**) is first used in Saigal (1972b). The definition given here is the one used in Stone (1986). The class of ***INS***-matrices is first defined in Stone (1981). The results on ***INS***-matrices given in Section 6.6 can be found in Stone (1981, 1986).

6.11.25 The idea of superfluous matrices can be found in both Stone (1981) and Garcia, Gould, and Turnbull (1984). The term *superfluous* is coined in Howe (1983a). It is in this latter paper that superfluous matrices of nonzero degree are first shown to exist. Indeed, Theorems **6.7.2** and

6.7.3 are special cases of results taken from Howe (1983). As discussed at the end of Section 6.7, an example of a superfluous matrix with degree equal to zero is given in Kelly and Watson (1979). Theorem **6.7.4** is a special case of a general theorem concerning degree and the direct sum of functions (see Howe and Stone (1983)).

6.11.26 Except for Lemmas **6.8.1** and **6.8.2**, the results in Section 6.8 are taken from Morris (1990a). Additional insight into the geometry of the polytope $\mathcal{P}_M$ can be gained by reading Morris (1990a) where Theorems **6.8.3** and **6.8.4** are proved using this geometry.

6.11.27 The results in Section 6.9 are taken from Doverspike and Lemke (1982). Pseudomanifolds have been used in a wide variety of mathematical disciplines including algebraic topology and complementary pivot theory. Several of the references given in this section use pseudomanifolds such as Saigal (1972b), Stone (1981), and, of course, Doverspike and Lemke (1982). The following is a sample of the works where the reader can find further information on this topic: Spanier (1966), Gould and Tolle (1983), Eaves (1984), and Freund (1984).

6.11.28 Exercise **6.10.5** is taken from part of Proposition 3.1 in Doverspike and Lemke (1982). The inspiration behind Exercise **6.10.17** is derived from Ha (1985). Part (b) of Exercise **6.10.23** appears as Theorem 2 of Cottle and Stone (1983). Exercise **6.10.28** is taken from material in Murty (1972). Exercise **6.10.29** appears as Corollary 4.5 of Stone (1986). Exercise **6.10.31** is taken from material in Olech, Parthasarathy, and Ravindran (1991). (For additional results on almost $\boldsymbol{N}$-matrices see Gowda (1991b).) Exercise **6.10.37** is derived from Lemma 4.1 of Morris (1990a).

6.11.29 Several works make use of spherical geometry in studying the LCP. Instead of working with a complementary cone directly, one works with the intersection of the complementary cone and the unit sphere. (If the complementary cone is nondegenerate, this intersection will be a nondegenerate spherical simplex.)

As pointed out in Section 2.9, nondegenerate homogeneous functions on R^n can be viewed as functions on S^{n-1}. Thus, in the context of the LCP, connections can be made between degree theory and spherical geometry.

We have not treated spherical geometry in this book, however, the interested reader is encouraged to consult the following sample of works taken from the literature: Kelly and Watson (1979), Cottle, von Randow, and Stone (1981), Garcia, Gould, and Turnbull (1983), Fredricksen, Watson, and Murty (1986), and Kelly (1990).

Chapter 7

SENSITIVITY AND STABILITY ANALYSIS

Sensitivity analysis of the linear complementarity problem (q, M) is concerned with the study of the behavior of the solution(s) of the problem when the data, i.e., the vector q and matrix M, are subject to change. Typically, the need for this kind of analysis is attributable to one of several factors. The foremost of these is the fact that when the LCP is used in modeling a practical application, the data, most often, are noisy; that is to say, they contain errors. This could be due to a number of possible reasons: inaccurate measurement, insufficient knowledge of the data, uncertainty, etc. To illustrate this, consider the LCP formulation of the market equilibrium problem discussed in Section 1.2. In this problem, part of the data is derived from the demand function (1.2.10). Typically, this function is obtained through an empirical process which is unequivocally subject to errors. As a result of the errors present in the data, the solution obtained from the LCP is at best an approximation of the true equilibrium. In order for such a solution to be of practical use, it is imperative that the modeler obtain some sensitivity information of the solution on the data. Similar considerations also arise from the other equilibrium problems (such as the traffic and the network equilibirum problems discussed in Sections 4.5 and 5.1, respectively).

Another instance in which sensitivity analysis plays an important role occurs when the data of the problem depend on certain parameters. We have seen an example of this type in the context of the implicit complementarity problem (cf. (1.5.4) in Section 1.5). There, each system (1.5.5) is equivalent to a linear complementarity problem in which the constant vector is parametrized by u. Another example of this type is the journal bearing problem discussed in Section 5.1. There, both the vector q and matrix M are defined in terms of the grid sizes Δz and Δx; in fact, q and M are nonlinear functions of Δz and Δx. These grid sizes are the key parameters in the finite difference LCP model which is designed to be a numerical approximation of the physical problem. Hence, it is vital to be able to analyze the solution of the discretized LCP as a function of the grid sizes. This kind of analysis is an important facet of the sensitivity study of the LCP.

Since it is generally impossible to compute an explicit solution of the LCP (q, M) for all values of the data within a domain of interest, sensitivity analysis offers the only avenue for the study of the problem when the data are subject to change. The main goal of such an analysis is to provide qualitative as well as quantitative information on the problem itself, or on a given solution of the problem, for a prescribed range of values of the data. This chapter presents several principal aspects of sensitivity analysis of the linear complementarity problem.

7.1 A Basic Framework

We introduce a basic framework within which the sensitivity analysis of the linear complementarity problem can be undertaken. Let $\mathcal{M} \subseteq R^{n\times n}$ be a subset of matrices, and $\mathcal{Q} \subseteq R^n$ a subset of vectors. Associated with each pair of vectors and matrices, $q \in \mathcal{Q}$ and $M \in \mathcal{M}$ respectively, we consider the LCP (q, M) and its (possibly empty) solution set SOL(q, M). Sensitivity analysis is concerned with the investigation of this solution set as q and M vary in $\mathcal{Q}$ and $\mathcal{M}$, respectively. Questions such as the nonemptiness of

$\mathrm{SOL}(q, M)$ for all $(q, M) \in \mathcal{Q} \times \mathcal{M}$, the (upper or lower) semicontinuity of the solution set $\mathrm{SOL}(q, M)$ as a multivalued mapping from $\mathcal{Q} \times \mathcal{M}$ into R^n, as well as the boundedness and some form of Lipschitz continuity of the solution map are of particular interest.

Many important special cases of the above general framework can be identified. The foremost of these is the instance where a fixed vector $\bar{q}$ and matrix $\bar{M}$ are given and the sets $\mathcal{Q}$ and $\mathcal{M}$ are certain neighborhoods of $\bar{q}$ and $\bar{M}$, respectively. This special case corresponds to the *stability analysis* of the given LCP $(\bar{q}, \bar{M})$ in which one is interested only in small changes of the data, and in how such changes affect the behavior of either a given solution or the entire solution set of the problem $(\bar{q}, \bar{M})$. Various notions of stability can be defined and two of these will be analyzed in Section 7.3.

Besides yielding useful information about the solution and the problem when the data are perturbed, the sensitivity results for the LCP often have important algorithmic implications. We shall later discuss two applications of these results in an algorithmic context; one of them concerns the basic splitting method given in **5.2.1** for solving the LCP (q, M), and the other is related to the local convergence of Newton's method for solving the nonlinear complementarity problem (1.2.22). In each case, we shall demonstrate how the sensitivity results are instrumental in establishing the convergence of the sequences of iterates produced by these methods, and how some results pertaining to the rates of convergence can also be derived.

Another special case of the general framework for sensitivity analysis of the LCP is the *multivariate parametric linear complementarity problem* in which both the constant vector and the defining matrix in the LCP are functions of a parameter vector. More specifically, this problem involves the family:

$$\{(q(\varepsilon), M(\varepsilon)) : \varepsilon \in \Lambda\} \tag{1}$$

where Λ (the parameter space) is a subset of R^m, and $q : \Lambda \to R^n$ and $M : \Lambda \to R^{n \times n}$ are given functions. The parametric linear complementarity problem introduced in Section 4.5 is a special case of this multivariate parametric problem in which there is one single parameter (i.e., $m = 1$), the function $q(\varepsilon)$ is affine in ε (i.e., $q(\varepsilon) = q + \varepsilon d$ where $q, d \in R^n$), and the function $M(\varepsilon)$ is a constant. The implicit complementarity problem in Section 1.5 provides another context in the which the parametrized family (1) arises naturally.

7.2 An Upper Lipschitzian Property

Consider the sensitivity analysis problem of the LCP (q, M) in which the matrix M is not subject to change and the vector q is slightly perturbed. In order to simplify the notation somewhat, we suppress the dependence on M in $\text{SOL}(q, M)$ and let $S(q)$ denote the solution set of (q, M). In this notation, $S(\cdot)$ defines a multivalued mapping from R^n into itself. Notice that $S(q)$ may be empty for some q.

The solution mapping $S(\cdot)$ possesses two elementary properties that are worth noting. One property is the *closedness* of this map; that is to say, if $\{q^\nu\}$ is a sequence of vectors in R^n converging to $\bar{q}$, and $\{z^\nu\}$ is a corresponding sequence of solutions with $z^\nu \in S(q^\nu)$ for all ν, and if $\{z^\nu\}$ converges to $\bar{z}$, then $\bar{z}$ is a solution of the LCP $(\bar{q}, M)$. This property is trivial to prove, yet it is the basis upon which several of the convergence results of the (iterative) splitting method studied in Chapter 5 are derived. (See for example the limiting argument in the proof of Theorem **5.3.3**.) The other special property of the solution map $S(\cdot)$ is that it is *polyhedral*; this means that its graph which is the set

$$\{(q, z) \in R^n \times R^n : z \in S(q)\},$$

is a finite union of convex polyhedra. As a matter of fact, this set is equal to

$$\bigcup_{\alpha} \{(q, z) \in R^n \times R^n_+ : q_\alpha + M_{\alpha\alpha} z_\alpha = 0, q_{\bar{\alpha}} + M_{\bar{\alpha}\alpha} z_\alpha \geq 0, z_{\bar{\alpha}} = 0\}$$

where the union ranges over all index subsets α of $\{1, \ldots, n\}$.

In essence, the polyhedrality of the solution map $S(\cdot)$ provides the key to the following result which establishes a very general continuity property of $S(q)$ as q varies in a neighborhood of a fixed vector $\bar{q}$. The validity of this result requires no assumption on the underlying matrix M. In stating the result, we use $\|\cdot\|$ to denote the Euclidean vector norm $\|\cdot\|_2$, and $\mathcal{B}$ the associated (closed) unit ball.

7.2.1 Theorem. Let $M \in R^{n \times n}$ and $\bar{q} \in R^n$ be given. Then, there exists a constant $c > 0$ and a neighborhood $V \subseteq R^n$ of $\bar{q}$ such that for all vectors $q \in V$,

$$S(q) \subseteq S(\bar{q}) + c\|q - \bar{q}\|\mathcal{B}. \tag{1}$$

Before proving the theorem, we clarify the meaning of the inclusion (1). First of all, this result applies regardless of whether the set $S(\bar{q})$ is empty or not. In the case where $S(\bar{q})$ is empty, the inclusion (1) simply asserts that for all vectors q sufficiently close to $\bar{q}$, the LCP (q, M) also has no solution. In the opposite case where the LCP $(\bar{q}, M)$ has a solution, the conclusion is that for all vectors q sufficiently close to $\bar{q}$, if $z(q)$ is an arbitrary solution of (q, M) (assuming it exists), then there must be some solution $\bar{z}$ of $(\bar{q}, M)$ (which depends on $z(q)$) such that

$$\|z(q) - \bar{z}\| \leq c\|q - \bar{q}\|.$$

This last inequality shows that the solutions of the perturbed LCP (q, M), if they exist, not only are not too far away from some solutions of the given LCP $(\bar{q}, M)$, but their distances are bounded by a quantity proportional to the magnitude of the change in $\bar{q}$. Borrowing terminology from the theory of multifunctions (i.e., multivalued mappings), we therefore conclude that the solution mapping $S(\cdot)$ is *locally upper Lipschitzian with modulus c.* Notice that if $S(q)$ consists of a singleton for each q in a neighborhood of the base vector $\bar{q}$, the expression (1) implies that this solution function possesses a kind of weak Lipschitz continuity property at the vector $\bar{q}$.

The proof of Theorem **7.2.1** relies on a Lipschitzian property of convex polyhedra. In order to state this property, we let $P_A(b)$ denote the polyhedral set

$$P_A(b) = \{x \in R^n : Ax = b, x \geq 0\},$$

for a given matrix $A \in R^{m \times n}$ and vector $b \in R^m$.

7.2.2 Lemma. Let $A \in R^{m \times n}$ and $\bar{b} \in \operatorname{pos} A$ be given. Then, there exist a constant $L > 0$ and a neighborhood $V \subseteq R^m$ of $\bar{b}$ such that for every $b \in V$ and every vector $x(b) \in P_A(b)$, there exists a vector $\bar{x} \in P_A(\bar{b})$ such that

$$\|x(b) - \bar{x}\| \leq L\|b - \bar{b}\|.$$

Proof. Without loss of generality, we may assume that A is not equal to the zero matrix. Let $\mathcal{F}$ be the collection of feasible bases of A associated with the polyhedron $P_A(\bar{b})$, i.e., $\mathcal{F}$ consists of submatrices B of A which themselves are comprised of linearly independent columns from A and for which the subsystem

$$Bx_B = \bar{b}, \quad x_B \geq 0 \tag{2}$$

is consistent. Since $\bar{b} \in \text{pos}\, A$, $\mathcal{F}$ is a nonempty collection. Let $\mathcal{I}$ be the collection of infeasible bases of A associated with $P_A(\bar{b})$; that is, $\mathcal{I}$ consists of submatrices B of A which themselves are comprised of linearly independent columns from A but the subsystem (2) is inconsistent. Notice that $\mathcal{I}$ is a finite (possibly empty) collection, and

$$\bar{b} \notin \bigcup_{B \in \mathcal{I}} \text{pos}\, B.$$

The latter union of finitely generated cones is a closed set (which is possibly empty). Hence, there exists a neighborhood V of $\bar{b}$ such that

$$V \cap (\bigcup_{B \in \mathcal{I}} \text{pos}\, B) = \emptyset.$$

Let $b \in V \cap \text{pos}\, A$ be given, and let $x(b) \in P_A(b)$. Then, by Goldman's resolution theorem, **2.6.23**, $x(b)$ is the sum of a convex combination of extreme points of $P_A(b)$ and a nonnegative combination of extreme rays of $P_A(b)$, i.e.,

$$x(b) = \sum_{i=1}^{r} \lambda_i u^i + \sum_{j=1}^{s} \mu_j v^j \tag{3}$$

where λ_i and μ_j are nonnegative scalars with $\sum_{i=1}^{r} \lambda_i = 1$, and each u^i (v^j) is an extreme point (ray vector) of $P_A(b)$.

Associated with each extreme point u of $P_A(b)$ is a basis B of A whose columns correspond to the positive components of u. By the choice of b, B must necessarily belong to the collection $\mathcal{F}$. Thus, there exists an extreme point $x \in P_A(\bar{b})$ with $x = (x_B, 0)$. We have

$$B(x_B - u_B) = \bar{b} - b$$

where $u = (u_B, 0)$. Since B has full column rank, the matrix $B^{\mathrm{T}}B$ is positive definite. Hence,

$$x_B - u_B = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}(\bar{b} - b),$$

which implies

$$\|x - u\| \leq \|(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}\| \; \|\bar{b} - b\|.$$

Define the constant

$$L = \max_{B \in \mathcal{F}} \|(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}\|$$

and apply the previous argument to each of the extreme points u^i appearing in the representation (3). For each $i = 1, \ldots, r$, let x^i be an extreme point of $P_A(\bar{b})$ that corresponds to u^i obtained through this argument. Define the vector

$$\bar{x} = \sum_{i=1}^{r} \lambda_i x^i + \sum_{j=1}^{s} \mu_j v^j$$

which must belong to $P_A(\bar{b})$. It follows that

$$\|x(b) - \bar{x}\| \leq L\|\bar{b} - b\|$$

as desired. □

Proof of 7.2.1. We first show that if the LCP $(\bar{q}, M)$ has no solution, then the same is true for the LCP (q, M) for q sufficiently close to $\bar{q}$. This is obvious because the assumption implies $\bar{q} \notin K(M)$; since $K(M)$ is a closed set, it follows that there exists a neighborhood V of $\bar{q}$ such that $q \notin K(M)$ for all $q \in V$. Hence, the LCP (q, M) is not solvable for all these vectors q.

Suppose that $S(\bar{q}) \neq \emptyset$. Let $\mathcal{C}$ be the collection of all complementary submatrices A of $(I, -M)$ such that $\bar{q} \in \operatorname{pos} A$. Note that $\mathcal{C}$ is a finite collection. By the same argument as above, we deduce that there exists a neighborhood V of $\bar{q}$ such that for all $q \in V$ for which the LCP (q, M) is solvable, we have

$$q \in \bigcup_{A \in \mathcal{C}} \operatorname{pos} A.$$

According to Lemma **7.2.2**, there is a constant $L_A > 0$ associated with each complementary submatrix $A \in \mathcal{C}$ such that for all vectors q sufficiently close to $\bar{q}$ and for each vector $u \geq 0$ such that $Au = q$, there exists a corresponding vector $\bar{u} \geq 0$ satisfying $A\bar{u} = \bar{q}$ and

$$\|u - \bar{u}\| \leq L_A\|q - \bar{q}\|.$$

By defining the constant c to be the largest of the constants L_A with A ranging over all members of the collection $\mathcal{C}$, and by restricting the neighborhood V if necessary, we easily deduce the desired inclusion (1). □

An immediate consequence of **7.2.1** is the following continuity property and boundedness of the solution map $S(\cdot)$ at the vector $\bar{q}$ if the set $S(\bar{q})$ is bounded.

7.2.3 Corollary. Let $M \in R^{n \times n}$. If $S(\bar{q})$ is bounded, then the multivalued solution map $S : R^n \to R^n$ is upper semicontinuous at $\bar{q}$; moreover, there exist a constant $c' > 0$ and a neighborhood V of $\bar{q}$ such that

$$\|z\| \leq c' \qquad \text{for all } z \in S(q) \text{ and } q \in V.$$

Proof. If $S(\bar{q})$ is empty, then by **7.2.1**, so is $S(q)$ for all q sufficiently close to $\bar{q}$. Hence the conclusions of the corollary are vacuously true in this case. If $S(\bar{q})$ is nonempty (and bounded by assumption), then it is compact because it is closed. Let V and c be as given in **7.2.1**. Then, if U is any open set containing (the compact) $S(\bar{q})$, by restricting the neighborhood V if necessary, it follows that U will contain the set $S(\bar{q}) + c\|q - \bar{q}\|\mathcal{B}$ for all vectors $q \in V$. By the inclusion (1), the upper semicontinuity of the map $S(\cdot)$ at $\bar{q}$ follows. The last conclusion about the existence of the constant c' also follows from this inclusion. □

The second conclusion of **7.2.3** states that if the LCP $(\bar{q}, M)$ has a (nonempty) bounded solution set, then the (nonempty) solution sets of all nearby LCP (q, M), with q sufficiently close to $\bar{q}$, are uniformly bounded.

Local solvability

Theorem **7.2.1** does not assert the solvability of the LCP (q, M) when q is close to $\bar{q}$. Obviously, a sufficient condition for $S(q)$ to be nonempty is $M \in \boldsymbol{Q}$. The next result shows that under an appropriate uniqueness condition, the validity of this kind of local solvability property can be characterized in terms of the $\boldsymbol{Q}$-property of the Schur complement of a certain principal submatrix in M.

7.2.4 Theorem. Let $M \in R^{n \times n}$ and $\bar{q}$ be given. Suppose that $\bar{z}$ is a locally unique solution of LCP $(\bar{q}, M)$ and $M_{\alpha\alpha}$ is nonsingular, where

$$\alpha = \{i : \bar{z}_i > 0 = (\bar{q} + M\bar{z})_i\}.$$

(a) If the Schur complement

$$M_{\beta\beta} - M_{\beta\alpha} M_{\alpha\alpha}^{-1} M_{\alpha\beta} \tag{4}$$

belongs to the class $\boldsymbol{Q}$, where

$$\beta = \{i : \bar{z}_i = 0 = (\bar{q} + M\bar{z})_i\},$$

then there exists a neighborhood V of $\bar{q}$ such that $S(q) \neq \emptyset$ for all $q \in V$.

(b) Conversely, if $\bar{z}$ is a globally unique solution of $(\bar{q}, M)$ and if the LCP (q, M) is solvable for all q sufficiently close to $\bar{q}$, then the Schur complement (4) is a $\boldsymbol{Q}$-matrix.

Proof. Let γ denote the complement of $\alpha \cup \beta$ in $\{1, \ldots, n\}$. Since $M_{\alpha\alpha}$ is nonsingular, we have

$$-M_{\alpha\alpha}^{-1}\bar{q}_\alpha > 0, \tag{5}$$

$$\bar{q}_\beta - M_{\beta\alpha}M_{\alpha\alpha}^{-1}\bar{q}_\alpha = 0, \tag{6}$$

$$\bar{q}_\gamma - M_{\gamma\alpha}M_{\alpha\alpha}^{-1}\bar{q}_\alpha > 0. \tag{7}$$

Let N denote the Schur complement given in (4). Since $\bar{z}$ is a locally unique solution of the LCP $(\bar{q}, M)$, Theorem **3.6.5** implies that the matrix $N \in \boldsymbol{R_0}$, i.e., the LCP $(0, N)$ has zero as the unique solution. By Theorem **7.2.1**, there exist a constant $c > 0$ and a scalar $\varepsilon > 0$ such that for every vector r_β with $\|r_\beta\| \leq \varepsilon$, we have

$$\|z_\beta\| \leq c\|r_\beta\| \tag{8}$$

for every solution z_β of the LCP (r_β, N).

Now, choose the neighborhood V of $\bar{q}$ such that for every $q \in V$, we have

$$\begin{aligned} z_\alpha &= -M_{\alpha\alpha}^{-1}(q_\alpha + M_{\alpha\beta}z_\beta) \geq 0 \\ w_\gamma &= q_\gamma + M_{\gamma\alpha}z_\alpha + M_{\gamma\beta}z_\beta \geq 0 \end{aligned} \tag{9}$$

for every solution z_β of the LCP (r_β, N) where

$$r_\beta = q_\beta - M_{\beta\alpha}M_{\alpha\alpha}^{-1}q_\alpha.$$

That we can choose this neighborhood V with these properties is a consequence of (5) – (8). To complete the proof of part (a), it suffices to observe that the LCP (r_β, N) has a solution by the assumption $N \in \boldsymbol{Q}$; if z_β is any such solution, the first equation in (9) defines the vector z_α, and the second equation in (9) implies that $(z_\alpha, z_\beta, 0)$ is a solution of the LCP (q, M) for $q \in V$.

To establish part (b), suppose that $\bar{z}$ is a globally unique solution of $(\bar{q}, M)$ with $M_{\alpha\alpha}$ nonsingular and that the LCP (q, M) has a solution for each q sufficiently close to $\bar{q}$. By the global uniqueness of $\bar{z}$, the inclusion (1) implies that for a vector q sufficiently close to $\bar{q}$, any solution of (q, M) must be close to $\bar{z}$. In particular, it follows that if q is close enough to $\bar{q}$ and if $z \in S(q)$ with $w = q + Mz$, then we have

$$w_i = 0 < z_i \quad \text{for all } \ i \in \alpha,$$

$$z_i = 0 < w_i \quad \text{for all } \ i \in \gamma.$$

Let r_β be an arbitrary vector. To show that the LCP (r_β, N) has a solution, consider the problem (q, M) with

$$q_\alpha = \bar{q}_\alpha, \qquad q_\beta = \bar{q}_\beta + \varepsilon r_\beta, \qquad q_\gamma = \bar{q}_\gamma,$$

where $\varepsilon > 0$. By the aforementioned observation, it follows that if ε is small enough, this latter LCP (q, M) has a solution z with corresponding $w = q + Mz$; moreover, we have $w_\alpha = 0$ and $z_\gamma = 0$. It is then an easy matter to see that z_β solves the LCP $(\varepsilon r_\beta, N)$. Since ε is positive, we conclude that (r_β, N) has a solution. □

In the notation introduced in **3.9.15**, the index sets α, β and γ appearing in the above theorem correspond to $\alpha(\bar{z})$, $\beta(\bar{z})$ and $\gamma(\bar{z})$ respectively. Here and in the remainder of this chapter, these sets play a central role in the analysis. Equally important in the subsequent results is the Schur complement given in expression (4) and denoted N in the proof of **7.2.4**.

The $\boldsymbol{Q}$-property of the Schur complement N is vacuously true if the index set β is empty, i.e., if $\bar{z}$ is a nondegenerate solution of the LCP $(\bar{q}, M)$; in this case, it follows that the LCP (q, M) is solvable for all vectors q sufficiently close to $\bar{q}$. Nevertheless, as the example below shows, this local property does not imply $M \in \boldsymbol{Q}$.

7.2.5 Example. Consider the data

$$\bar{q} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \qquad M = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ -2 & 2 & 1 \end{bmatrix}.$$

The vector $\bar{z} = (0, 0, 1)^{\mathrm{T}}$ is a locally unique, nondegenerate solution of $(\bar{q}, M)$ with $\alpha = \{3\}$. It is easily seen that $M \notin \boldsymbol{Q}$.

As illustrated by the next example, it is essential that $\bar{z}$ be a unique solution of $(\bar{q}, M)$ in order for part (b) of **7.2.4** to hold.

7.2.6 Example. Consider the data

$$\bar{q} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \quad M = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}.$$

The matrix M is nondegenerate and belongs to class $\boldsymbol{Q}$. Two solutions of the LCP $(\bar{q}, M)$ are:

$$z^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad z^2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

The index sets α and β corresponding to the solution z^1 are

$$\alpha = \{1\} \quad \text{and} \quad \beta = \{2\}.$$

Since M is nondegenerate, z^1 is a locally unique solution of LCP $(\bar{q}, M)$ (see Theorem **3.6.3**). The Schur complement given in (4) is equal to -1, hence is not in $\boldsymbol{Q}$.

Theorem **7.2.4** is related to **6.5.5** which treats the case where the vector $\bar{q}$ is nonnegative and the solution $\bar{z} = 0$ (see also Corollary **6.5.6**). The assumptions in the latter theorem imply the local uniqueness of the zero solution. One noteworthy point is that in this special case (of a nonnegative q), the global uniqueness assumption in part (b) of **7.2.4** is replaced by a certain covering property of the map f_M.

An application: convergence of splitting methods

Theorem **7.2.1** has many applications. One of these occurs in the study of the splitting method described in **5.2.1** for solving the LCP (q, M). This method generates a sequence $\{z^\nu\}$ with the property that each $z^{\nu+1}$ is a solution of $(q + Cz^\nu, B)$ where (B, C) is a splitting of M. Clearly, $z^{\nu+1}$ is also a solution of $(q + C(z^\nu - z^{\nu+1}), M)$. Now, if the sequence $\{z^\nu - z^{\nu+1}\}$ converges to zero, then the latter LCP becomes a perturbation of (q, M) for all ν sufficiently large, and **7.2.1** is therefore applicable.

7.2.7 Theorem. Let $M \in R^{n\times n}$ and $q \in R^n$ be such that the LCP (q, M) has a finite number of solutions. Let (B, C) be a Q-splitting of M, and $\{z^\nu\}$ be any sequence produced by the splitting method **5.2.1**. The following two statements are equivalent.

(a) The sequence $\{z^\nu - z^{\nu+1}\}$ converges to zero as $\nu \to \infty$.

(b) The sequence $\{z^\nu\}$ converges to a solution of (q, M).

Proof. Clearly, it suffices to show (a) $\Rightarrow$ (b). Since $q + C(z^\nu - z^{\nu+1}) \to q$ as $\nu \to \infty$ and the perturbed problem $(q+C(z^\nu - z^{\nu+1}), M)$ has a solution, **7.2.1** implies that (q, M) must have a solution; moreover, there exists a constant $c > 0$ such that for all ν large enough and for each solution u^ν of $(q + C(z^\nu - z^{\nu+1}), M)$, there exists a solution y^ν of (q, M) satisfying

$$\|u^\nu - y^\nu\| \leq c\|z^\nu - z^{\nu+1}\|.$$

Since (q, M) has only a finite number of solutions, there are finitely many y^ν vectors. Hence, the above inequality implies that the solution sets of the LCPs $(q + C(z^\nu - z^{\nu+1}), M)$ are uniformly bounded for all ν sufficiently large; in particular, the sequence $\{z^{\nu+1}\}$ is bounded. Moreover, by the closedness property of the solution map $S(\cdot)$, every accumulation point of $\{z^\nu\}$ must be a solution of (q, M). But since SOL(q, M) is a finite set, the sequence $\{z^\nu\}$ possesses the following three properties: (i) it is bounded, (ii) it has a finite number of accumulation points, and (iii) $\|z^\nu - z^{\nu+1}\| \to 0$. Hence, by Theorem **2.1.10**, the sequence $\{z^\nu\}$ converges; its limit must be a solution of (q, M). □

7.2.8 Remark. Observe how the upper Lipschitzian result of **7.2.1** is employed to establish the boundedness of the sequence $\{z^\nu\}$; the existence of the Lipschitzian constant c is essential for this purpose.

7.2.9 Remark. We recall that a necessary and sufficient condition for the LCP (q, M) to have a finite number of solutions for all vectors q is when the matrix M is nondegenerate (see Theorem **3.6.3**).

Theorem **7.2.7** can be compared to **5.3.8**. There, the matrix M was assumed symmetric and the splitting (B, C) regular (in addition to being a Q-splitting); here, these two assumptions are not imposed. In a way, one can view **7.2.7** as providing the underlying justification for the validity of **5.3.8**. In turn, this can be traced back to the upper Lipschitzian property asserted in Theorem **7.2.1**.

We now switch our attention to the symmetric LCP (q, M) and demonstrate how Theorem **7.2.1** is again instrumental in establishing the convergence of the splitting method. As we can expect, the quadratic function

$$f(z) = q^{\mathrm{T}}z + \tfrac{1}{2}z^{\mathrm{T}}Mz$$

plays a central role in this analysis. The following theorem is the analog of **7.2.7** for the LCP (q, M) with a symmetric matrix M. This theorem provides the ultimate convergence result for the splitting method in **5.2.1** for solving a symmetric LCP.

7.2.10 Theorem. Let $M \in R^{n\times n}$ be a symmetric matrix, $q \in R^n$ be arbitrary, and (B, C) be a regular Q-splitting of M. The following two statements are equivalent.

(a) The quadratic function $f(z)$ is bounded below for $z \geq 0$.

(b) For any starting vector $z^0 \geq 0$, any sequence $\{z^\nu\}$ produced by the splitting method **5.2.1** converges to a solution of (q, M).

In order to prove **7.2.10**, we summarize the key facts about the sequence $\{z^\nu\}$ which we already know from the results developed in Section 5.3. Suppose that (a) holds. Let $\{z^\nu\}$ be a sequence produced by **5.2.1**. Then,

(i) the sequence $\{f(z^\nu)\}$ is nonincreasing and converges, say, to $\bar{f}$;

(ii) there exists a scalar $\eta > 0$ such that for all ν,

$$\|z^\nu - z^{\nu+1}\|_2^2 \leq \eta(f(z^\nu) - f(z^{\nu+1})); \qquad (10)$$

(iii) every accumulation point of $\{z^\nu\}$, if it exists, solves (q, M).

Another useful property is the fact that if M is symmetric, the quadratic function $f(z)$ attains only a finite number of values on the set $\mathrm{SOL}(q, M)$ (cf. **3.12.22**).

Proof of 7.2.10. (a) $\Rightarrow$ (b). The strategy of the proof is somewhat similar to that of **5.4.6**. Let

$$\sigma_\nu = f(z^\nu) - \bar{f} \geq 0.$$

The expression (10) implies

$$\|z^\nu - z^{\nu+1}\|^2 \leq \eta\sigma_\nu \tag{11}$$

where, for the sake of simplifying the notation somewhat, we have omitted the subscript 2 in the norm. Our goal is to derive a rate of convergence result for the sequence $\{\sigma_\nu\}$ that is analogous to the expression (5.4.4) in the case of a positive semi-definite M.

As in the proof of **7.2.7**, we deduce the existence of a constant $c > 0$ such that for all ν sufficiently large, there exists a solution $\bar{z}^\nu$ of (q, M) satisfying

$$\|z^{\nu+1} - \bar{z}^\nu\| \leq c\|z^{\nu+1} - z^\nu\|. \tag{12}$$

This inequality implies

$$\lim_{\nu\to\infty} f(\bar{z}^\nu) = \lim_{\nu\to\infty} f(z^\nu) = \bar{f}.$$

Since $f(z)$ attains only finitely many values on SOL(q, M), it follows that for all ν large enough, $f(\bar{z}^\nu) = \bar{f}$. As in the proof of **5.3.2**, we have for ν sufficiently large,

$$\begin{aligned}
&\sigma_{\nu+1} = f(z^{\nu+1}) - f(\bar{z}^\nu)\\
&= (z^{\nu+1} - \bar{z}^\nu)^{\mathrm{T}}(q + Cz^{\nu+1} + B\bar{z}^\nu) + \tfrac{1}{2}(z^{\nu+1} - \bar{z}^\nu)^{\mathrm{T}}(B - C)(z^{\nu+1} - \bar{z}^\nu)\\
&= (z^{\nu+1} - \bar{z}^\nu)^{\mathrm{T}}(q + Cz^\nu + Bz^{\nu+1}) + (z^{\nu+1} - \bar{z}^\nu)^{\mathrm{T}}C(z^{\nu+1} - z^\nu)\\
&\quad -\tfrac{1}{2}(z^{\nu+1} - \bar{z}^\nu)^{\mathrm{T}}M(z^{\nu+1} - \bar{z}^\nu)\\
&\leq (z^{\nu+1} - \bar{z}^\nu)^{\mathrm{T}}C(z^{\nu+1} - z^\nu) - \tfrac{1}{2}(z^{\nu+1} - \bar{z}^\nu)^{\mathrm{T}}M(z^{\nu+1} - \bar{z}^\nu)\\
&\leq (c\|C\| + c^2\|M\|/2)\|z^{\nu+1} - z^\nu\|^2
\end{aligned}$$

where the last inequality follows from the Cauchy-Schwartz inequality and (12). By (10), we deduce

$$\sigma_{\nu+1} \leq \rho'(\sigma_\nu - \sigma_{\nu+1})$$

where

$$\rho' = \eta(c\|C\| + c^2\|M\|/2).$$

Consequently, for all ν large enough,

$$\sigma_{\nu+1} \leq \rho\sigma_{\nu}, \tag{13}$$

where $\rho = \rho'/(1+\rho') < 1$. This last inequality gives the desired geometric convergence rate for the sequence $\{\sigma_\nu\}$. In light of this and the inequality (11), we can complete the proof of convergence of the sequence $\{z^\nu\}$ by the same argument as in the proof of **5.4.6**. This establishes the implication (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a). Fix an arbitrary nonnegative vector z^0. Let $\{z^\nu\}$ be a sequence of iterates generated by the splitting method in **5.2.1** with z^0 as the starting vector. By assumption (b), $\{z^\nu\}$ converges to a solution $\bar{z}$ of (q, M). By Lemma **5.3.2**, we obtain

$$f(z^0) \geq f(\bar{z}).$$

Since there are only finitely many functional values $f(\bar{z})$ for $\bar{z} \in \text{SOL}(q, M)$, it follows that $f(z^0)$ is bounded below by the smallest of such values. Since z^0 is arbitrary, part (b) is established. □

According to Proposition **3.7.14**, the matrix M is strictly copositive if and only if for all $q \in R^n$, the quadratic function $f(z)$ is bounded below for $z \geq 0$. Hence, combining this observation with Theorem **7.2.10**, we deduce the following corollary.

7.2.11 Corollary. Let $M \in R^{n\times n}$ be symmetric. Suppose that M has a regular Q-splitting (B, C) . Then, M is strictly copositive if and only if for all vectors $q \in R^n$ and all $z^0 \in R^n_+$, any sequence $\{z^\nu\}$ produced by the splitting method in **5.2.1** converges to a solution of (q, M). □

The above corollary generalizes Theorem **5.3.5** where the strict copositivity of M was characterized in terms of a related universal, subsequential convergence property of the splitting method.

The proof of the implication [(a) $\Rightarrow$ (b)] in Theorem **7.2.10** can be extended to establish the geometric rate of convergence for the splitting method of **5.2.1**.

7.2.12 Corollary. Let $M \in R^{n\times n}$ be a symmetric matrix, and (B, C) be a regular Q-splitting of M. Suppose that the quadratic function $f(z)$ is bounded below for $z \geq 0$. Let $\{z^\nu\}$ be a sequence produced by **5.2.1**, and let z^* be its limit. Then, there exist positive scalars c_1 and c_2 with $c_2 < 1$ such that

$$\|z^\nu - z^*\| \leq c_1(c_2)^\nu$$

for all ν sufficiently large. Furthermore, if z^* is a locally unique solution of (q, M), then there exist a constant $c' > 0$ and an integer $\bar{\nu} \geq 0$ such that for $\nu \geq \bar{\nu}$,

$$\|z^{\nu+1} - z^*\| \leq c'\|z^\nu - z^*\|.$$

Proof. The first conclusion follows from a general property of sequences that satisfy the kind of inequalities given by (10) and (13); Exercise **7.6.12** gives the precise statement of this property, and the reader is asked to supply a proof. We now prove the second assertion of the corollary. Following the notation in the proof of Theorem **7.2.10**, we have $\bar{f} = f(z^*)$ and

$$\sigma_\nu = f(z^\nu) - f(z^*) = \nabla f(z^*)^{\mathrm{T}}(z^\nu - z^*) + \tfrac{1}{2}(z^\nu - z^*)^{\mathrm{T}} M(z^\nu - z^*).$$

Since $\{z^\nu\} \to z^*$, it follows that for all ν sufficiently large, we must have

$$\nabla f(z^*)^{\mathrm{T}} z^\nu = (q + Mz^*)^{\mathrm{T}} z^\nu = 0.$$

As $\nabla f(z^*)^{\mathrm{T}} z^* = 0$, we deduce

$$\sigma_\nu = \tfrac{1}{2}(z^\nu - z^*)^{\mathrm{T}} M(z^\nu - z^*) \leq \tfrac{\|M\|}{2}\|z^\nu - z^*\|^2. \tag{14}$$

Consider the sequence of solutions $\{\bar{z}^\nu\}$ of (q, M) for which the inequality (12) holds for ν sufficiently large. Obviously, $\{\bar{z}^\nu\}$ converges to z^*. But since z^* is an isolated solution, we must have $\bar{z}^\nu = z^*$ for all but finitely many ν's. Hence, combining (11), (12) and (14), we obtain for all ν large enough,

$$\|z^{\nu+1} - z^*\|^2 \leq c^2\eta\sigma_\nu \leq (c')^2\|z^\nu - z^*\|^2$$

where $(c')^2 = c^2\eta\|M\|/2$. □

7.2.13 Remark. In general, the two conclusions of the above corollary pertain to different properties of the sequence $\{z^\nu\}$. If $c' < 1$, then clearly, the second conclusion implies the first. In Exercise **7.6.13**, the reader is asked to verify that these two properties do not generally imply one another.

7.3 Solution Stability

The results of the last section are concerned with the *stability* of the linear complementarity problem when the defining matrix is not subject to change; they provide conditions under which the perturbed problems are solvable when the constant vector is changed slightly (see Theorem **7.2.4**), and they establish the (upper Lipschitz) continuity of the solution set (see Theorem **7.2.1**). The fact that the matrix M is kept fixed is essential for these results to hold. For instance, the upper Lipschitzian property of the solution set (cf. expression (7.2.1)) easily fails for an LCP with a $\boldsymbol{P_0}$-matrix M which is subject to perturbation; if one perturbs M by adding an arbitrarily small positive quantity to the diagonal entries, the resulting perturbed matrix becomes a $\boldsymbol{P}$-matrix, so the left-hand side of (7.2.1) is a nonempty set for any q, whereas the set in the right-hand side may be empty.

In this section, we take up the issue of stability of the linear complementarity problem at a given solution when both the constant vector and the defining matrix of the problem are perturbed. Here, our focus is on the change of this particular solution when the data of the problem are slightly altered.

7.3.1 Definition. Let $q \in R^n$ and $M \in R^{n \times n}$ be given. A solution z^* of the LCP (q, M) is said to be *stable* if there are neighborhoods V of z^* and U of the pair (q, M) such that

(i) for all $(\bar{q}, \bar{M}) \in U$, the set

$$S_V(\bar{q}, \bar{M}) = \text{SOL}(\bar{q}, \bar{M}) \cap V$$

is nonempty,

(ii) $\sup\{\|y - z^*\| : y \in S_V(\bar{q}, \bar{M})\} \to 0$ as $(\bar{q}, \bar{M})$ approaches (q, M).

If, in addition to the above conditions, the set $S_V(\bar{q}, \bar{M})$ is a singleton, then the solution z^* is said to be *strongly stable.*

Obviously, solution stability is a desirable feature for an LCP. This is because when a stable solution of the LCP is at hand, one is sure that a slight change of the data will not have a drastic effect on the perturbed problem; and in fact, the perturbed problem will have a solution which is near the given solution of the unperturbed problem. On the other hand, if one had an unstable solution of an LCP, then it would be more difficult to predict the behavior of the given problem resulting from a slight change of the data.

Condition (ii) in **7.3.1** has two important implications. First, it implies that if z^* is stable, then z^* must be locally unique. Thus, local uniqueness of a solution is necessary for its stability. Second, condition (ii) implies a certain stability of the positive variables of the solutions to the perturbed LCP that are close to z^*. In turn, this latter property leads to a reduction of the problem, allowing one to ignore certain well-behaved variables and concentrate on the remaining variables as one analyzes the change of the solution z^*.

In essence, the reduction that we are about to explain has already been carried out in Section 6.5. There, in order to perform the local analysis of the LCP (q, M) with $q \in \mathcal{K}(M)$, a related LCP of smaller dimension is constructed and used as the principal tool in the analysis. In what follows, we motivate the latter (reduced) LCP from an algebraic point of view. For this purpose, we let $\alpha = \alpha(z^*)$, $\beta = \beta(z^*)$ and $\gamma = \gamma(z^*)$ be the three index sets associated with the solution z^*, see **3.9.15**. Then, condition (ii) of **7.3.1** implies that for all pairs $(\bar{q}, \bar{M})$ sufficiently close to (q, M), if $y \in S_V(\bar{q}, \bar{M})$, we must have

$$y_\alpha > 0 \quad \text{and} \quad w_\gamma = (\bar{q} + \bar{M}y)_\gamma > 0. \tag{1}$$

By complementarity, it follows

$$w_\alpha = (\bar{q} + \bar{M}y)_\alpha = 0 \quad \text{and} \quad y_\gamma = 0. \tag{2}$$

Hence, with the exception of the variables correponding to the indices in the set β, we know exactly which variables are positive and which are zero for the perturbed LCP $(\bar{q}, \bar{M})$. Consequently, as far as the stability of the solution z^* is concerned, it is sufficient to understand the change of the β-variables as they are the main uncertainty for the perturbed problems.

In Theorem **7.2.4**, we have seen that the Schur complement, denoted N and given in expression (7.2.4), plays an important role in the sensitivity of the LCP when the matrix M is not changed. The above discussion has hinted that this matrix N might remain a central element in the solution stability of the LCP (q, M) when M is perturbed. As the subsequent results show, N is the decisive factor that distinguishes the two notions of stability defined in **7.3.1**.

In what follows, we establish necessary and sufficient conditions for the two kinds of solution stability to hold. In both cases, the derivation relies on the reduction of the LCP (q, M) to the homogeneous LCP $(0, N)$ that is defined by the Schur complement N. This reduction has two noteworthy features: (i) the size of N is typically much smaller than that of M, and (ii) the constant vector in the reduced problem is zero. That such a reduction is possible is due to the local nature of the analysis which is being made entirely in a neighborhood of the solution z^* under consideration.

7.3.2 Theorem. Let z^* be a solution of the LCP (q, M). Suppose that $M_{\alpha\alpha}$ is nonsingular. The following statements are equivalent.

(a) z^* is stable for the LCP (q, M).

(b) The zero vector is stable for the LCP $(0, N)$.

(c) $N \in \text{int}(\boldsymbol{Q}) \cap \boldsymbol{R_0}$.

Proof. We first derive the aforementioned reduction of the LCP (q, M) to the homogeneous problem $(0, N)$. To do this, we express the problem (q, M) in terms of the index sets α, β and γ as

$$w_\alpha = q_\alpha + M_{\alpha\alpha} z_\alpha + M_{\alpha\beta} z_\beta + M_{\alpha\gamma} z_\gamma \geq 0,$$

$$w_\beta = q_\beta + M_{\beta\alpha} z_\alpha + M_{\beta\beta} z_\beta + M_{\beta\gamma} z_\gamma \geq 0,$$

$$w_\gamma = q_\gamma + M_{\gamma\alpha} z_\alpha + M_{\gamma\beta} z_\beta + M_{\gamma\gamma} z_\gamma \geq 0,$$

$$z \geq 0 \quad \text{and} \quad w^{\mathrm{T}} z = 0.$$

At the solution z^*, we have

$$z^*_\alpha = -M^{-1}_{\alpha\alpha} q_\alpha > 0, \quad z^*_\beta = 0, \quad z^*_\gamma = 0,$$

and

$$w^*_\gamma = (q + Mz^*)_\gamma = q_\gamma + M_{\gamma\alpha} z^*_\alpha > 0.$$

Consequently, there are neighborhoods $\tilde{U}$ of (q, M) and $\tilde{V} = \tilde{V}_\alpha \times \tilde{V}_\beta \times \tilde{V}_\gamma$ of z^* such that if $z \in \tilde{V}$ and $(\bar{q}, \bar{M}) \in \tilde{U}$, then $\bar{M}_{\alpha\alpha}$ is nonsingular, and

$$z_\alpha > 0, \quad w_\gamma = \bar{q}_\gamma + \bar{M}_{\gamma\alpha} z_\alpha + \bar{M}_{\gamma\beta} z_\beta + \bar{M}_{\gamma\gamma} z_\gamma > 0,$$

$$0 < -\bar{M}^{-1}_{\alpha\alpha}(\bar{q}_\alpha + \bar{M}_{\alpha\beta} z_\beta) \in \tilde{V}_\alpha.$$

For $(\bar{q}, \bar{M}) \in \tilde{U}$, the problem of finding a solution $z \in \tilde{V}$ is reduced to the problem of finding $(z_\alpha, z_\beta) \in \tilde{V}_\alpha \times \tilde{V}_\beta$ satisfying the system

$$w_\alpha = \bar{q}_\alpha + \bar{M}_{\alpha\alpha} z_\alpha + \bar{M}_{\alpha\beta} z_\beta = 0,$$

$$w_\beta = \bar{q}_\beta + \bar{M}_{\beta\alpha} z_\alpha + \bar{M}_{\beta\beta} z_\beta \geq 0,$$

$$z_\beta \geq 0, \quad z_\beta^{\mathrm{T}} w_\beta = 0.$$

Since $\bar{M}_{\alpha\alpha}$ is nonsingular, we can eliminate the z_α variables using the first equation and then substitute the resulting expression into the other inequality, thus obtaining an LCP in the variable $z_\beta \in \tilde{V}_\beta$

$$\begin{gathered} w_\beta = (\bar{q}_\beta - \bar{M}_{\beta\alpha}\bar{M}_{\alpha\alpha}^{-1}\bar{q}_\alpha) + (\bar{M}_{\beta\beta} - \bar{M}_{\beta\alpha}\bar{M}_{\alpha\alpha}^{-1}\bar{M}_{\alpha\beta}) z_\beta, \\ z_\beta \geq 0, \quad w_\beta \geq 0, \quad z_\beta^{\mathrm{T}} w_\beta = 0. \end{gathered} \tag{3}$$

By letting

$$\bar{r}_\beta = \bar{q}_\beta - \bar{M}_{\beta\alpha}\bar{M}_{\alpha\alpha}^{-1}\bar{q}_\alpha \quad \text{and} \quad \bar{N} = \bar{M}_{\beta\beta} - \bar{M}_{\beta\alpha}\bar{M}_{\alpha\alpha}^{-1}\bar{M}_{\alpha\beta},$$

we recognize the problem (3) as the LCP $(\bar{r}_\beta, \bar{N})$. For $(\bar{q}, \bar{M}) = (q, M)$, we have $(\bar{r}_\beta, \bar{N}) = (0, N)$, and this is the reduced homogeneous LCP that we mentioned above.

We are now ready to prove the equivalence of the three statements (a), (b) and (c).

(a) $\Rightarrow$ (b). Suppose that z^* is a stable solution for the LCP (q, M). Let V and U be the neighborhoods specified in **7.3.1**. By restricting the neighborhood U if necessary, we may assume—as a result of condition (ii)—that $V = \tilde{V}$. We may further assume that the two neighborhoods U and $\tilde{U}$ also coincide.

Let W be a neighborhood of the pair $(0, N)$ such that $(r_\beta, \bar{N}) \in W$ implies $(\bar{q}, \bar{M}) \in \tilde{U}$ where

$$\bar{q}_\alpha = q_\alpha, \quad \bar{q}_\beta = q_\beta + r_\beta, \quad \bar{q}_\gamma = q_\gamma,$$

and $\bar{M} = M$ except for the principal submatrix $\bar{M}_{\beta\beta}$ which is equal to $M_{\beta\alpha}M_{\alpha\alpha}^{-1}M_{\alpha\beta} + \bar{N}$. It is then a simple matter to verify that $\tilde{V}_\beta$ and W are the required neighborhoods for zero to be a stable solution of the LCP $(0, N)$. This establishes (b).

(b) $\Rightarrow$ (c). Suppose that zero is a stable solution of the LCP $(0, N)$. Then, there exists a neighborhood W_N of N such that for all vectors r sufficiently small, the LCP $(r, \bar{N})$ has a solution for all $\bar{N} \in W_N$. Hence, by means of a scaling argument, it follows that such a matrix $\bar{N}$ must belong to $\boldsymbol{Q}$. Consequently, $N \in \text{int}(\boldsymbol{Q})$. The same scaling argument also shows that zero must be the (globally) unique solution of the homogeneous problem $(0, N)$ because of its local uniqueness. Hence $N \in \boldsymbol{R_0}$ as asserted by (c).

(c) $\Rightarrow$ (a). Suppose that $N \in \text{int}(\boldsymbol{Q}) \cap \boldsymbol{R_0}$. We need to construct the neighborhoods V and U required in **7.3.1**. Let W_N be a neighborhood of N such that $\bar{N} \in W_N$ implies $\bar{N} \in \boldsymbol{Q}$. Restrict the neighborhood $\tilde{U}$ described above so that $(\bar{q}, \bar{M}) \in \tilde{U}$ implies $\bar{M}_{\beta\beta} - \bar{M}_{\beta\alpha}\bar{M}_{\alpha\alpha}^{-1}\bar{M}_{\alpha\beta} \in W_N$. Then, the reduced problem (3) has a solution z_β whenever $(\bar{q}, \bar{M}) \in \tilde{U}$. In turn, this solution induces a solution $(z_\alpha, z_\beta, 0)$ to the problem $(\bar{q}, \bar{M})$ via

$$z_\alpha = -\bar{M}_{\alpha\alpha}^{-1}(\bar{q}_\alpha + \bar{M}_{\alpha\beta} z_\beta).$$

To complete the proof, it remains to be shown that all solutions z_β of the problem (3) approach zero as $(\bar{q}, \bar{M})$ tends to (q, M). Suppose that this is not true. Let $\{(r_\beta^\nu, \bar{N}_\nu)\}$ be a sequence of LCPs of the form (3) such that $r_\beta^\nu \to 0$ and $\bar{N}_\nu \to N$ as $\nu \to \infty$, and let $\{z_\beta^\nu\}$ be a corresponding sequence of vectors with each $z_\beta^\nu \in \text{SOL}(r_\beta^\nu, \bar{N}_\nu)$ and

$$\|z_\beta^\nu\| \geq \varepsilon \quad \text{for all } \nu$$

where ε is some positive scalar. It is then easy to verify that any accumulation point of the normalized sequence $\{z_\beta^\nu / \|z_\beta^\nu\|\}$ must be a nonzero solution of the problem $(0, N)$. Since such an accumulation point exists, we obtain a contradiction to the assumption $N \in \boldsymbol{R_0}$. □

7.3.3 Remark. Part (c) of Theorem **7.3.2** is vacuously true if z^* is a nondegenerate solution of the LCP (q, M) (because the index set β is empty). Consequently, any nondegenerate solution z^* of (q, M) must be stable, provided that $M_{\alpha\alpha}$ is nonsingular. As a matter of fact, Theorem **7.3.7** shows that such a solution must be strongly stable.

Theorem **7.3.2** shows that the stability of a solution z^* of an LCP (q, M) can be characterized in terms of a certain matrix property of the Schur complement N, provided that the principal submatrix $M_{\alpha\alpha}$ is nonsingular. Nevertheless, as the example below shows, this latter matrix can be singular at a stable solution.

7.3.4 Example. Consider the data

$$q = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & -2 & 2 \\ 1 & 1 & 2 \\ -1 & -1 & 0 \end{bmatrix}.$$

The LCP (q, M) has a unique solution $z^* = (1, 0, 0)$ with $\alpha = \{1\}$ and $\beta = \{2, 3\}$. The principal submatrix $M_{\alpha\alpha} = 0$ is singular, but the solution z^* is stable. The reader is asked to verify the latter statement in Exercise **7.6.2**.

As Theorem **7.3.2** shows, the matrix class $\mathrm{int}(\boldsymbol{Q}) \cap \boldsymbol{R_0}$ plays a central role in the solution stability of the LCP. In the next result, we provide several characterizations of such a matrix within the class of semimonotone matrices.

7.3.5 Theorem. Let $M \in R^{n\times n} \cap \boldsymbol{E_0}$. The following statements are equivalent.

(a) $M \in \boldsymbol{R_0}$.

(b) $M \in \boldsymbol{R}$.

(c) $M \in \mathrm{int}(\boldsymbol{Q}) \cap \boldsymbol{R_0}$.

(d) $M \in \boldsymbol{Q} \cap \boldsymbol{R_0}$.

Proof. (a) $\Rightarrow$ (b). This is the inclusion (3.9.9).

(b) $\Rightarrow$ (c). This is Exercise **3.12.25**.

(c) $\Rightarrow$ (d) $\Rightarrow$ (a). These are obvious. □

The implication [(c) $\Rightarrow$ (b)] in Theorem **7.3.2** says that if a matrix M belongs to the interior of the class $\boldsymbol{Q}$, then the homogeneous LCP $(0, M)$ is stable at the zero solution provided that zero is the unique solution. The next result generalizes this conclusion to the inhomogeneous LCP (q, M) when it has a unique solution.

7.3.6 Proposition. Let $M \in \text{int}(\boldsymbol{Q}) \cap \boldsymbol{R_0}$. If the LCP (q, M) has a unique solution z^*, then (q, M) is stable at z^*.

Proof. It suffices to show all solutions of the LCP $(\bar{q}, \bar{M})$ converge to z^* as $(\bar{q}, \bar{M})$ approaches (q, M). Assume the contrary. Then, there exists an $\varepsilon > 0$ and a sequence $\{(q^\nu, M^\nu)\} \to (q, M)$ such that for each ν, the LCP (q^ν, M^ν) has a solution z^ν satisfying $\|z^\nu - z^*\| \geq \varepsilon$. (Note that $M^\nu \in \boldsymbol{Q}$ by the interior assumption of M.) The sequence $\{z^\nu\}$ is bounded; otherwise a subsequential limit of the normalized sequence $\{z^\nu/\|z^\nu\|\}$ would yield a nonzero solution of the homogeneous LCP $(0, M)$ which would contradict the assumption $M \in \boldsymbol{R_0}$. Now that $\{z^\nu\}$ is bounded, let $\tilde{z}$ be any limit point. It is easy to show that $\tilde{z} \in \text{SOL}(q, M)$. But $\|\tilde{z} - z^*\| \geq \varepsilon$ contradicting the uniqueness assumption on z^*. □

The stability notion defined in **7.3.1** allows the matrix M to change. This accounts for the membership of the matrix N in the interior of the class $\boldsymbol{Q}$ in condition (c) of Theorem **7.3.2**. If one is merely interested in a more restricted kind of stability in which the perturbation occurs only in the vector q, then the interiority condition can be dropped. In essence, this latter case is covered by the local results developed in Section 6.5.

Strong stability

A stable solution of the LCP need not be strongly stable. This can be seen from Theorem **7.3.7** below which establishes that the principal submatrix $M_{\alpha\alpha}$ associated with a strongly stable solution z^* of (q, M) must be nonsingular; as illustrated by Example **7.3.4**, the submatrix $M_{\alpha\alpha}$ may be singular if z^* is just stable.

The next result provides necessary and sufficient conditions for the strong stability of a given solution of an LCP. This characterization resembles Theorem **7.3.2** but with two salient differences: (i) the nonsingularity of $M_{\alpha\alpha}$ is no longer an assumption, but rather, is a consequence of strong stability, and (ii) the matrix class to which the Schur complement N belongs, is $\boldsymbol{P}$. As we see from the proof of Theorem **7.3.2**, the stability of a solution z^* for the LCP (q, M) is closely related to the homogeneous LCP $(0, N)$ under perturbation; in essence, the strong stability of z^* is completely dictated by the unique solvability of the problems (r, N) for all vectors $r \in R^{|\beta|}$; the latter is clearly equivalent to the $\boldsymbol{P}$-property of N (cf. Section 3.3).

7.3.7 Theorem. Let z^* be a solution of the LCP (q, M). The following statements are equivalent:

(a) z^* is strongly stable for the LCP (q, M).

(b) $M_{\alpha\alpha}$ is nonsingular, and the zero vector is strongly stable for the LCP $(0, N)$.

(c) $M_{\alpha\alpha}$ is nonsingular, and $N \in \boldsymbol{P}$.

Proof. (a) $\Rightarrow$ (b). Suppose z^* is strongly stable. Let U and V be the two neighborhoods as specified in Definition **7.3.1**. We establish first the nonsingularity of $M_{\alpha\alpha}$. Assume the contrary. Let u_α be a nonzero vector such that

$$M_{\alpha\alpha} u_\alpha = 0.$$

Choose $r_\beta \geq 0$ such that

$$r_\beta + M_{\beta\alpha} u_\alpha \geq 0.$$

Consider the perturbed LCP (q^θ, M) where

$$q^\theta_\alpha = q_\alpha, \quad q^\theta_\beta = q_\beta + \theta r_\beta, \quad q^\theta_\gamma = q_\gamma.$$

It is easily seen that for all $\theta > 0$ sufficiently small, this perturbed problem (q^θ, M) has two distinct solutions, z^* and z^θ where

$$z^\theta_\alpha = z^*_\alpha + \theta u_\alpha, \quad z^\theta_\beta = 0, \quad z^\theta_\gamma = 0.$$

Moreover, if we choose $\theta > 0$ small enough, the pair (q^θ, M) and the solution z^θ fall within the neighborhood U of (q, M) and V of z^* respectively. But this contradicts the strong stability of z^* which requires the set $S_V(\bar{q}, \bar{M})$ be a singleton for all $(\bar{q}, \bar{M})$ sufficiently close to (q, M).

Now, by Theorem **7.3.2**, it follows that the zero vector is a stable solution of the homogeneous LCP $(0, N)$. To deduce the strong stability of the zero solution, it suffices to observe that there is a one-to-one correspondence between the vectors in SOL$(r_\beta, \bar{N}) \cap \tilde{V}_\beta$ and those in SOL$(\bar{q}, \bar{M}) \cap \tilde{V}$ (see the notation used in the proof of [(a) $\Rightarrow$ (b)] in **7.3.2**).

(b) $\Rightarrow$ (c). It suffices to show $N \in \boldsymbol{P}$. Fix an arbitrary vector $r \in R^{|\beta|}$. Since zero is a strongly stable solution for the homogeneous problem $(0, N)$, there exist two scalars $\varepsilon_1, \varepsilon_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1]$ the LCP $(\varepsilon r, N)$ has a unique solution v satisfying $\|v\| \leq \varepsilon_2$. We argue that this latter LCP has a globally unique solution for all sufficiently small ε. According to Theorem **7.3.2**, the homogeneous LCP $(0, N)$ has a unique solution. Hence, applying Theorem **7.2.1** to the latter LCP, we deduce the existence of scalars $\varepsilon_3 > 0$ and $c > 0$ such that for all $\varepsilon \in (0, \varepsilon_3]$ and for all solutions v of the LCP $(\varepsilon r, N)$, we have

$$\|v\| \leq c\varepsilon\|r\|.$$

Let

$$\bar{\varepsilon} = \min(\varepsilon_1, \varepsilon_3, \frac{\varepsilon_2}{c\|r\|}).$$

Then, it follows that for all $\varepsilon \in (0, \bar{\varepsilon})$, the LCP $(\varepsilon r, N)$ must have a unique solution; consequently, the same is true for (r, N). This establishes $N \in \boldsymbol{P}$.

(c) $\Rightarrow$ (a). Suppose that $M_{\alpha\alpha}$ is nonsingular and $N \in \boldsymbol{P}$. Then, provided that $\bar{M}$ is sufficiently close to M, the principal submatrix $\bar{M}_{\alpha\alpha}$ remains nonsingular and the corresponding Schur complement

$$\bar{N} = \bar{M}_{\beta\beta} - \bar{M}_{\beta\alpha}\bar{M}_{\alpha\alpha}^{-1}\bar{M}_{\alpha\beta} \in \boldsymbol{P}.$$

Consequently, in the notation of Theorem **7.3.2**, the LCP given in (3) has a unique solution for all pairs $(\bar{q}, \bar{M})$ with $\bar{M}$ sufficiently close to M. By the reduction argument made in the proof of this previous theorem, it is an easy matter to establish the strong stability of the solution z^* for the LCP (q, M). □

7.3.8 Remark. The two conditions in part (c) of **7.3.7** are precisely the requirement for the solution z^* to be a strongly regular vector with respect to the function $H_{q,M}(\cdot)$ (cf. Definition **5.8.3**). Consequently, if z^* is a solution of the LCP (q, M), then z^* is strongly stable if and only if it is a strongly regular vector with respect to the "min" function.

The strong stability of a solution z^* of the LCP (q, M) ensures that any slightly perturbed LCP will have a unique solution that is close to z^*. This does not imply, however, that z^* is the (globally) unique solution of (q, M), nor does it imply that the perturbed problems have (globally) unique solutions. The data $(\bar{q}, M)$ given in Example **7.2.5** illustrate this. By the characterization in **7.3.7**, the solution $\bar{z}$ exhibited there is strongly stable for the given LCP. Yet, the same LCP has another solution, namely, $(0, 1, 0)$.

Some Lipschitzian results

If z^* is a strongly stable solution of the LCP (q, M), then the mapping $S_V : U \to R^n$ defined by

$$S_V(\bar{q}, \bar{M}) = \mathrm{SOL}(\bar{q}, \bar{M}) \cap V.$$

is a single-valued function. The next result, **7.3.9**, shows that this function $S_V(\cdot)$ is Lipschitzian when the neighborhood U is properly restricted. Notice the distinction between the two theorems, **7.2.1** and **7.3.9**. In the former result, the matrix M is not perturbed, and the upper Lipschitzian property pertains to the entire solution set of the perturbed LCP; in the latter result, the matrix M is permitted to vary, and the Lipschitzian property refers to that (unique) solution of the perturbed LCP lying in the neighborhood V; solutions outside of V—if they exist—do not necessarily obey this property.

7.3.9 Theorem. Suppose that z^* is a strongly stable solution of the LCP (q, M). Then, there exist neighborhoods $\bar{U}$ of (q, M) and $\bar{V}$ of z^*, and a Lipschitzian function $z : \bar{U} \to \bar{V}$ such that $z(q, M) = z^*$ and for all $(\bar{q}, \bar{M}) \in \bar{U}$, the vector $z(\bar{q}, \bar{M})$ is the unique solution of the LCP $(\bar{q}, \bar{M})$ that belongs to $\bar{V}$.

Only the Lipschitzian property in the above theorem requires a proof. In turn, this proof relies on the lemma below which asserts two Lipschitzian properties of the (unique) solution of an LCP with a $\boldsymbol{P}$-matrix. In turn, this lemma makes use of the fundamental constant $c(M)$ of a $\boldsymbol{P}$-matrix M defined in (5.10.6).

7.3.10 Lemma. Let $M \in R^{n \times n} \cap \boldsymbol{P}$.

(a) For any two vectors q and q' in R^n,

$$\|z^* - z'\|_\infty \leq c(M)^{-1} \|q - q'\|_\infty \tag{4}$$

where z^* and z' denote the unique solutions of the LCPs (q, M) and (q', M) respectively.

(b) For each vector $q \in R^n$, there exists a neighborhood U of the pair (q, M) and a constant $c' > 0$ such that for any $(q^i, M^i) \in U$ $(i = 1, 2)$, $M^i \in \boldsymbol{P}$ and

$$\|z^1 - z^2\|_\infty \leq c'(\|q^1 - q^2\|_\infty + \|M^1 - M^2\|_\infty)$$

where z^i denotes the unique solution of (q^i, M^i) for $i = 1, 2$.

Proof. The proof of part (a) is rather similar to that of Propositions **5.10.5** and **5.10.7**. For each $i = 1, \ldots, n$, we have

$$0 \geq (z_i^* - z_i')(w_i^* - w_i') = (z_i^* - z_i')[(q - q') + M(z^* - z')]_i. \tag{5}$$

Hence,

$$\max_{1 \leq i \leq n} (z^* - z')_i (M(z^* - z'))_i \leq \max_{1 \leq i \leq n} |(z^* - z')_i (q - q')_i|$$
$$\leq \|z^* - z'\|_\infty \|q - q'\|_\infty.$$

By the definition of the constant $c(M)$, we have

$$\max_{1 \leq i \leq n} (z^* - z')_i (M(z^* - z'))_i \geq c(M) \|z^* - z'\|_\infty^2.$$

Combining this with the previous inequalities, and cancelling one factor $\|z - z'\|_\infty$, we obtain the desired inequality (4).

To prove part (b), we first note that the following inequality holds for any two $\boldsymbol{P}$-matrices M^1 and M^2:

$$|c(M^1) - c(M^2)| \leq \|M^1 - M^2\|_\infty. \tag{6}$$

Hence, it follows from Proposition **5.10.7** that there exist a neighborhood U of the pair (q, M), a constant $\tilde{c} > 0$ and a scalar $\delta \in (0, c(M))$ such that for any $(q', M') \in U$, the matrix $M' \in \boldsymbol{P}$, $\|M' - M\|_\infty \leq \delta$ and

$$\|z'\|_\infty \leq \tilde{c} \tag{7}$$

where z' is the unique solution of the LCP (q', M').

Now, take two pairs (q^1, M^1) and (q^2, M^2) in the neighborhood U. As in the expression (5), we have

$$0 \geq (z^1 - z^2)_i[(q^1 - q^2) + (M^1 z^1 - M^2 z^2)]_i.$$

By applying an argument similar to that used in part (a) above, we deduce

$$c(M^1)\|z^1 - z^2\|_\infty \leq \|q^1 - q^2\|_\infty + \|M^1 - M^2\|_\infty \|z^2\|_\infty.$$

By (6),

$$c(M^1) \geq c(M) - \|M^1 - M\|_\infty \geq c(M) - \delta.$$

Hence, combining the last two inequalities with (7), we obtain

$$\|z^1 - z^2\|_\infty \leq (c(M) - \delta)^{-1} \max(1, \tilde{c})[\ \|q^1 - q^2\|_\infty + \|M^1 - M^2\|_\infty]$$

which establishes the Lipschitzian property of the solution of the LCP as the data vary in the neighborhood U. □

7.3.11 Remark. Lemma **7.3.10** shows that when $M \in \boldsymbol{P}$, the unique solution of (q, M) is a globally Lipschitzian function of the vector q. Moreover, the inequality (6) shows that the constant $c(M)$ associated with a $\boldsymbol{P}$-matrix M is a Lipschitzian function of M with a modulus equal to one.

Lemma **7.3.10** can be extended to the mixed LCP (1.5.1) with a nonsingular matrix A and the Schur complement $B - DA^{-1}C \in \boldsymbol{P}$. This extension can be used to complete the proof of the desired Lipschitzian property of the solution function $z(\cdot)$ in Theorem **7.3.9**. The reader is asked to supply the omitted details of the proof in Exercise **7.6.4**.

When z^* is a strongly stable solution of the LCP (q, M), Theorem **7.3.9** ensures the existence of a Lipschitzian solution function $z(\cdot)$ defined on a neighborhood of the data pair (q, M) and having values in a neighborhood of z^*. In the next theorem, we derive a weaker version of this result by assuming that z^* is a stable solution of (q, M). In essence, the conclusion is a strengthening of the condition (ii) in the definition **7.3.1** of stability and is a kind of weak Lipschitzian property at the solution z^*.

7.3.12 Theorem. Let z^* be a stable solution of the LCP (q, M). Then, there exist a constant $c > 0$ and neighborhoods U of (q, M) and V of z^* such that

(i) for all $(\bar{q}, \bar{M}) \in U$, the set $S_V(\bar{q}, \bar{M})$ is nonempty,

(ii) $\sup\{\|y - z^*\| : y \in S_V(\bar{q}, \bar{M})\} \leq c(\|\bar{q} - q\| + \|\bar{M} - M\|)$.

Proof. Let U and $\bar{V}$ be the neighborhoods as given in **7.3.1**. By Theorem **7.2.1**, there exist a neighborhood $\mathcal{Q}$ of the vector q and a constant $c' > 0$ such that for all vectors $\tilde{q} \in \mathcal{Q}$, we have

$$\mathrm{SOL}(\tilde{q}, M) \subseteq \mathrm{SOL}(q, M) + c'\|\tilde{q} - q\|\mathcal{B}. \tag{8}$$

If $y \in S_{\bar{V}}(\bar{q}, \bar{M})$, then $y \in \mathrm{SOL}(\tilde{q}, M)$ where

$$\tilde{q} = \bar{q} + (\bar{M} - M)y.$$

Moreover, if $\varepsilon > 0$ is the radius of the neighborhood $\bar{V}$, then the vectors y in $S_{\bar{V}}(\bar{q}, \bar{M})$ are uniformly bounded by $\varepsilon + \|z^*\|$. Hence, by restricting the neighborhood U if necessary, it follows that the vector $\tilde{q} \in \mathcal{Q}$. By (8), there exists a solution z of (q, M) such that

$$\|y - z\| \leq c'(\|\bar{q} - q\| + \|\bar{M} - M\| \, \|y\|). \tag{9}$$

By restricting the vector y to be chosen from $S_V(\bar{q}, \bar{M})$ where $V \subseteq \bar{V}$ is a smaller neighborhood of z^*, and by restricting the neighborhood U even further if necessary, we can be ensured that the solution z must lie in the neighborhood $\bar{V}$. But since z^* is the only solution of (q, M) within $\bar{V}$, we must have $z = z^*$. Assertion (ii) now follows easily from (9) in view of the boundedness of the solutions $y \in S_V(\bar{q}, \bar{M})$. □

It is natural to ask whether the Lipschitzian property (ii) in Theorem **7.3.12** can be strengthened to the following inequality

$$\sup\{\|y^1 - y^2\| : y^i \in S_V(q^i, M^i), i = 1, 2\} \leq c(\|q^1 - q^2\| + \|M^1 - M^2\|)$$

for any $(q^i, M^i) \in U$ $(i = 1, 2)$. The example below shows that this strengthening is generally not possible.

7.3.13 Example. Let $M \in R^{2 \times 2}$ be the matrix with all entries equal to 1. The zero vector is the (unique) solution of the homogeneous LCP $(0, M)$, and it is stable (because $M \in \mathrm{int}(\boldsymbol{Q}) \cap \boldsymbol{R_0}$). Consider the vectors

$$q^1 = \begin{bmatrix} \varepsilon^2 - \varepsilon \\ -\varepsilon \end{bmatrix}, \quad \text{and} \quad q^2 = \begin{bmatrix} -\varepsilon \\ \varepsilon^2 - \varepsilon \end{bmatrix}.$$

For $\varepsilon > 0$ sufficiently small, the unique solutions of the LCPs, (q^1, M) and (q^2, M), are given, respectively, by

$$z^1 = \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix}, \quad z^2 = \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix}.$$

Hence,

$$\frac{\|z^1 - z^2\|_\infty}{\|q^1 - q^2\|_\infty} = \frac{1}{\varepsilon} \to \infty \quad \text{as} \quad \varepsilon \downarrow 0.$$

In Section 5.10, we have discussed the issues of residues and error bounds for the linear complementarity problem. There, the results obtained all rely on the blanket assumption that the matrix M either belongs to the class $\boldsymbol{P}$ or is positive semi-definite; and they are applicable to an arbitrary vector in R^n. In particular, Proposition **5.10.5** establishes an upper bound for the distance $\|z - x\|_\infty$ between a vector $x \in R^n$ and the unique solution z of (q, M) in terms of the residue $\|\min(x, q + Mx)\|_\infty$. In what follows, we derive a local version of this result by considering a stable solution $z \in \text{SOL}(q, M)$ and vectors x that are close to z. This result justifies the use of the quantity $\|\min(x, q + Mx)\|_\infty$ as a measure of goodness for x to be an approximate solution of (q, M).

7.3.14 Proposition. Suppose z is a stable solution of the LCP (q, M). Then, there exist a constant $\mu > 0$ and a neighborhood V of z such that for every $x \in V$,

$$\|x - z\| \leq \mu \|\min(x, q + Mx)\|.$$

Proof. By Theorem **7.3.12**, there exist neighborhoods $\mathcal{Q}$ of q and V' of z and a constant $c > 0$ such that for every $q' \in \mathcal{Q}$ and every $y \in S_{V'}(q', M)$,

$$\|y - z\| \leq c\|q - q'\|.$$

Since the "min" function is continuous, there exists a neighborhood V of z such that for every vector $x \in V$, we have $q' \in \mathcal{Q}$ and $x - u \in V'$ where

$$q' = q + (M - I)u, \quad \text{and} \quad u = \min(x, q + Mx).$$

Let x be an arbitrary vector in this neighborhood V. Then, it is easy to see that the vector $v = x - u \in \text{SOL}(q', M)$. Hence, we have

$$\|z - v\| \leq c\|(M - I)u\|,$$

from which it follows that

$$\|z - x\| \leq (1 + c\|M - I\|)\|u\|.$$

Consequently, the proposition is established with $\mu = 1 + c\|M - I\|$. □

Besides being instrumental in the proof of the above local error bound result (Proposition **7.3.14**), Theorem **7.3.12** can be used to establish a rate of convergence property of the splitting method (Algorithm **5.2.1**) for solving a general asymmetric LCP. This latter property is in contrast to that asserted in Corollary **7.2.12** which is applicable only to a symmetric LCP.

7.3.15 Theorem. Let (B, C) be a splitting of the matrix $M \in R^{n\times n}$ with $B \in \boldsymbol{Q}$. For a given $z^0 \in R^n_+$, let $\{z^\nu\}$ be a sequence of iterates produced by Algorithm **5.2.1**. Suppose that $\{z^\nu\}$ converges to a solution z^* of (q, M) and that z^* is a stable solution of the LCP $(q + Cz^*, B)$. Then there exist a constant $\theta > 0$ and an integer $\bar{\nu} \geq 0$ such that for all $\nu \geq \bar{\nu}$,

$$\|z^{\nu+1} - z^*\| \leq \theta\|z^\nu - z^*\|.$$

Proof. As $z^{\nu+1} \in \text{SOL}(q+Cz^\nu, B)$ and the sequence $\{z^\nu\}$ converges to z^*, the stability assumption of z^* and Theorem **7.3.12** together readily yield the desired conclusion asserted by the theorem. □

7.3.16 Remark. According to Theorem **7.3.2**, z^* is a stable solution for the LCP $(q + Cz^*, B)$ if $B_{\alpha\alpha}$ is nonsingular and the Schur complement $B_{\beta\beta} - B_{\beta\alpha}B^{-1}_{\alpha\alpha}B_{\alpha\beta}$ belongs to $\text{int}(\boldsymbol{Q}) \cap \boldsymbol{R_0}$ where

$$\alpha = \{i : z^* > 0 = (q + Mz^*)\} \quad \text{and} \quad \beta = \{i : z^* = 0 = (q + Mz^*)\}.$$

7.4 Solution Differentiability

In this section, we study the multivariate parametric linear complementarity problem (7.1.1). Throughout the discussion, we assume that $q : \Lambda \to R^n$ and $M : \Lambda \to R^{n\times n}$ are Lipschitz continuous functions of the parameter vector $\varepsilon \in \Lambda \subseteq R^m$. Suppose that a solution z^* of the LCP $(q(\varepsilon^*), M(\varepsilon^*))$ is given that corresponds to a specific value ε^* of the parameter. We are interested in the behavior of this solution when ε is perturbed around the base vector ε^*. Thus, in this analysis, the parameter set Λ is a suitable neighborhood of ε^*.

According to Theorem **7.3.9**, if z^* is a strongly stable solution, then there exist neighborhoods Λ of ε^* and V of z^*, and a Lipschitzian function $z : \Lambda \to V$ such that $z(\varepsilon^*) = z^*$ and for each $\varepsilon \in \Lambda$, $z(\varepsilon)$ is the unique solution of the LCP $(q(\varepsilon), M(\varepsilon))$ that lies in V. By assuming an additional differentiability property on the data functions $q(\cdot)$ and $M(\cdot)$, we establish below that the solution $z(\cdot)$ is directionally differentiable at ε^*; furthermore, we provide a necessary and sufficient condition for this solution function to be Fréchet differentiable at z^*.

Before deriving these differentiability results, we use the parametric LCP with a single parameter to illustrate the main idea.

7.4.1 Example. Consider the PLCP (4.5.1) with parameter $\lambda \in R$ and data

$$q = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \quad d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

The unique solution of the complete parametric problem is given by

$$\begin{aligned} \lambda \geq 2 \quad &\Rightarrow \quad (z_1(\lambda), z_2(\lambda)) = (0, 0), \\ 2 \geq \lambda \geq 4/3 \quad &\Rightarrow \quad (z_1(\lambda), z_2(\lambda)) = (0, 1 - \lambda/2), \\ 4/3 \geq \lambda \quad &\Rightarrow \quad (z_1(\lambda), z_2(\lambda)) = (4/3 - \lambda, 5/3 - \lambda). \end{aligned}$$

It is easily seen that the left and right slopes of the function $z_2(\lambda)$ are different at the breakpoints $\lambda = 2$ and $4/3$. Hence, $z_2(\lambda)$ fails to be differentiable at these critical values of λ. Note, however, that $z_1(\lambda)$ is identically equal to 0 for $\lambda \in [4/3, \infty)$; in particular, it is differentiable at the value $\lambda = 2$. At values other than these two, both $z_1(\lambda)$ and $z_2(\lambda)$ are continuously differentiable.

The above example illustrates several important points. First of all, left and right derivatives of the solution function $z(\lambda)$ exist at all values λ. The LCP $(q + \lambda d, M)$ has a unique nondegenerate solution at all values of λ except at the two breakpoints $\lambda = 2, 4/3$. At the nondegenerate values of λ, the solution $z(\lambda)$ is F-differentiable. At the degenerate value $\lambda = 2$, there is a change of basis and the solution $z(2) = (0, 0)$ is degenerate; yet one variable (namely, z_1) is differentiable whereas the other variable is not.

The nondifferentiable variable z_2 is the degenerate variable which is also the one that causes the basis to change. At the other critical value $\lambda = 4/3$, both variables become nondifferentiable.

In essence, given the parameter vector ε^* and the solution function $z(\cdot)$ of the multivariate parametric linear complementarity problem (7.1.1) around ε^*, those components $z_i(\cdot)$ for $i \in \gamma(z(\varepsilon^*))$ must be F-differentiable at ε^*; as a matter of fact, these variables $z_i(\varepsilon)$ are identically equal to zero for ε in a small neighborhood of ε^*. The differentiability of the remaining components $z_i(\cdot)$ (for $i \in \alpha(z(\varepsilon^*)) \cup \beta(z(\varepsilon^*))$) at ε^* is completely determined by the degenerate variables z_i for $i \in \beta(z(\varepsilon^*))$. The result to follow makes precise the central role played by these degenerate variables in the differentiability of $z(\cdot)$ at ε^*. In stating the result, we write

$$w(\varepsilon, z) = q(\varepsilon) + M(\varepsilon)z$$

and let $\nabla_\varepsilon w^*_\kappa$ denote the Jacobian matrix $(\partial w_i(\varepsilon^*, z^*)/\partial \varepsilon_j)$ for (i, j) in $\kappa \times \{1, \ldots, m\}$ where $\kappa \subseteq \{1, \ldots, n\}$.

7.4.2 Theorem. Suppose that z^* is a strongly stable solution of the LCP $(q(\varepsilon^*), M(\varepsilon^*))$, and that the functions $q(\cdot)$ and $M(\cdot)$ are Lipschitz continuous in a neighborhood of the vector $\varepsilon^* \in R^m$. Suppose further, that the function $w(\cdot, z^*)$ is F-differentiable at ε^*. Then there exist neighborhoods Λ of ε^* and V of z^*, and a Lipschitzian function $z : \Lambda \to V$ such that $z(\epsilon^*) = z^*$ and for each $\varepsilon \in \Lambda$, $z(\varepsilon)$ is the unique solution of the LCP $(q(\varepsilon), M(\varepsilon))$ in V. Moreover,

(a) the function $z(\cdot)$ is directionally differentiable at ε^*, the directional derivative $z'(\varepsilon^*, \eta)$ along a direction $\eta \in R^m$ is the unique solution z to the following complementarity system:

$$\begin{aligned} \nabla_\varepsilon w^*_\alpha \eta + M(\varepsilon^*)_{\alpha\alpha} z_\alpha + M(\varepsilon^*)_{\alpha\beta} z_\beta &= 0, \\ \nabla_\varepsilon w^*_\beta \eta + M(\varepsilon^*)_{\beta\alpha} z_\alpha + M(\varepsilon^*)_{\beta\beta} z_\beta &\geq 0, \\ z_\beta^T [\nabla_\varepsilon w^*_\beta \eta + M(\varepsilon^*)_{\beta\alpha} z_\alpha + M(\varepsilon^*)_{\beta\beta} z_\beta] &= 0, \\ z_\beta \geq 0, \quad z_\gamma &= 0 \end{aligned} \tag{1}$$

where α, β and γ are the index sets associated with the solution z^*;

(b) the following stronger limit property holds:

$$\lim_{\varepsilon \to \varepsilon^*} \frac{z(\varepsilon) - z(\varepsilon^*) - z'(\varepsilon^*, \varepsilon - \varepsilon^*)}{\|\varepsilon - \varepsilon^*\|} = 0; \tag{2}$$

(c) the function $z(\cdot)$ is F-differentiable at ε^* if and only if either β is empty or

$$\nabla_\varepsilon w^*_\beta - M(\varepsilon^*)_{\beta\alpha} M(\varepsilon^*)^{-1}_{\alpha\alpha} \nabla_\varepsilon w^*_\alpha = 0; \tag{3}$$

in this case, the F-derivative of $z(\cdot)$ at ε^* is given by

$$\nabla z_\alpha(\varepsilon^*) = -M(\varepsilon^*)^{-1}_{\alpha\alpha} \nabla_\varepsilon w^*_\alpha, \ \nabla z_\beta(\varepsilon^*) = 0, \ \nabla z_\gamma(\varepsilon^*) = 0; \tag{4}$$

(d) if the function $w(\cdot, z^*)$ has a strong F-derivative at ε^* and if $z(\cdot)$ is F-differentiable at ε^*, then the F-derivative of $z(\cdot)$ at ε^* is also strong.

Proof. In light of the discussion that precedes Example **7.4.1**, it suffices to establish the four differentiability assertions (a) – (d).

The system (1) is essentially a mixed linear complementarity problem in the variables (z_α, z_β). According to the characterization of strong stability (cf. **7.3.7**), this system has a unique solution for all vectors $\eta \in R^m$ (see **3.12.5**). In order to show that such a solution is equal to the directional derivative $z'(\varepsilon^*, \eta)$, it suffices to verify the limit expression (2). For this purpose, let $y(\varepsilon)$ denote the unique solution of the system (1) corresponding to $\eta = \varepsilon - \varepsilon^*$. Since $M(\varepsilon^*)_{\alpha\alpha}$ is nonsingular, we have

$$y_\alpha(\varepsilon) = -M(\varepsilon^*)^{-1}_{\alpha\alpha}(\nabla_\varepsilon w^*_\alpha(\varepsilon - \varepsilon^*) + M(\varepsilon^*)_{\alpha\beta} y_\beta(\varepsilon)),$$

and $y_\beta(\varepsilon)$ is the unique solution of the LCP $(s_\beta(\varepsilon), N(\varepsilon^*))$ where

$$N(\varepsilon^*) = M(\varepsilon^*)_{\beta\beta} - M(\varepsilon^*)_{\beta\alpha} M(\varepsilon^*)^{-1}_{\alpha\alpha} M(\varepsilon^*)_{\alpha\beta} \in \boldsymbol{P},$$

$$s_\beta(\varepsilon) = (\nabla_\varepsilon w^*_\beta - M(\varepsilon^*)_{\beta\alpha} M(\varepsilon^*)^{-1}_{\alpha\alpha} \nabla_\varepsilon w^*_\alpha)(\varepsilon - \varepsilon^*).$$

Consequently, it follows from Lemma **7.3.10** that there must exist a constant $c > 0$ such that for all ε,

$$\|(y_\alpha(\varepsilon), y_\beta(\varepsilon))\| \leq c\|\varepsilon - \varepsilon^*\|.$$

By letting ε be sufficiently close to ε^*, we can therefore be assured that the vector $z(\varepsilon^*) + y(\varepsilon)$ is a solution of the LCP $(\tilde{q}(\varepsilon), M(\varepsilon^*))$ where

$$\tilde{q}(\varepsilon) = q(\varepsilon^*) + \nabla_\varepsilon w^*(\varepsilon - \varepsilon^*);$$

moreover, $z(\varepsilon^*) + y(\varepsilon)$ can be made arbitrarily close to $z(\varepsilon^*)$ by further restricting ε if necessary.

By definition, $z(\varepsilon)$ is a solution of the LCP $(q(\varepsilon), M(\varepsilon))$; equivalently, $z(\varepsilon)$ is a solution of the LCP $(\bar{q}(\varepsilon), M(\varepsilon^*))$ where

$$\bar{q}(\varepsilon) = q(\varepsilon) + (M(\varepsilon) - M(\varepsilon^*))z(\varepsilon^*) + (M(\varepsilon) - M(\varepsilon^*))(z(\varepsilon) - z(\varepsilon^*)).$$

Since both vectors $\tilde{q}(\varepsilon)$ and $\bar{q}(\varepsilon)$ can be made arbitrarily close to $q(\varepsilon^*)$ by restricting ε, it follows from **7.3.9** that there exists a constant $L > 0$ such that for all ε sufficiently close to ε^*, we have

$$\| z(\varepsilon) - z(\varepsilon^*) - y(\varepsilon) \| \leq L \| \bar{q}(\varepsilon) - \tilde{q}(\varepsilon) \|.$$

By the F-differentiability of $w(\cdot, z^*)$ at ε^*, it is easy to deduce

$$\lim_{\varepsilon \to \varepsilon^*} \frac{\bar{q}(\varepsilon) - \tilde{q}(\varepsilon)}{\| \varepsilon - \varepsilon^* \|} = 0,$$

which implies,

$$\lim_{\varepsilon \to \varepsilon^*} \frac{z(\varepsilon) - z(\varepsilon^*) - y(\varepsilon)}{\| \varepsilon - \varepsilon^* \|} = 0.$$

This establishes the limit condition (2) which in turn yields the conclusion that the unique solution to the system (1) is equal to the directional derivative $z'(\varepsilon^*, \eta)$.

To prove part (c), suppose that $z(\cdot)$ is F-differentiable at ε^* and that $\beta \neq \emptyset$. Then, we have

$$z'(\varepsilon^*, \eta) + z'(\varepsilon^*, -\eta) = 0 \quad \text{for all } \eta \in R^m.$$

Since the β-components of these directional derivatives are nonnegative, it follows that

$$z'_\beta(\varepsilon^*, \eta) = 0.$$

But $z'_\beta(\varepsilon^*, \eta)$ is the (unique) solution of the LCP $(r_\beta(\eta), N(\varepsilon^*))$ where

$$r_\beta(\eta) = (\nabla_\varepsilon w^*_\beta - M(\varepsilon^*)_{\beta\alpha} M(\varepsilon^*)^{-1}_{\alpha\alpha} \nabla_\varepsilon w^*_\alpha)\eta,$$

we conclude $r_\beta(\eta) \geq 0$ for all $\eta \in R^m$, from which (3) follows easily.

Conversely, if $\beta \neq \emptyset$ and (3) holds, then for all vectors $\eta \in R^m$, we have

$$z'_\alpha(\varepsilon^*, \eta) = -M(\varepsilon^*)^{-1}_{\alpha\alpha} \nabla_\varepsilon w^*_\alpha \eta, \quad z'_\beta(\varepsilon^*, \eta) = 0, \quad z'_\gamma(\varepsilon^*, \eta) = 0.$$

Thus, the directional derivative $z'(\varepsilon^*, \eta)$ is a linear function in the direction η. By the limit condition (2), it follows that $\nabla z(\varepsilon^*)$ exists and is given by (4). The case $\beta = \emptyset$ is proved in a similar way.

Finally, to prove part (d), it suffices to verify

$$\lim_{(\varepsilon,\varepsilon')\to(\varepsilon^*,\varepsilon^*)} \frac{z(\varepsilon) - z(\varepsilon') - \nabla z(\varepsilon^*)(\varepsilon - \varepsilon')}{\|\varepsilon - \varepsilon'\|} = 0$$

assuming $\nabla z(\varepsilon^*)$ exists. The proof of this limit condition is analogous to that of (2). The reader is asked to supply the omitted details in Exercise **7.6.8**. □

7.4.3 Remark. In **7.4.2**, it is assumed that the function $w(\cdot, z^*)$ has an F-derivative at ε^*. This is a fairly weak differentiability assumption; it implies neither the F-differentiability of $q(\cdot)$ nor that of $M(\cdot)$ at ε^*, nor does it imply the F-differentiability of $w(\cdot, z)$ at a vector $z \neq z^*$. The application to Newton's method will clarify these points (cf. Remark **7.4.6**).

As a function in the direction η, the directional derivative $z'(\varepsilon^*, \eta)$, being the unique solution of the complementarity system (1), is piecewise linear (see Proposition **1.4.6**); moreover, $z'(\varepsilon^*, \eta)$ is Lipschitzian in η (see Exercise **7.6.4**). Since there is no particular functional form assumed on the functions $q(\cdot)$ and $M(\cdot)$, the solution $z(\varepsilon)$ in Theorem **7.4.2** is, in general, a nonlinear function in ε. The limit condition (2) allows us to write

$$z(\varepsilon) = z(\varepsilon^*) + z'(\varepsilon^*, \varepsilon - \varepsilon^*) + o(\|\varepsilon - \varepsilon^*\|).$$

This says that the solution function $z(\varepsilon)$ can be approximated by a Lipschitz continuous piecewise affine function in the parameter vector ε.

Besides being a mixed LCP, the system (1) has another noteworthy point: namely, it involves only the α and β variables; thus, its size is less than that of the family of parametric LCP (7.1.1) (unless of course γ is empty). In the event where β is empty—which correponds to the case where $z(\varepsilon^*)$ is a nondegenerate solution of the LCP $(q(\varepsilon^*), M(\varepsilon^*))$—the system (1) reduces to the system of linear equations in the z_α variables:

$$\nabla_\varepsilon w_\alpha^* \eta + M(\varepsilon^*)_{\alpha\alpha} z_\alpha = 0.$$

Incidentally, in this latter nondegenerate case, the condition (3) is vacuously satisfied; hence, the solution function $z(\cdot)$ is trivially F-differentiable at ε^* if $z(\varepsilon^*)$ is a nondegenerate solution of $(q(\varepsilon^*), M(\varepsilon^*))$.

An application: Newton's method

In Section 1.2, the nonlinear complementarity problem was introduced as a source problem for the LCP, and a solution strategy for the former problem was briefly sketched. Interestingly, the convergence of a basic iterative method for solving the nonlinear complementarity problem can be deduced from the solution differentiability results of the multivariate parametric LCP established in Theorem **7.4.2**. The main objective of this subsection is to explain how this can be accomplished.

There is a rich theory of iterative methods for solving the nonlinear complementarity problem. To present the full details of this theory is beyond the scope of this book. In the sequel, we develop the local convergence theory for the most fundamental of such iterative schemes; namely, the *Newton method.* There are several motivations for this development. First, the theory is important in its own right. Second, the development provides an illustration of how the solution differentiability results of the multivariate parametric LCP can be useful in an algorithmic context. Third, we have the opportunity to expand the discussion started in Section 1.2 in order to give further evidence of the important role played by the LCP in the study of the nonlinear complementarity (and the variational inequality) problem.

Consider the nonlinear complementarity problem defined in (1.2.22)

$$z \geq 0, \quad f(z) \geq 0, \quad \text{and} \quad z^{\mathrm{T}} f(z) = 0. \tag{5}$$

Here $f : R^n \to R^n$ is assumed to be Lipschitz continuously differentiable. Newton's method attempts to solve this problem in the following way.

7.4.4 Algorithm. (Newton's Method)

Step 0. *Initialization.* Let $z^0 \geq 0$ be given, set $\nu = 0$.

Step 1. *General step.* Let $z^{\nu+1}$ be some solution of the LCP $(q^\nu, \nabla f(z^\nu))$ where

$$q^\nu = f(z^\nu) - \nabla f(z^\nu) z^\nu.$$

(The choice of the solution $z^{\nu+1}$ will be made precise subsequently.)

Step 2. *Termination test.* If $z^{\nu+1}$ satisfies a prescribed stopping rule, terminate. Otherwise, repeat the general step with ν replaced by $\nu + 1$.

The well-definedness and convergence of the above method is local in nature. This means that in order for the sequence $\{z^\nu\}$ to be well defined and for it to converge to a desired solution z^* of the problem (5), the initial iterate z^0 is required to be chosen from a suitable neighborhood of z^*. As we see shortly, the key property we need is the strong stability of z^* as a solution to a certain linear complementarity problem.

We now formally establish the convergence of Newton's method by using the sensitivity results for the LCP. The first step is to define the family of multivariate parametric LCPs

$$\{(q(v), M(v)) : v \in V\} \tag{6}$$

where V is a certain domain in R^n to be specified and

$$q(v) = f(v) - \nabla f(v)v, \qquad M(v) = \nabla f(v).$$

Note that the vector v plays the role of the parameter vector ε in the notation of (7.1.1).

Suppose that z^* is a solution of the problem (5). Clearly, z^* solves the LCP $(q(z^*), M(z^*))$. We suppose that z^* is a strongly stable solution of the latter LCP; according to Theorem **7.3.7**, this is equivalent to the following two conditions:

$$\nabla_\alpha f_\alpha(z^*) \text{ is nonsingular,} \tag{7}$$

$$\nabla_\beta f_\beta(z^*) - \nabla_\alpha f_\beta(z^*) \nabla_\alpha f_\alpha(z^*)^{-1} \nabla_\beta f_\alpha(z^*) \in \boldsymbol{P}, \tag{8}$$

where

$$\alpha = \{i : z_i^* > 0 = f_i(z^*)\}, \qquad \beta = \{i : z_i^* = 0 = f_i(z^*)\}.$$

Then, by considering z^* both as the base parameter vector ε^* as well as the solution, we deduce, by **7.4.2**, the existence of two neighborhoods V_1 and V_2 of z^* such that for each vector $v \in V_1$, the LCP $(q(v), M(v))$ has a unique solution $G(v)$ that belongs to the neighborhood V_2; moreover,

$$G(z^*) = z^*, \tag{9}$$

and as a function of $v \in V_1$, the solution $G(v)$ is Lipschitzian and directionally differentiable. The neighborhood V_1 serves as the domain of definition for the family (6).

The equation (9) says that the solution z^* is a fixed point of the mapping $G : V_1 \to V_2$. With this function G, we may rephrase Newton's method **7.4.4** as the fixed-point iteration

$$z^{\nu+1} = G(z^\nu).$$

As such, Theorem **2.5.9** can be applied to establish the well-definedness and the convergence of the sequence of iterates produced by the method. In order for this theorem to be applicable, it is essential for the mapping G to have an F-derivative at the fixed point z^* and for $\rho(\nabla G(z^*)) < 1$. The following result, which is an application of **7.4.2**, furnishes these requirements. (This result is reminiscent of Exercise **2.10.13** that asserts the same conclusion, but in the context of Newton's method for systems of nonlinear equations.)

7.4.5 Proposition. Let $f : R^n \to R^n$ be Lipschitz continuously differentiable. Suppose that z^* is a solution of the problem (5) satisfying the conditions (7) and (8). Let $G : V_1 \to R^n$ be as defined above. Then $\nabla G(z^*)$ exists and $\nabla G(z^*) = 0$.

Proof. In order to apply **7.4.2**, we first establish that the function

$$w(v, z^*) = q(v) + M(v)z^* = f(v) - \nabla f(v)(v - z^*) \tag{10}$$

has an F-derivative at $v = z^*$; as a matter of fact, we claim

$$\nabla_v w(z^*, z^*) = 0. \tag{11}$$

For this, it suffices to verify the limit condition

$$\lim_{h \to 0} \frac{w(z^* + h, z^*) - w(z^*, z^*)}{\|h\|} = 0.$$

Substituting the expression (10) into the numerator, we see that the left-hand side of the above equation reduces to

$$\lim_{h \to 0} \frac{f(z^* + h) - f(z^*) - \nabla f(z^* + h)h}{\|h\|}$$

which is equal to zero by the continuous differentiability of f. Hence, (11) follows. This in turn implies that the condition (3) is trivially satisfied. Hence, $\nabla G(z^*)$ exists. Finally, the conclusion $\nabla G(z^*) = 0$ is an immediate consequence of (4). □

7.4.6 Remark. The above proposition serves to clarify Remark **7.4.3**. Since the function f is assumed only once differentiable, neither $q(v)$ nor $M(v)$ is F-differentiale at an arbitrary vector v. Yet, the proof in **7.4.5** demonstrates that the function $w(v, z^*)$ is F-differentiable at $v = z^*$. The differentiability assumption of f in **7.4.5** can be further weakened. Indeed, this result holds if f is F-differentiable in a neighborhood of z^* and has a strong F-derivative at z^*.

Proposition **7.4.5** is all that is required for the applicability of Theorem **2.5.9**. The convergence of Algorithm **7.4.4** is summarized in the result below. The first conclusion of the result requires no further proof as it is a straightforward consequence of **7.4.5** and **2.5.9**; the second conclusion shows that Newton's method possesses a quadratic rate of convergence. The proof of this rate property is also based on the previous sensitivity results of the LCP.

7.4.7 Theorem. Let f and z^* satisfy the assumptions in **7.4.5**. Then, there exists a neighborhood V of z^* such that whenever the initial iterate z^0 is chosen in V, a sequence of vectors $\{z^\nu\}$ can be defined such that each $z^{\nu+1}$ solves the LCP $(q^\nu, \nabla f(z^\nu))$, and $\{z^\nu\}$ converges to z^*. Moreover, there exists a constant $c > 0$ such that for all ν sufficiently large,

$$\|z^{\nu+1} - z^*\| \le c\|z^\nu - z^*\|^2. \tag{12}$$

Proof. In light of the preceding discussion, we prove only the quadratic rate property (12). By its choice, $z^{\nu+1} \in \text{SOL}(q(z^\nu), M(z^\nu))$; or equivalently, $z^{\nu+1}$ is a solution of $(\bar{q}^\nu, M(z^*))$ where

$$\begin{aligned}\bar{q}^\nu &= f(z^\nu) - \nabla f(z^\nu)z^\nu \\ &+ \ (\nabla f(z^\nu) - \nabla f(z^*))z^* + (\nabla f(z^\nu) - \nabla f(z^*))(z^{\nu+1} - z^*).\end{aligned}$$

Since z^* is a strongly stable solution of the LCP $(q(z^*), M(z^*))$, Theorem **7.3.9** implies that there exists a constant $c' > 0$ such that for all ν

sufficiently large,

$$\|z^{\nu+1} - z^*\| \leq c'\|\bar{q}^\nu - q(z^*)\|.$$

Substituting the definitions of $\bar{q}^\nu$ and $q(z^*)$, we have,

$$\bar{q}^\nu - q(z^*) =$$
$$-(f(z^*) - f(z^\nu) - \nabla f(z^\nu)(z^* - z^\nu)) + (\nabla f(z^\nu) - \nabla f(z^*))(z^{\nu+1} - z^*);$$

hence, we deduce

$$(1-c'\|\nabla f(z^\nu)-\nabla f(z^*)\|)\|z^{\nu+1}-z^*\| \leq c'\|f(z^*)-f(z^\nu)-\nabla f(z^\nu)(z^*-z^\nu)\|.$$

Since $\{z^\nu\}$ converges to z^*, there exists a scalar $\varepsilon > 0$ such that for all ν sufficiently large,

$$1 - c'\|\nabla f(z^\nu) - \nabla f(z^*)\| \geq \varepsilon.$$

Moreover, by the Lipschitzian property of the F-derivative $\nabla f(\cdot)$, it follows from the mean value inequality (2.1.6) that there exists a constant $\varepsilon' > 0$ such that for ν large enough,

$$\|f(z^*) - f(z^\nu) - \nabla f(z^\nu)(z^* - z^\nu)\| \leq \varepsilon'\|z^* - z^\nu\|^2.$$

Consequently, combining the last three inequalities, we easily derive the existence of a constant $c > 0$ for which (12) holds for all ν large enough. □

In the above proof of convergence of Newton's method **7.4.4**, we have used part (c) of the differentiability result **7.4.2**. Without going into the details, we mention that part (d) of this latter result is useful in establishing the feasibility of a continuation scheme for enlarging the domain of convergence of the same method. See Note **7.7.8**.

7.5 Stability Under Copositivity

Section 7.3 has analyzed the stability of the linear complementarity problem at a given solution. The analysis is local in nature in that it is made in a neighborhood of the solution. In this section, we broaden the analysis by focusing on the change of the solution set of the LCP (q, M) as a whole, and not on any one particular solution. Nevertheless, unlike the results in Section 7.2, the treatment herein allows both the vector q and the matrix M to vary. This broader treatment entails the restriction of M to a certain matrix class.

In order to provide the necessary background for the results to follow, we recall Theorem **3.8.6** which states that the LCP (q, M) has a solution whenever M is copositive and q belongs to the dual cone of the solution set $\mathrm{SOL}(0, M)$ of the homogeneous LCP $(0, M)$ associated with M. To simplify the notation somewhat, let $S = \mathrm{SOL}(0, M)$. Clearly, $S \subseteq R^n_+$. This implies that the dual cone S^* of S must have a nonempty interior; as a matter of fact, $\operatorname{int} S^*$ contains all positive vectors. It is also easy to see that $q \in \operatorname{int} S^*$ if and only if

$$[\,0 \neq v \geq 0,\ Mv \geq 0,\ v^{\mathrm{T}}Mv = 0\,] \quad \Rightarrow \quad [\,v^{\mathrm{T}}q > 0\,]. \tag{1}$$

With this equivalence at hand, we may state the following stability result for an LCP (q, M) under a copositive perturbation. Part of the significance of this result is that no assumption is imposed on the matrix M; the conclusions of the theorem concerns those perturbed LCPs of (q, M) that are defined by copositive matrices.

7.5.1 Theorem. Let $M \in R^{n\times n}$ be an arbitrary matrix, and let $q \in \operatorname{int} S^*$ where S is the solution set of $(0, M)$. Then there exist positive scalars ε, c and L such that for all $(\tilde{q}, \tilde{M}) \in R^n \times R^{n\times n}$ with $\tilde{M}$ copositive and satisfying $\|q - \tilde{q}\| + \|\tilde{M} - M\| \leq \varepsilon$, the following statements hold:

(a) the LCP $(\tilde{q}, \tilde{M})$ is solvable,

(b) for all $\tilde{z} \in \mathrm{SOL}(\tilde{q}, \tilde{M})$, $\|\tilde{z}\| \leq c$,

(c) $\mathrm{SOL}(\tilde{q}, \tilde{M}) \subseteq \mathrm{SOL}(q, M) + L(\|q - \tilde{q}\| + \|\tilde{M} - M\|)\mathcal{B}$.

Proof. We claim that there exist neighborhoods $\mathcal{Q}$ of q and $\mathcal{M}$ of M such that for any $(\tilde{q}, \tilde{M}) \in \mathcal{Q} \times \mathcal{M}$, we have $\tilde{q} \in \operatorname{int} \tilde{S}^*$ where $\tilde{S} = \mathrm{SOL}(0, \tilde{M})$. Suppose not. Then, there exist sequences $\{q^k\} \subset R^n$, $\{M_k\} \subset R^{n\times n}$, and $\{v^k\} \subset R^n$ such that $v^k \in \mathrm{SOL}(0, M_k)$ and $\|v^k\| = 1$ for each k, $\{q^k\}$ and $\{M_k\}$ converge to q and M respectively, and

$$(v^k)^{\mathrm{T}}q^k \leq 0.$$

Let $\tilde{v}$ be an accumulation point of the sequence $\{v^k\}$. Then $\tilde{v}$ is a nonzero solution of $(0, M)$ and satisfies $\tilde{v}^{\mathrm{T}}q \leq 0$. But this contradicts the assumption $q \in \operatorname{int} S^*$. Hence, the existence of $\mathcal{Q}$ and $\mathcal{M}$ follows.

If $(\tilde{q}, \tilde{M}) \in \mathcal{Q} \times \mathcal{M}$ and $\tilde{M}$ is copositive, Theorem **3.8.6** implies that the LCP $(\tilde{q}, \tilde{M})$ must have a solution. Hence part (a) is established.

Part (b) is proved by contradiction. Suppose that no such constant c exists. Then there exist sequences $\{\|z^k\|\} \to \infty$, $(q^k, M_k) \to (q, M)$ such that for each k, M_k is copositive, and

$$w^k = q^k + M_k z^k \geq 0, \; 0 \neq z^k \geq 0, \; (w^k)^{\mathrm{T}} z^k = 0.$$

It is easy to show that if v is a subsequential limit of the normalized sequence $\{z^k/\|z^k\|\}$, then v violates the implication (1). Indeed, we have

$$\frac{w^k}{\|z^k\|} = \frac{q^k}{\|z^k\|} + M_k \frac{z^k}{\|z^k\|} \geq 0$$

$$0 = (q^k)^{\mathrm{T}} \frac{z^k}{\|z^k\|} + \frac{(z^k)^{\mathrm{T}} M_k z^k}{\|z^k\|} \geq (q^k)^{\mathrm{T}} \frac{z^k}{\|z^k\|}$$

$$0 = \frac{(q^k)^{\mathrm{T}}}{\|z^k\|} \frac{z^k}{\|z^k\|} + \frac{(z^k)^{\mathrm{T}}}{\|z^k\|} M_k \frac{z^k}{\|z^k\|}.$$

Passing to the limit $k \to \infty$ in these expressions establishes the claim on the limit point v. This contradiction completes the proof of (b).

Finally, we prove (c). For this purpose, let $\tilde{z}$ be an arbitrary solution of $(\tilde{q}, \tilde{M})$. Then $\tilde{z} \in \mathrm{SOL}(\bar{q}, M)$ where

$$\bar{q} = \tilde{q} + (\tilde{M} - M)\tilde{z}.$$

Since the solutions of the LCP $(\tilde{q}, \tilde{M})$ are uniformly bounded for all $(\tilde{q}, \tilde{M})$ as given, the vector $\bar{q}$ can be made arbitrarily close to q by restricting ε if necessary. Hence, it follows from Theorem **7.2.1** that there exists a constant $c' > 0$ such that for any solution $\tilde{z}$ of such a pair $(\tilde{q}, \tilde{M})$, there exists a solution z of (q, M) satisfying

$$\|\tilde{z} - z\| \leq c' \|\bar{q} - q\|.$$

With $L = c' \max(1, c)$ where $c > 0$ is the constant obtained in part (b), the desired inclusion of part (c) follows. □

As noted in the proof of part (c), part (b) provides a uniform bound for the solutions of the perturbed LCPs that are close to the given problem

(q, M). It is important to point out that such a bound is valid only for the LCP $(\tilde{q}, \tilde{M})$ with $\tilde{M}$ copositive. As an example, consider the case $n = 1$, with $M = 0$ and $q = 1$. The perturbed LCP (q, M_ε) where $M_\varepsilon = -\varepsilon < 0$ has a solution $z_\varepsilon = 1/\varepsilon$ which clearly is unbounded as ε tends to zero. In a similar way, the restriction that the perturbed matrix $\tilde{M}$ be copositive is also essential for part (c) to hold. Note, however, the perturbed problem $(\tilde{q}, M_\varepsilon)$ remains solvable as long as $\tilde{q}$ is sufficiently close to $q = 1$. This suggests that in general, part (a) in Theorem **7.5.1** might remain valid under a weaker assumption on the perturbed matrix $\tilde{M}$.

An application of Theorem **7.5.1** has appeared in Lemma **5.3.4** which asserts the boundedness of a sequence $\{z^\nu\}$ produced by the splitting method in **5.2.1** for solving a symmetric LCP. Assumption (a) in this lemma implies that M is copositive, and (b) is exactly the condition that q belongs to $\operatorname{int} S^*$. Since for each ν sufficiently large, $z^{\nu+1}$ can be looked upon as a solution of the perturbed LCP $(q + C(z^\nu - z^{\nu+1}), M)$, the boundedness of $\{z^\nu\}$ therefore follows easily from Theorem **7.5.1**.

By restricting M to be a copositive-plus matrix, we can derive the following consequence of **7.5.1** which gives several equivalent conditions for the stability of the LCP (q, M) with an arbitrary vector q.

7.5.2 Corollary. Let $M \in R^{n \times n}$ be a copositive-plus matrix, and $q \in R^n$ be an arbitrary vector. The following statements are equivalent.

(a) There exists a scalar $\delta > 0$ such that the system

$$\tilde{q} + Mz \geq 0, \quad z \geq 0,$$

has a solution z for each $\tilde{q} \in R^n$ satisfying $\|q - \tilde{q}\| \leq \delta$.

(b) There exists a scalar $\varepsilon > 0$ such that for each $(\tilde{q}, \tilde{M}) \in R^n \times R^{n \times n}$ satisfying $\|q - \tilde{q}\| + \|\tilde{M} - M\| \leq \varepsilon$, the system

$$\tilde{q} + \tilde{M}z \geq 0, \quad z \geq 0,$$

has a solution.

(c) The system

$$M^{\mathrm{T}}u \leq 0, \quad q^{\mathrm{T}}u \leq 0, \quad 0 \neq u \geq 0 \tag{2}$$

has no solution $u \in R^n$.

(d) The solution set of the LCP (q, M) is nonempty and bounded.

(e) The system

$$q + Mz > 0, \quad z \geq 0 \tag{3}$$

has a solution.

(f) The system

$$Mv \geq 0, \quad q^{\mathrm{T}}v \leq 0, \quad (M + M^{\mathrm{T}})v = 0, \quad 0 \neq v \geq 0 \tag{4}$$

has no solution.

(g) There exists a scalar $\delta > 0$ such that the LCP $(\tilde{q}, M)$ is solvable for all $\tilde{q} \in R^n$ satisfying $\|q - \tilde{q}\| \leq \delta$.

(h) There exist scalars $\varepsilon > 0$ and $c > 0$ such that the LCP $(\tilde{q}, \tilde{M})$ is solvable for all $(\tilde{q}, \tilde{M}) \in R^n \times R^{n \times n}$ with $\tilde{M}$ copositive and satisfying $\|q - \tilde{q}\| + \|\tilde{M} - M\| \leq \varepsilon$; and furthermore, $\|\tilde{z}\| \leq c$ for all $\tilde{z}$ in $\mathrm{SOL}(\tilde{q}, \tilde{M})$.

Proof. We establish the equivalence of these statements by verifying a series of implications.

(b) $\Rightarrow$ (a). This is trivial.

(a) $\Rightarrow$ (c). Suppose the system in (c) has a solution u which, we may assume, satisfies $\|u\| = 1$. Let $\tilde{q} = q - \delta u$. Then, the existence of a solution to the system in (a) with this vector $\tilde{q}$ easily yields a contradiction to the assumption about u.

(c) $\Leftrightarrow$ (e). The two systems in (c) and (e) are dual of one another. Thus, the infeasibility of one is equivalent to the feasibility of the other.

(c) $\Leftrightarrow$ (f). Clearly, if (2) has no solution, then (4) can not have a solution. Conversely, if the former system has a solution u, then by copositivity, we have $u^{\mathrm{T}}M^{\mathrm{T}}u = 0$ which implies $(M + M^{\mathrm{T}})u = 0$ by copositivity-plus. Thus, (4) also has a solution.

(c) $\Rightarrow$ (h). The inconsistency of the system (2) implies the validity of the implication (1). Thus, (h) follows from Theorem **7.5.1**.

(g) $\Rightarrow$ (a). This is obvious.

(e) $\Rightarrow$ (b). If the system (3) has a solution, then that solution will satisfy the perturbed system

$$\tilde{q} + \tilde{M}z > 0, \quad z \geq 0$$

wherever the pair $(\tilde{q}, \tilde{M})$ is within a certain neighborhood of (q, M). Hence (b) follows.

[(h) $\Rightarrow$ (g)] and [(h) $\Rightarrow$ (d)]. These are obvious.

At this point, we have established the equivalence of the statements (a), (b), (c), (e), (f), (g) and (h).

Finally, to complete the proof, it suffices to verify (d) $\Rightarrow$ (f). We claim that if z solves (q, M) and v is a solution of the system (4), then the vector $z + \theta v$ is a solution of (q, M) for all $\theta \geq 0$. Clearly, $z + \theta v$ is feasible; moreover,

$$0 \leq (z + \theta v)^{\mathrm{T}}(q + Mz + \theta Mv) = \theta q^{\mathrm{T}} v \leq 0.$$

Hence, $z + \theta v \in \mathrm{SOL}(q, M)$ for all $\theta \geq 0$. Since v is nonzero, this contradicts the boundedness of the solution set of (q, M). Consequently, all eight statements (a) – (h) are equivalent. □

7.5.3 Remark. The statement (e) is noteworthy because it is equivalent to the strict feasibility condition of the LCP (q, M).

Solution rays

Among other things, Corollary **7.5.2** establishes that for a solvable LCP (q, M) with a copositive-plus matrix M, the boundedness of the solution set of (q, M) plays an important role in the stability of the problem; in turn, the latter boundedness property is equivalent to the non-existence of a vector u satisfying the system (2) (or (4)). Moreover, the proof of the implication [(d) $\Rightarrow$ (f)] in **7.5.2** shows that any such vector u, if it exists, has the property that for any solution z of (q, M), $z + \theta u$ remains a solution of (q, M) for all $\theta \geq 0$. This latter property motivates the following general definition.

7.5.4 Definition. The LCP (q, M) is said to have a *solution ray* at z with *generator* v if v is nonzero and $z + \theta v$ solves (q, M) for all $\theta \geq 0$. We say that a nonzero vector v *generates a solution ray* of (q, M) if (q, M) has a solution ray at some z with v as the generator.

It is not difficult to see that for a general LCP (q, M), $\mathrm{SOL}(q, M)$ is unbounded if and only if (q, M) has a solution ray; moreover, a nonzero vector v generates a solution ray for (q, M) at z if and only if

(i) z solves (q, M), and v solves $(0, M)$,

(ii) $v^{\mathrm{T}}(q + Mz) = 0$, and $z^{\mathrm{T}}Mv = 0$.

Geometrically, the LCP (q, M) has a solution ray if and only if q lies in a strongly degenerate complementary cone. . (The reader is asked to supply the proof of the above statements in Exercise **7.6.5**.) Thus, the generators of the solution rays of (q, M) are nonzero solutions of the homogeneous problem $(0, M)$. In particular, if $M \in \boldsymbol{R_0}$, then (q, M) has no solution ray for all $q \in R^n$. This conclusion is not surprising because according to Proposition **3.9.23**, such LCPs must have bounded solution sets, and hence, can not have solution rays.

In general, the generator v of a solution ray of (q, M) is dependent on the base vector z. The following example illustrates this point.

7.5.5 Example. Consider the LCP (q, M) with

$$q = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

The vector $v = (1, 0)$ is a generator of a solution ray of (q, M) at the zero solution; however, v is not a generator of a solution ray at the solution $(1, 3)$.

As noted in the preceding discussion, when the matrix M is copositive-plus, then a vector v generates a solution ray of the LCP (q, M) if v is a solution of the system (2) or (4); moreover, any member of $\mathrm{SOL}(q, M)$ can serve as the base solution z. Thus, any one of the conditions in **7.5.2** is necessary for the non-existence of a solution ray of (q, M) when M is copositive-plus. It turns out that these conditions are sufficient as well; moreover, a geometric interpretation can be given in terms of the cone $K(M)$.

7.5.6 Proposition. Let $M \in R^{n \times n}$ be copositive-plus and $q \in K(M)$ be arbitrary. The following statements are equivalent:

(a) $q \in \operatorname{int} K(M)$.

(b) LCP (q, M) has no solution ray.

(c) LCP (q, M) is strictly feasible.

(d) q is not contained in any strongly degenerate complementary cone.

Proof. (a) $\Rightarrow$ (b). Since a copositive-plus matrix is copositive-star, we have $K(M) = (\mathrm{SOL}(0, M))^*$ by Theorem **3.8.13**. Hence, assumption (a) implies $q \in \operatorname{int} S^*$ where $S = \mathrm{SOL}(0, M)$. By **7.5.1**, it follows that $\mathrm{SOL}(q, M)$ is bounded. Consequently, (q, M) cannot have a solution ray.

(b) $\Rightarrow$ (c). This has been noted in the above discussion.

(c) $\Rightarrow$ (a). The strict feasibility of (q, M) implies $q \in \operatorname{int}(\operatorname{pos}(I, -M))$. By **3.8.13** again, (a) follows readily.

Finally, the equivalence of the two statements (b) and (d) for a general LCP has been observed previously; see **7.6.5**. $\square$

7.5.7 Remark. The implication (a) $\Rightarrow$ (d) in **7.5.6**, proved here by invoking the perturbation result of Theorem **7.5.1**, was previously proved as Theorem **6.3.14** for the (broader) class of $\boldsymbol{L}$-matrices.

The concept of a solution ray is closely related to that of a recession direction (defined in Exercise **2.10.18**). Indeed, a vector v generates a solution ray of (q, M) if and only if $v \neq 0$ and $v \in 0^+ \mathrm{SOL}(q, M)$. Moreover, according to a characterization of a solution ray mentioned earlier, it is easy to see that (with $S = \mathrm{SOL}(0, M)$)

$$0^+ \mathrm{SOL}(q, M) = \bigcup_{z \in \mathrm{SOL}(q,M)} \{v \in R^n : v^{\mathrm{T}}(q + Mz) = z^{\mathrm{T}} M v = 0\} \cap S.$$

In the event that $\mathrm{SOL}(q, M)$ is convex (this holds if, for instance, M is a column sufficient matrix), the latter expression simplifies to

$$0^+ \mathrm{SOL}(q, M) = \{v \in R^n : v^{\mathrm{T}}(q + Mz) = z^{\mathrm{T}} M v = 0\} \cap \mathrm{SOL}(0, M)$$

for any $z \in \mathrm{SOL}(q, M)$.

To close this discussion, we return to a question left open by Theorem **7.5.1**. This question has to do with the consideration that whether under the stated assumption of q and an appropriate assumption of M, the perturbed LCP $(\tilde{q}, \tilde{M})$ will have a solution when the copositivity assumption on $\tilde{M}$ is removed. A partial answer to this question is provided by Theorem **6.4.4**. This result states that if $M \in \boldsymbol{L}$, then statement (d) in Corollary

7.5.2 implies the solvability of the perturbed LCP $(\tilde{q}, \tilde{M})$ as long as $(\tilde{q}, \tilde{M})$ is close enough to the given pair (q, M); there need be no restriction on $\tilde{M}$. Since this solvability property of the perturbed LCPs clearly implies statement (g) in **7.5.2**, it therefore can be added to the list of equivalent conditions for the stability of a copositive-plus LCP.

In the sequel, we establish a result that provides another affirmative answer to the question raised above in the case of a copositive matrix M. The proof of this result resembles that of Theorem **6.4.4** which uses Theorem **6.4.3**.

7.5.8 Theorem. Let $M \in R^{n\times n}$ be copositive and $S = \text{SOL}(0, M)$. Suppose that $q \in \text{int}\, S^*$. Then there exists a neighborhood $U \subseteq R^n \times R^{n\times n}$ of the pair (q, M) such that $\text{SOL}(q', M') \neq \emptyset$ for all $(q', M') \in U$.

Proof. Part (b) of Theorem **7.5.1** implies that $\text{SOL}(q, M)$ is bounded. Hence, (q, M) has no solution ray; equivalently, q is not contained in any strongly degenerate complementary cone. As we have pointed out previously, $R^n_{++} \subseteq \text{int}\, S^*$. Hence, similar to the proof of **6.4.4**, we can deduce that q lies in the closure of the set of all points with a well-defined nonzero local degree. The proof can be completed by invoking Theorem **6.4.3**. □

In Exercise **7.6.6**, the reader is asked to establish a generalization of Theorem **7.5.8** beyond the class of copositive matrices.

An application: quadratic programs

We discuss a specialization of the stability results **7.5.1** and **7.5.2** to the convex quadratic program (1.2.1):

$$\begin{aligned} \text{minimize} \quad & c^{\mathrm{T}}x + \tfrac{1}{2}x^{\mathrm{T}}Qx \\ \text{subject to} \quad & Ax \geq b \\ & x \geq 0 \end{aligned} \tag{5}$$

where $Q \in R^{n\times n}$ is symmetric positive semi-definite, $A \in R^{m\times n}$, $c \in R^n$ and $b \in R^m$. The specialization is based on the equivalent formulation of (5) as the LCP (q, M) with q and M given in (1.2.3):

$$q = \begin{bmatrix} c \\ -b \end{bmatrix}, \quad M = \begin{bmatrix} Q & -A^{\mathrm{T}} \\ A & 0 \end{bmatrix}. \tag{6}$$

In the result below, we use y to denote a vector of multipliers associated with the constraint $Ax \geq b$.

7.5.9 Proposition. Let $Q \in R^{n \times n}$ be symmetric positive semi-definite, and $A \in R^{m \times n}$, $c \in R^n$ and $b \in R^m$ all be arbitrary. The following statements are equivalent.

(a) The quadratic program (5) has an optimal solution, and the set of Karush-Kuhn-Tucker pairs (x, y) is bounded.

(b) The system

$$\begin{aligned} &Qu = 0, \quad Au \geq 0, \quad A^{\mathrm{T}}v \leq 0 \\ &c^{\mathrm{T}}u - b^{\mathrm{T}}v \leq 0, \quad 0 \neq (u, v) \geq 0 \end{aligned} \tag{7}$$

has no solution.

(c) There exist positive scalars ε and L such that for every $(\tilde{Q}, \tilde{A}, \tilde{c}, \tilde{b})$ satisfying

$$\|(\tilde{Q}, \tilde{A}, \tilde{c}, \tilde{b}) - (Q, A, c, b)\| \leq \varepsilon,$$

the associated quadratic program (5) with the data $(\tilde{Q}, \tilde{A}, \tilde{c}, \tilde{b})$ has a Karush-Kuhn-Tucker pair $(\tilde{x}, \tilde{y})$; moreover, if $\tilde{Q}$ is positive semi-definite, then, for any Karush-Kuhn-Tucker pair $(\tilde{x}, \tilde{y})$ of the perturbed quadratic program, there exists a corresponding pair (x, y) of Karush-Kuhn-Tucker vectors of the unperturbed program (5) such that

$$\|(\tilde{x}, \tilde{y}) - (x, y)\| \leq L\|(\tilde{Q}, \tilde{A}, \tilde{c}, \tilde{b}) - (Q, A, c, b)\|.$$

Proof. The matrix M in (6) is positive semi-definite, hence copositive-plus. Consequently, in light of Theorem **7.5.1** and Corollary **7.5.2** (see the discussion following **7.5.7**), it suffices to note that the system (7) is a specialization of (4) to the pair (q, M) given in (6). □

When the matrix Q is positive definite, any sufficiently small perturbation matrix $\tilde{Q}$ of Q will remain positive definite. In this case, the above proposition specializes to yield the following sharper result.

7.5.10 Corollary. Let $Q \in R^{n \times n}$ be symmetric positive definite, and let $A \in R^{m \times n}$, $c \in R^n$ and $b \in R^m$ all be arbitrary. The following statements are equivalent.

(a) The quadratic program (5) has a unique optimal solution x, and the set of optimal multiplier vectors y is bounded.

(b) The system

$$A^{\mathrm{T}}v \leq 0, \quad b^{\mathrm{T}}v \geq 0, \quad 0 \neq v \geq 0 \tag{8}$$

has no solution.

(b′) The system

$$Ax > b \quad x \geq 0 \tag{9}$$

has a solution.

(c) There exist positive scalars ε and L such that for every $(\tilde{Q}, \tilde{A}, \tilde{c}, \tilde{b})$ satisfying

$$\|(\tilde{Q}, \tilde{A}, \tilde{c}, \tilde{b}) - (Q, A, c, b)\| \leq \varepsilon,$$

the associated quadratic program (5) with the data $(\tilde{Q}, \tilde{A}, \tilde{c}, \tilde{b})$ has a unique optimal solution $\tilde{x}$, and for any optimal multiplier vector $\tilde{y}$ of the perturbed quadratic program, there exists a corresponding optimal multiplier vector y of the unperturbed program (5) such that

$$\|(\tilde{x}, \tilde{y}) - (x, y)\| \leq L\|(\tilde{Q}, \tilde{A}, \tilde{c}, \tilde{b}) - (Q, A, c, b)\|. \quad \square$$

7.5.11 Remark. We note that condition (9) is equivalent to the strict feasibility of the quadratic program (5).

In the last corollary, some necessary and sufficient conditions for the stability of the strictly convex quadratic program (5) are derived. These conditions involve either the optimal multiplier vectors or the strict feasibility of the program. In Exercise **7.6.10**, the reader is asked to establish a (global) Lipschitzian property of the (unique) optimal solution of (5) under no such assumptions.

7.6 Exercises

7.6.1 Let $M \in R^{n \times n} \cap \boldsymbol{R_0}$ and $q \in R^n$.

(a) Show that there exist a neighborhood U of the pair (q, M) and a constant $c > 0$ such that for all $(\bar{q}, \bar{M}) \in U$,

$$\|z\| \leq c \quad \text{for all } z \in \mathrm{SOL}(\bar{q}, \bar{M}).$$

(b) Deduce from (a) that there exist a neighborhood U of (q, M) and a constant $L > 0$ such that for all $(\bar{q}, \bar{M}) \in U$,

$$\mathrm{SOL}(\bar{q}, \bar{M}) \subseteq \mathrm{SOL}(q, M) + L(\|q - \bar{q}\| + \|M - \bar{M}\|)\mathcal{B}.$$

7.6.2 Verify that the solution z^* given in Example **7.3.4** is stable for the given LCP.

7.6.3 In the convergence analysis of the splitting algorithm **5.2.1**, the boundedness of the sequence $\{z^\nu\}$ and the limit condition

$$\lim_{\nu \to \infty} \|z^{\nu+1} - z^\nu\| = 0 \tag{1}$$

have played a major role. This exercise shows how these two properties are related under certain assumptions on the matrix B and the pair (q, M).

(a) Suppose $\mathrm{SOL}(q, M)$ is bounded. Show that if (1) holds, then $\{z^\nu\}$ is bounded, and every accumulation point of $\{z^\nu\}$ solves the LCP (q, M).

(b) Conversely, suppose $B \in \boldsymbol{P}$. Show that if $\{z^\nu\}$ is bounded and if every accumulation point of $\{z^\nu\}$ solves (q, M), then (1) holds.

7.6.4 State a result analogous to Lemma **7.3.10** for a mixed LCP. Then use this result to (i) complete the proof of Theorem **7.3.9**, and (ii) show that the directional derivative $z'(\varepsilon^*, \eta)$ in Theorem **7.4.2** is a Lipschitzian function in the direction η.

7.6.5 Let $q \in R^n$ and $M \in R^{n \times n}$ be given.

(a) Show that the following statements are equivalent:

(i) $\mathrm{SOL}(q, M)$ is unbounded,

(ii) (q, M) has a solution ray,

(iii) q belongs to a strongly degenerate complementary cone.

(b) Show that a nonzero vector v generates a solution ray for the (q, M) at z if and only if

(i) z solves (q, M) and v solves $(0, M)$

(ii) $v^{\mathrm{T}}(q + Mz) = z^{\mathrm{T}}Mv = 0$.

7.6.6 We say that $M \in R^{n\times n}$ is a $\boldsymbol{G}$-matrix if $\mathrm{SOL}(d, M) = \{0\}$ for some n-vector $d > 0$. Let $S = \mathrm{SOL}(0, M)$. A $\boldsymbol{G}$-matrix M is said to be $\boldsymbol{G}$*-sharp* if M satisfies the implication (4.11.1); i.e., if

$$x \in S \quad \Rightarrow \quad (M + M^{\mathrm{T}})x \geq 0. \tag{2}$$

(a) Show that every copositive matrix is $\boldsymbol{G}$-sharp.

(b) Consider the matrix

$$M = \begin{bmatrix} 0 & -1 & 2 \\ 2 & 1 & -4 \\ 2 & 1 & 1 \end{bmatrix}.$$

Show that M is not copositive and $M \notin \boldsymbol{R}_0 \cup \boldsymbol{P}_0$, but M is $\boldsymbol{G}$-sharp.

(c) Show that the conclusion of Theorem **7.5.8** remains valid if M is a $\boldsymbol{G}$-sharp matrix and $q \in \mathrm{int}\, S^*$.

(d) Prove or disprove: suppose that M is a $\boldsymbol{G}$-matrix and $q \in \mathrm{int}\, S^*$, then $\mathrm{SOL}(q, M) \neq \emptyset$.

(e) Suppose that M satisfies the implication (2) (but M is not necessarily a $\boldsymbol{G}$-matrix). Let W be an arbitrary bounded subset of R^n. Show that if $q \in \mathrm{int}\, S^*$, then there exists a neighborhood V of q such that the set

$$\{(z, w) \in R^n_+ \times R^n_+ : w - Mz \in V, z * w \in W\}$$

(where $*$ denotes the Hadamard product), is bounded. Deduce that $\mathrm{SOL}(q, M)$ is bounded, if it is nonempty.

7.6.7 Let $M \in R^{n\times n}$ be positive semi-definite and $q \in R^n$ be arbitrary. Let $x \in \mathrm{FEA}(q, M)$. Show that (a) if i is an index such that $(q + Mx)_i > 0$, then for any solution z of (q, M),

$$z_i \leq \frac{x^{\mathrm{T}}(q + Mx)}{(q + Mx)_i},$$

and (b) if i is an index such that $x_i > 0$, then

$$w_i \leq \frac{x^{\mathrm{T}}(q + Mx)}{x_i}$$

for any solution z of (q, M) with $w = q + Mz$.

7.6.8 Complete the proof of part (d) in Theorem **7.4.2**.

7.6.9 The fact that the directional derivative $z'(\varepsilon^*, \varepsilon - \varepsilon^*)$ satisfies the limit condition (7.4.2) is not surprising. Show that in general, if the function $f : \mathcal{D} \to R^m$ where $\mathcal{D}$ is an open subset of R^n, is directionally differentiable and locally Lipschitzian at a point $x \in \mathcal{D}$, then the limit condition below must hold:

$$\lim_{y \to x} \frac{f(y) - f(x) - f'(x, y - x)}{\|y - x\|} = 0.$$

7.6.10 Consider the quadratic program (7.5.5) where the matrix Q is assumed symmetric positive definite. Let $x(b, c)$ denote the unique solution of (7.5.5) if it exists. Show that there exists a constant $L > 0$ such that for any two pairs of vectors, (b, c) and (b', c'),

$$\|x(b, c) - x(b', c')\| \le L\|(b, c) - (b', c')\|.$$

7.6.11 Consider an LCP of the form (Pc, PAP^{T}) where $P \in R^{m \times n}$, $A \in R^{n \times n}$ is positive definite (but asymmetric) and $c \in R^n$. Let $w(c)$ denote the unique w-variable of any solution (w, z) of this problem. Show that there exists a constant $L > 0$ such that for all vectors $c, c' \in R^n$,

$$\|w(c) - w(c')\| \le L\|c - c'\|.$$

7.6.12 Let $\{z^\nu\}$ be a sequence of vectors in R^n and $\{a_\nu\}$ be a sequence of nonnegative scalars. Suppose that there exist two constants $\gamma > 0$ and $\rho \in (0, 1)$ such that for all ν sufficiently large,

$$\|z^\nu - z^{\nu+1}\| \le \gamma(a_\nu - a_{\nu+1}) \quad \text{and} \quad a_{\nu+1} \le \rho a_\nu.$$

Show that the sequence $\{z^\nu\}$ converges; moreover, if z^* denotes its limit, then there exist constants $c_1 > 0$ and $c_2 \in (0, 1)$ such that for all ν sufficiently large

$$\|z^\nu - z^*\| \le c_1(c_2)^\nu.$$

7.6.13 Consider the sequence of scalars $\{a_\nu\}$ defined by

$$a_\nu = \begin{cases} (1/8)^\nu & \nu \text{ even} \\ (1/4)^\nu & \nu \text{ odd.} \end{cases}$$

Show that for this sequence, there exists a scalar $c_2 \in (0,1)$ such that for all $\nu \geq 0$,

$$a_\nu \leq (c_2)^\nu$$

but the subsequence $\{a_{\nu+1}/a_\nu : \nu \text{ even}\}$ is unbounded. Conversely, consider the sequence $\{a'_\nu\}$ defined by

$$a'_\nu = 1/\nu.$$

Show that for this latter sequence, the sequence $\{a'_{\nu+1}/a'_\nu\}$ is bounded, but there can not exist a scalar $c_2 \in (0,1)$ such that for all ν sufficiently large,

$$a'_\nu \leq (c_2)^\nu.$$

7.7 Notes and References

7.7.1 The systematic study of sensitivity analysis in nonlinear programming starts with the seminal work of Fiacco and McCormick (1968) in which the classical implicit function theorem was used to obtain the first perturbation properties of parametric nonlinear programs. This subject has since developed into a very fruitful area of research within mathematical programming. A large body of literature is available, and several texts have been written, with Bank, Guddat, Klatte, Kummer and Tammer (1983) and Fiacco (1983) being two of the more recent additions. The former text contains some discussion of the sensitivity issues pertaining specifically to the LCP.

7.7.2 In the study of sensitivity analysis of nonlinear programming and its extensions, S. M. Robinson has made numerous influencial contributions. The most significant departure of Robinson's research from the classic results of Fiacco and McCormick is that the assumption of strict complementary slackness is removed. This is a major contribution because with this assumption in place, sensitivity analysis of nonlinear programming (including complementarity problems) can typically be carried out by a straightforward application of the implicit function theorem, whereas in the absence of strict complementarity, such a routine exercise is no longer feasible and new techniques need to be introduced for the analysis. Robinson's work has opened up a new avenue for the subject area that allows general results to be derived under very mild assumptions.

7.7.3 Although much of Robinson's work is not directly cast in the framework of the LCP, many of the sensitivity results presented in this chapter are derived as special cases of his discoveries. In particular, Theorem **7.2.1** is a specialization of an upper Lipschitzian property of general polyhedral multifunctions obtained in Robinson (1981). Lemma **7.2.2** on which the proof of this theorem is based is drawn from Walkup and Wets (1969).

7.7.4 Throughout the study of the sensitivity and stability of a solution of the LCP, the Schur complement N defined by the expression (7.2.4) has played a major role. It appears that Robinson (1980a) and Mangasarian (1980) are the earliest references to have identified the importance of this matrix N in the local analysis of the LCP.

7.7.5 Several of the convergence results of the splitting methods described in Section 7.2 appeared in the literature only recently. The implication (a) $\Rightarrow$ (b) in Theorem **7.2.10** was obtained by Iusem (1990a) and Tseng and Luo (1990). The first conclusion in Corollary **7.2.12** was also obtained in the latter reference.

7.7.6 Definition **7.3.1** is due to Ha (1985) who established the characterization of a strongly stable solution in Theorem **7.3.7**. Although Ha obtained a number of useful results pertaining to a stable solution, (among these is Example **7.3.4**) he fell short of establishing the complete characterization in Theorem **7.3.2**; the latter result was obtained in Gowda and Pang (1992a) who relied heavily on Ha's development. In the related paper of Jansen and Tijs (1987), the concept of a *robust solution* of the LCP was introduced as a refinement of that of a stable solution.

7.7.7 In Theorem **7.3.9**, the Lipschitzian property of the solution function $z(\cdot,\cdot)$ is established as a consequence of the strong stability of the solution z^*. In Robinson's original definition of a strongly regular solution of a generalized equation (whose connection with a strongly stable solution is demonstrated in Theorem **7.3.7**), this Lipschitzian property is part of the requirement (see Robinson (1980a)). The fact that this requirement can be waived in the definition of a strongly stable solution in the case of the LCP has a great deal to do with the polyhedrality structure of this problem. Lemma **7.3.10** on which **7.3.9** is based, was obtained in Mangasarian and Shiau (1987) by a different approach.

7.7.8 Beginning with the work of Robinson (1985) and Kyparisis (1986), several authors have made important contributions concerning the solution differentiability of the parametric nonlinear complementarity and variational inequality problems; see Kyparisis (1988, 1990), Qiu and Magnanti (1989, 1992), and Pang (1988, 1990b). Our presentation in Section 7.4 is largely based on the work of Pang (1990b). The latter reference also provides a demonstration of the feasibility of continuation of Newton's method for solving the linearly constrained variational inequality problem that includes the nonlinear complementarity problem as a special case.

7.7.9 In essence, parts (a) and (b) of Theorem **7.4.2** establishes that the solution function $z(\varepsilon)$ is B-differentiable at ε^*. In general, a function f satisfying the assumption in Exercise **7.6.9** is said to be *B(ouligand)-differentiable* at the point $x \in \mathcal{D}$. This terminology was introduced in Robinson (1987). Shapiro (1990) has studied the relationships between various concepts of directional differentiability; in particular, the result in **7.6.9** is due to him.

7.7.10 Using Robinson's notion of a strongly regular solution, Josephy (1979a, 1979b) was the first person to establish the convergence of Newton's method and the quasi-Newton methods for solving a generalized equation. Inexact versions of these methods for the nonlinear complementarity and variational inequality problems were described in Pang (1986b). The latter reference also contains an extension of Proposition **7.3.14** to the nonlinear complementarity problem.

7.7.11 The stability analysis of the LCP under the assumption of a copositive matrix is the generalization of the stability study of linear and convex quadratic programs. Robinson (1977) established a characterization of stability in linear programming; Daniel (1973b) and Guddat (1976) studied the stability of convex quadratic programs. Robinson (1979) extended all these results to the context of a generalized equation and discussed the specialization to the linear case. (Incidentally, the linear generalized equation is essentially equivalent to a positive semi-definite LCP of a particular kind). Doverspike (1982) established some perturbation results for the copositive-plus LCP. Mangasarian (1982) extended Doverspike's results and derived a long list of equivalent conditions for the stability of such an LCP. Most of these conditions are stated in Corollary **7.5.2**.

7.7.12 Our discussion of solution rays of the LCP is based on Cottle (1974b). This study was inspired by some questions arising from structural mechanics that were raised in a private communication by G. Maier to Cottle in October, 1973.

7.7.13 As it is evident from Corollary **7.5.2**, the stability of a copositive-plus LCP is intimately related to the boundedness of the solution set of the problem. Mangasarian (1985) derived some interesting bounds for the solutions of a positive semi-definite LCP and applied these results to a linear program. Exercise **7.6.7** pertains to a result of this kind. Mangasarian and McLinden (1985) extended the work of Mangasarian (1985) to the nonlinear case and to convex programs. In a related work, Kojima, Mizuno and Yoshise (1990) investigate properties of ellipsoids that contain all solutions of a positive semi-defininte LCP with bounded solution set. The latter investigation is motivated by the family of interior-point methods discussed in Section 5.9.

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