Zero-Sum Games and Linear Programming Duality

Bernhard von Stengel

Department of Mathematics
London School of Economics

John von Neumann (1903–1957)

- set theory
- mathematics of quantum mechanics
- minimax theorem [1928], game theory
- stored-program computer
- self-replicating automata

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from *The Man from the Future (2021):*

"Von Neumann would carry on a conversation with my three-year-old son, and the two of them would talk as equals, and I sometimes wondered if he used the same principle when he talked to the rest of us." Edward Teller, 1966

3 October 1947: Dantzig meets von Neumann

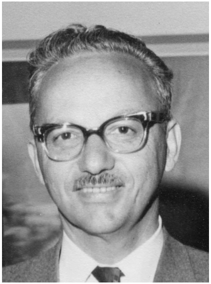
GD: In under one minute I slapped on the blackboard a geometric and algebraic version of the linear programming problem.

Von Neumann stood up and said, “Oh, that!”

[gives eye-popping lecture on LP duality]

JvN: . . . I have recently completed a book with Oskar Morgenstern on the theory of games. I conjecture that the two problems are equivalent.

GD: Thus I learned about Farkas’s Lemma and about duality for the first time.



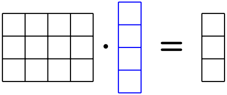
George Dantzig (1914–2005)

Notation, treat vectors and scalars as matrices

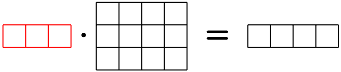
All vectors are column vectors. $\boldsymbol{A}^\top$ = matrix $\boldsymbol{A}$ transposed.

$\mathbf{0} = (0, \ldots, 0)^\top, \quad \mathbf{1} = (1, \ldots, 1)^\top.$

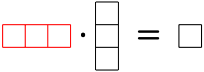
$\boldsymbol{A}\boldsymbol{x}$ = linear combination of columns of $\boldsymbol{A}$



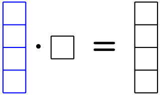
$\boldsymbol{y}^\top\boldsymbol{A}$ = linear combination of rows of $\boldsymbol{A}$



$\boldsymbol{y}^\top\boldsymbol{b}$ = scalar product of $\boldsymbol{y}$ and $\boldsymbol{b}$



$\boldsymbol{x}\alpha$ = (column) vector $\boldsymbol{x}$ scaled by α



$\alpha\boldsymbol{y}^\top$ = row vector $\boldsymbol{y}$ scaled by α

Primal and dual linear programs

Primal LP:

$$\begin{aligned} \textbf{maximize} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\,, \\ & \boldsymbol{x} \geq \boldsymbol{0}\,. \end{aligned}$$

Dual LP:

$$\begin{aligned} \textbf{minimize} \quad & \boldsymbol{y}^\top \boldsymbol{b} \\ \text{subject to} \quad & \boldsymbol{y} \geq \boldsymbol{0}\,, \\ & \boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{c}^\top. \end{aligned}$$

Primal and dual linear programs

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Weak LP duality: For any **feasible** primal $\boldsymbol{x}$, dual $\boldsymbol{y}$:

$$\boldsymbol{c}^\top \boldsymbol{x} \leq \boldsymbol{y}^\top \boldsymbol{b}$$

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$$(\boldsymbol{c}^\top)\boldsymbol{x} \le (\boldsymbol{y}^\top \boldsymbol{A})\boldsymbol{x} = \boldsymbol{y}^\top(\boldsymbol{A}\boldsymbol{x}) \le \boldsymbol{y}^\top(\boldsymbol{b})$$

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Strong LP duality: If both primal and dual LP are feasible, then they have (optimal) solutions $\boldsymbol{x}$ and $\boldsymbol{y}$ with $\boldsymbol{c}^\top \boldsymbol{x} = \boldsymbol{y}^\top \boldsymbol{b}$.

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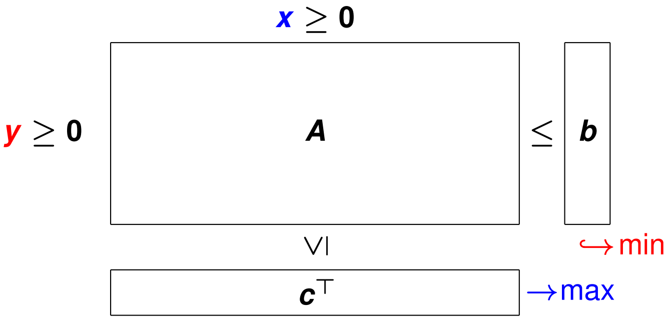
Tucker diagram

Primal LP: **maximize** $\mathbf{c}^\top \mathbf{x}$ subject to $\mathbf{A}\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}.$

Dual LP: **minimize** $\mathbf{y}^\top \mathbf{b}$ subject to $\mathbf{y}^\top \mathbf{A} \geq \mathbf{c}^\top, \quad \mathbf{y} \geq \mathbf{0}.$



LP duality proved with Lemma of Farkas [1902]

Equalities with nonnegative variables

$$\nexists\, \boldsymbol{x} \;:\; \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\,,\;\; \boldsymbol{x} \geq \mathbf{0} \quad\Leftrightarrow\quad \exists\, \boldsymbol{y} \;:\; \boldsymbol{y}^\top \boldsymbol{A} \geq \mathbf{0}^\top,\;\; \boldsymbol{y}^\top \boldsymbol{b} < 0$$

Inequalities with nonnegative variables

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Inequalities only, get $\mathbf{0} \leq -\mathbf{1}$ from infeasible $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$:

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Lemma of Farkas [1902]

IV. Grundsatz der einfachen Ungleichungen.

Es sei

$$(1.)\qquad \begin{cases} A_{11}u_1 + A_{12}u_2 + \cdots + A_{1n}u_n \equiv \Theta_1 \geqq 0, \\ A_{21}u_1 + A_{22}u_2 + \cdots + A_{2n}u_n \equiv \Theta_2 \geqq 0, \\ \dots\dots\dots\dots \end{cases}$$

das gegebene System von Ungleichungen, und in jeder Lösung desselben möge

$$(2.)\qquad A_1u_1 + A_2u_2 + \cdots + A_nu_n \equiv \vartheta \geqq 0$$

bestehen.

Es giebt immer solche nicht-negativen, von den Variablen u unabhängigen Multiplicatoren λ, dass

$$(3.)\qquad \vartheta \equiv \lambda_1\Theta_1 + \lambda_2\Theta_2 + \cdots$$

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This is the equality version of the Lemma in dual form:

$$\forall \boldsymbol{x}\ \ (\boldsymbol{A}\boldsymbol{x} \geq \mathbf{0} \ \Rightarrow\ \boldsymbol{c}^\top\boldsymbol{x} \geq \mathbf{0}) \quad \Rightarrow \quad \exists\, \boldsymbol{y} \geq \mathbf{0} \ :\ \boldsymbol{y}^\top\boldsymbol{A} = \boldsymbol{c}^\top$$

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$$\nexists\, \boldsymbol{y} \geq \boldsymbol{0} \ :\ \boldsymbol{y}^\top \boldsymbol{A} = \boldsymbol{c}^\top \quad \Rightarrow \quad \exists\, \boldsymbol{x} \ :\ \boldsymbol{Ax} \geq \boldsymbol{0},\ \ \boldsymbol{c}^\top \boldsymbol{x} < \boldsymbol{0}$$

Zero-sum games

Game matrix $A \in \mathbb{R}^{m \times n}$

maximizing row player chooses row $i \in [m] = \{1, \ldots, m\}$

minimizing column player chooses column $j \in [n] = \{1, \ldots, n\}$

payoff a_{ij} to row player = cost to column player

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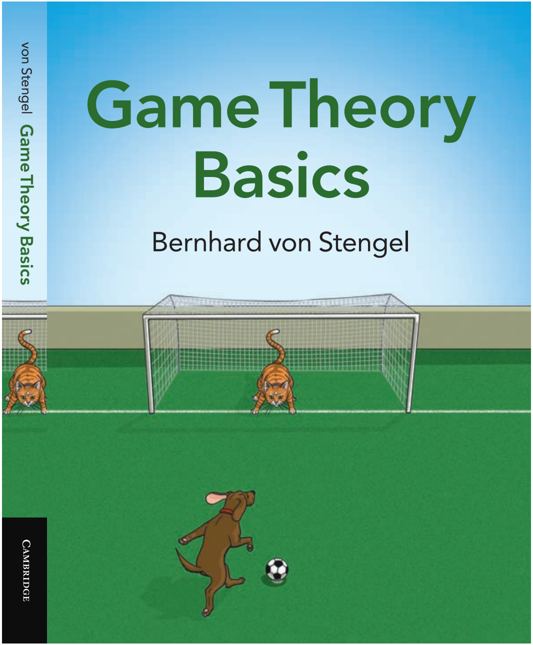
Mixed-strategy sets

$$Y = \{y \in \mathbb{R}^m \mid y \geq 0,\ 1^\top y = 1\},$$
$$X = \{x \in \mathbb{R}^n \mid x \geq 0,\ 1^\top x = 1\},$$

expected payoff / cost: $y^\top A x$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Best responses

Let $\boldsymbol{x} \in \boldsymbol{X}$. $(\boldsymbol{A}\boldsymbol{x})_{\boldsymbol{i}}$ = expected payoff in row $\boldsymbol{i}$.

A **best response** $\boldsymbol{y} \in \boldsymbol{Y}$ to $\boldsymbol{x}$ maximizes $\boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{x}$.

$$\begin{aligned}
& \max\{\boldsymbol{y}^\top(\boldsymbol{A}\boldsymbol{x}) \mid \boldsymbol{y} \in \boldsymbol{Y}\} \\
= \; & \max\{(\boldsymbol{A}\boldsymbol{x})_1, \ldots, (\boldsymbol{A}\boldsymbol{x})_m\} \\
= \; & \min\{\boldsymbol{v} \in \mathbb{R} \mid (\boldsymbol{A}\boldsymbol{x})_1 \le \boldsymbol{v}, \ldots, (\boldsymbol{A}\boldsymbol{x})_m \le \boldsymbol{v}\} \\
= \; & \min\{\boldsymbol{v} \in \mathbb{R} \mid \boldsymbol{A}\boldsymbol{x} \le \mathbf{1}\boldsymbol{v}\}
\end{aligned}$$

max-min and min-max strategies

min-max strategy $\hat{x} \in X$:

$$\max_{y \in Y} y^\top A \hat{x} = \min_{x \in X} \max_{y \in Y} y^\top A x$$

$$= \min_{x \in X} \{ v \in \mathbb{R} \mid Ax \leq \mathbf{1} v \}$$

max-min strategy $\hat{y} \in Y$:

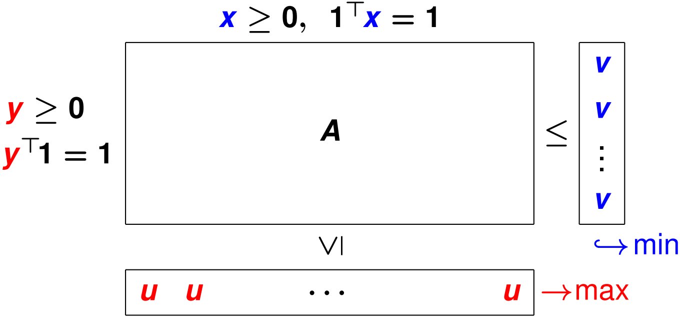
$$\min_{x \in X} \hat{y}^\top A x = \max_{y \in Y} \min_{x \in X} y^\top A x$$

$$= \max_{y \in Y} \{ u \in \mathbb{R} \mid y^\top A \geq u \mathbf{1}^\top \}$$

Written as general LP

Minimizer: **minimize** $\boldsymbol{v}$ subject to $\boldsymbol{Ax} \leq \mathbf{1}\boldsymbol{v}$, $\boldsymbol{x} \in \boldsymbol{X}$.

Maximizer: **maximize** $\boldsymbol{u}$ subject to $\boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{u}\mathbf{1}^\top$, $\boldsymbol{y} \in \boldsymbol{Y}$.



Written as general LP

Minimizer: **minimize** v subject to $Ax \leq \mathbf{1}v$, $x \in X$.

Maximizer: **maximize** u subject to $y^\top A \geq u\mathbf{1}^\top$, $y \in Y$.

	$x \geq \mathbf{0}$			v			
$y \geq \mathbf{0}$	A			-1 ⋮ -1	$\leq$	0 ⋮ 0	
u	-1	$\cdots$	-1	0	$=$	-1	
		$\geq$		$=$		$\hookrightarrow$ min	
	0	$\cdots$	0	-1	$\rightarrow$ max		

von Neumann's minimax theorem

Every zero-sum game A has a **value** v :

$$\boxed{\max_{y \in Y} \min_{x \in X} y^\top A x = v = \min_{x \in X} \max_{y \in Y} y^\top A x}$$

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also, with max-min strategy $\hat{\boldsymbol{y}}$ and min-max strategy $\hat{\boldsymbol{x}}$:

$$\min_{\boldsymbol{x}\in \boldsymbol{X}} \hat{\boldsymbol{y}}^\top \boldsymbol{A}\boldsymbol{x} = \hat{\boldsymbol{y}}^\top \boldsymbol{A}\hat{\boldsymbol{x}} = \max_{\boldsymbol{y}\in \boldsymbol{Y}} \boldsymbol{y}^\top \boldsymbol{A}\hat{\boldsymbol{x}}$$

$$\Leftrightarrow \quad \forall \boldsymbol{x} \in \boldsymbol{X},\ \boldsymbol{y} \in \boldsymbol{Y} : \quad \hat{\boldsymbol{y}}^\top \boldsymbol{A}\boldsymbol{x} \geq \hat{\boldsymbol{y}}^\top \boldsymbol{A}\hat{\boldsymbol{x}} \geq \boldsymbol{y}^\top \boldsymbol{A}\hat{\boldsymbol{x}}$$

$\Leftrightarrow$ $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{x}})$ is a **Nash equilibrium** (exists via fixed point theorem).

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The minimax theorem is a consequence of strong LP duality.

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What about the converse?

Dantzig's game [1951]

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} & -\boldsymbol{b} \\ -\boldsymbol{A}^\top & \boldsymbol{0} & \boldsymbol{c} \\ \boldsymbol{b}^\top & -\boldsymbol{c}^\top & 0 \end{bmatrix}$$

Dantzig's game [1951]

$$B = \begin{bmatrix} 0 & A & -b \\ -A^\top & 0 & c \\ b^\top & -c^\top & 0 \end{bmatrix}$$

$B = -B^\top \;\Rightarrow\;$ symmetric game with value $\mathbf{0}$ (by minimax theorem),

$\exists$ optimal $z = (y, x, t) \geq \mathbf{0}$ with $\boxed{Bz \leq \mathbf{0} \text{ and } z^\top B \geq \mathbf{0}^\top}$:

$$Ax - bt \leq \mathbf{0}\,, \qquad -A^\top y + ct \leq \mathbf{0}\,, \qquad b^\top y - c^\top x \leq 0\,.$$

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If $t > 0$: $x\frac{1}{t}$ primal optimal and $y\frac{1}{t}$ dual optimal.

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If $t > 0$: $x\frac{1}{t}$ primal optimal and $y\frac{1}{t}$ dual optimal.

If $t = 0$ $\boxed{\text{and}}$ $b^\top y < c^\top x$ then $b^\top y < 0$ or $0 < c^\top x$

(otherwise $b^\top y \geq 0 \geq c^\top x$), and $Ax \leq \mathbf{0}$ and $y^\top A \geq \mathbf{0}^\top$.

Unbounded rays

Suppose for some $\bar{\boldsymbol{x}}$:

$$\boldsymbol{A}\bar{\boldsymbol{x}} \leq \boldsymbol{b}\,, \quad \bar{\boldsymbol{x}} \geq \boldsymbol{0}\,,$$

and $\boldsymbol{0} < \boldsymbol{c}^\top \boldsymbol{x}$, $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{0}$ for some $\boldsymbol{x} \geq \boldsymbol{0}$.

Then $\boldsymbol{A}(\bar{\boldsymbol{x}} + \boldsymbol{x}\alpha) \leq \boldsymbol{b}$, $\bar{\boldsymbol{x}} + \boldsymbol{x}\alpha \geq \boldsymbol{0}$,

$$\boldsymbol{c}^\top(\bar{\boldsymbol{x}} + \boldsymbol{x}\alpha) \;=\; \boldsymbol{c}^\top\bar{\boldsymbol{x}} + (\boldsymbol{c}^\top\boldsymbol{x})\alpha \to \infty$$

as $\alpha \to \infty$.

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as $\alpha \to \infty$. $\Rightarrow$ (by weak duality): dual LP infeasible.

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$\Rightarrow$ **Strong LP duality theorem**

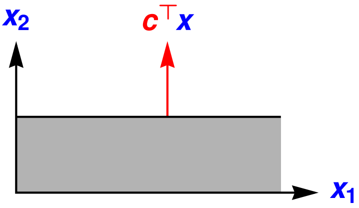
Either primal and dual LP are feasible and then have optimal solutions with equal objective functions,

or **(infeasibility certificate)** at least one LP is infeasible and the other (if feasible) is unbounded (with an unbounded ray).

But what if $t = 0$ and $b^\top y = c^\top x$?

Example

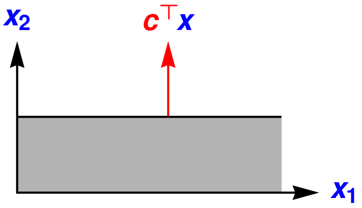
maximize		x_2
subject to		$x_2 \le 1$
	x_1 ,	$x_2 \ge 0$



But what if $t = 0$ and $b^\top y = c^\top x$?

Example

$$\begin{aligned} \textbf{maximize} \quad & x_2 \\ \text{subject to} \quad & x_2 \leq 1 \\ & x_1 \,,\; x_2 \geq 0 \end{aligned}$$



$$B = \begin{bmatrix} 0 & A & -b \\ -A^\top & 0 & c \\ b^\top & -c^\top & 0 \end{bmatrix}$$

$$\begin{array}{cccc} y_1 & x_1 & x_2 & t \end{array}$$

$$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{array}{c} \leq 0 \\ \leq 0 \\ \leq 0 \\ \leq 0 \end{array}$$

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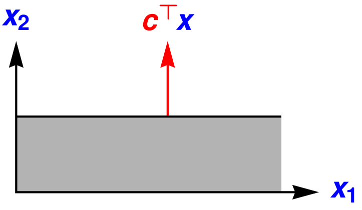

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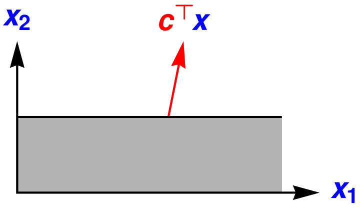
But what if $t = 0$ and $b^\top y = c^\top x$?

Example

$$\begin{aligned} \textbf{maximize} \quad & \varepsilon x_1 + x_2 \\ \text{subject to} \quad & x_2 \le 1 \\ & x_1, \; x_2 \ge 0 \end{aligned}$$



$$B = \begin{bmatrix} 0 & A & -b \\ -A^\top & 0 & c \\ b^\top & -c^\top & 0 \end{bmatrix}$$

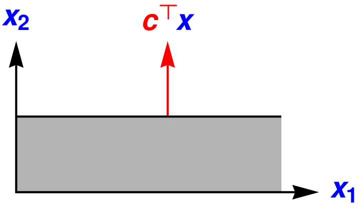
$$\begin{array}{cccc} y_1 & x_1 & x_2 & t \\ 0 & 1 & 0 & 0 \end{array}$$

$$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \varepsilon \\ -1 & 0 & 0 & 1 \\ 1 & -\varepsilon & -1 & 0 \end{bmatrix} \begin{array}{l} = 0 \\ = 0 \\ = 0 \\ < 0 \end{array}$$

But what if $t = 0$ and $b^\top y = c^\top x$?

Example

$$\begin{array}{lrl} \textbf{maximize} & x_2 & \\ \text{subject to} & x_2 & \le 1 \\ & x_1 \;,\quad x_2 & \ge 0 \end{array}$$



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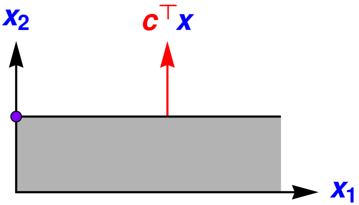

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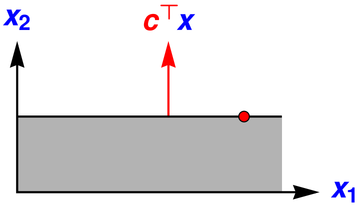

$$B = \begin{bmatrix} 0 & A & -b \\ -A^\top & 0 & c \\ b^\top & -c^\top & 0 \end{bmatrix}$$

$$\begin{array}{cccc} y_1 & x_1 & x_2 & t \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} & \frac{1}{6} \end{array}$$

$$\begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{array}{l} = 0 \\ = 0 \\ = 0 \\ = 0 \end{array}$$

If $\boldsymbol{t} = \mathbf{0}$ $\boxed{\text{and}}$ $\boldsymbol{b}^\top \boldsymbol{y} = \boldsymbol{c}^\top \boldsymbol{x}$ then

Dantzig's game gives no information about the LP!

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{0} & \boldsymbol{A} & -\boldsymbol{b} \\ -\boldsymbol{A}^\top & \mathbf{0} & \boldsymbol{c} \\ \boldsymbol{b}^\top & -\boldsymbol{c}^\top & 0 \end{bmatrix}$$

This means an unused best response and thus violates **strict complementarity.** This only occurs in degenerate cases.

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want $\quad \exists\, \boldsymbol{z} \geq \mathbf{0},\ \ \boldsymbol{B}\boldsymbol{z} \leq \mathbf{0},\ \ z_k - (\boldsymbol{B}\boldsymbol{z})_k > 0$

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$$\boldsymbol{B} = -\boldsymbol{B}^\top \in \mathbb{R}^{\boldsymbol{k}\times\boldsymbol{k}} \quad \Rightarrow \quad \exists\, \boldsymbol{z} \geq \mathbf{0},\ \boldsymbol{Bz} \leq \mathbf{0},\ \boxed{\boldsymbol{z_k} - (\boldsymbol{Bz})_{\boldsymbol{k}} > \mathbf{0}}\,.$$

Tucker's Lemma [1956]

$$\boldsymbol{B} = -\boldsymbol{B}^\top \in \mathbb{R}^{k\times k} \quad \Rightarrow \quad \exists\, \boldsymbol{z} \geq \mathbf{0},\ \boldsymbol{Bz} \leq \mathbf{0},\ \boxed{z_k - (\boldsymbol{Bz})_k > 0}\,.$$

Applied to

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{0} & \boldsymbol{A} & -\boldsymbol{b} \\ -\boldsymbol{A}^\top & \mathbf{0} & \mathbf{0} \\ \boldsymbol{b}^\top & \mathbf{0}^\top & 0 \end{bmatrix}$$

$$\Rightarrow \quad \exists\ \boldsymbol{z} = (\boldsymbol{y}, \boldsymbol{x}, t) \geq \mathbf{0} \ \text{ with}$$

$$\boldsymbol{Ax} - \boldsymbol{b}t \leq \mathbf{0}, \quad -\boldsymbol{A}^\top \boldsymbol{y} \leq \mathbf{0}, \quad \boldsymbol{b}^\top \boldsymbol{y} \leq 0, \quad \boxed{t - \boldsymbol{b}^\top \boldsymbol{y} > 0}\,.$$

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$$B = -B^\top \in \mathbb{R}^{k \times k} \quad \Rightarrow \quad \exists\, z \geq 0,\ Bz \leq 0,\ \boxed{z_k - (Bz)_k > 0}.$$

Applied to

$$B = \begin{bmatrix} 0 & A & -b \\ -A^\top & 0 & 0 \\ b^\top & 0^\top & 0 \end{bmatrix}$$

$$\Rightarrow \quad \exists\ z = (y, x, t) \geq 0 \quad \text{with}$$

$$Ax - bt \leq 0\,, \quad -A^\top y \leq 0\,, \quad b^\top y \leq 0\,, \quad \boxed{t - b^\top y > 0}.$$

$$\Rightarrow \quad \text{if } t = 0 : \quad \exists\, y \geq 0,\ \ y^\top A \geq 0^\top,\ \ y^\top b < 0$$

$$\text{if } t > 0 : \quad \exists\, x\tfrac{1}{t} \geq 0,\ \ Ax\tfrac{1}{t} \leq b$$

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= Lemma of Farkas!

Variants of Tucker's Lemma [1956]

For $\boldsymbol{B} = -\boldsymbol{B}^\top \in \mathbb{R}^{\boldsymbol{k}\times\boldsymbol{k}}$, $\boldsymbol{A} \in \mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}}$:

$$\exists\, \boldsymbol{z} \geq \mathbf{0}\,,\quad \boldsymbol{Bz} \leq \mathbf{0}\,,\quad \boldsymbol{z_k} - (\boldsymbol{Bz})_{\boldsymbol{k}} > \mathbf{0}$$

$$\exists\, \boldsymbol{x} \geq \mathbf{0}\,,\ \boldsymbol{y} \geq \mathbf{0}\ \ :\ \ \boldsymbol{y}^\top\boldsymbol{A} \geq \mathbf{0}^\top,\quad \boldsymbol{Ax} \leq \mathbf{0}\,,\quad \boldsymbol{x_n} + (\boldsymbol{y}^\top\boldsymbol{A})_{\boldsymbol{n}} > \mathbf{0}$$

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$$\Downarrow: \quad \boldsymbol{B} = \begin{bmatrix} 0 & \boldsymbol{A} \\ -\boldsymbol{A}^\top & 0 \end{bmatrix}, \boldsymbol{z} = \binom{\boldsymbol{y}}{\boldsymbol{x}}. \qquad \Uparrow: \quad \boldsymbol{B} = \boldsymbol{A}\,, \ \boldsymbol{z} = \boldsymbol{y} + \boldsymbol{x}$$

$$\exists\, \boldsymbol{x} \geq \boldsymbol{0}\,, \ \boldsymbol{y} \geq \boldsymbol{0} \ : \ \boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{0}^\top, \quad \boldsymbol{Ax} \leq \boldsymbol{0}\,, \quad x_n + (\boldsymbol{y}^\top \boldsymbol{A})_n > 0$$

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$$\exists\, z \geq 0\,, \quad Bz \leq 0\,, \quad z_k - (Bz)_k > 0$$

$$\exists\, x \geq 0\,,\; y \geq 0 \quad : \quad y^\top A \geq 0^\top, \quad Ax \leq 0\,, \quad x_n + (y^\top A)_n > 0$$

$$\Downarrow : \quad Ax \leq 0\,, \quad -Ax \leq 0 \qquad\qquad \Uparrow : \quad I_{m \times m} s + Ax = 0$$

$$\exists\, x \geq 0\,,\; y \quad : \quad y^\top A \geq 0^\top, \quad Ax = 0\,, \quad x_n + (y^\top A)_n > 0$$

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Lemma of Farkas $\Rightarrow$ Lemma of Tucker

Lemma of Farkas :

$$\nexists\, \boldsymbol{x} \geq \mathbf{0} \;:\; \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \quad \Leftrightarrow \quad \exists\, \boldsymbol{y} \;:\; \boldsymbol{y}^\top \boldsymbol{A} \geq \mathbf{0}^\top, \;\; \boldsymbol{y}^\top \boldsymbol{b} < 0\,.$$

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$\boldsymbol{A} = [\boldsymbol{A}_1 \cdots \boldsymbol{A}_n]$:

either $\exists\, \boldsymbol{z} \in \mathbb{R}^{n-1} : \boldsymbol{z} \geq \mathbf{0}, \;\; \sum_{j=1}^{n-1} \boldsymbol{A}_j \boldsymbol{z}_j = -\boldsymbol{A}_n$:

let $\boldsymbol{x} = \binom{\boldsymbol{z}}{1}, \quad \boldsymbol{y} = \mathbf{0}$

or $\exists\, \boldsymbol{y} : \boldsymbol{y}^\top \boldsymbol{A}_j \geq 0 \;\; (1 \leq j \leq n-1), \quad \boldsymbol{y}^\top(-\boldsymbol{A}_n) < 0$:
let $\boldsymbol{x} = \mathbf{0}$.

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$\Rightarrow$ $\boldsymbol{x} \geq \mathbf{0}, \;\; \boldsymbol{y}^\top \boldsymbol{A} \geq \mathbf{0}^\top, \;\; \boldsymbol{A}\boldsymbol{x} = \mathbf{0}\,, \;\; x_n + (\boldsymbol{y}^\top \boldsymbol{A})_n > 0$

= Lemma of Tucker

Dantzig's assumption

... assumes Tucker's Lemma and hence the Lemma of Farkas,

which proves LP duality directly.

The minimax theorem is not of much use here!

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Next: we fix this.

Distilled from Adler [2013].

Tucker's Theorem

For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$

$$\exists\, \boldsymbol{x} \geq \mathbf{0},\ \boldsymbol{y} \ :\ \boldsymbol{x} \geq \mathbf{0},\ \ \boldsymbol{y}^\top \boldsymbol{A} \geq \mathbf{0}^\top,\ \ \boldsymbol{A}\boldsymbol{x} = \mathbf{0},\ \ \boxed{\boldsymbol{x}^\top + \boldsymbol{y}^\top \boldsymbol{A} > \mathbf{0}^\top}$$

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Also: Tucker's Theorem $\Rightarrow$ Tucker's Lemma

Stiemke [1915], Gordan [1873]

Stiemke's Theorem

$$\nexists\, \mathbf{y} \;:\; \mathbf{y}^\top \mathbf{A} \geq \mathbf{0}^\top,\;\; \mathbf{y}^\top \mathbf{A} \neq \mathbf{0}^\top \quad \Leftrightarrow \quad \exists\, \mathbf{x} \;:\; \mathbf{A}\mathbf{x} = \mathbf{0},\;\; \mathbf{x} > \mathbf{0}$$

Stiemke [1915], Gordan [1873]

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Gordan's Theorem

$$\nexists\, \mathbf{x} \;:\; \mathbf{A}\mathbf{x} = \mathbf{0},\;\; \mathbf{x} \geq \mathbf{0},\;\; \mathbf{x} \neq \mathbf{0} \quad \Leftrightarrow \quad \exists\, \mathbf{y} \;:\; \mathbf{y}^\top \mathbf{A} > \mathbf{0}^\top$$

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$$\exists\, \mathbf{x}, \mathbf{y} \;:\; \mathbf{x} \geq \mathbf{0}\,,\;\; \mathbf{y}^\top \mathbf{A} \geq \mathbf{0}^\top,\;\; \mathbf{A}\mathbf{x} = \mathbf{0}\,,\;\; \mathbf{x}^\top + \mathbf{y}^\top \mathbf{A} > \mathbf{0}^\top$$

Gordan, Ville [1938], minimax theorem

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minimax theorem

$$\exists\, \boldsymbol{x} \in \boldsymbol{X},\; \boldsymbol{y} \in \boldsymbol{Y},\; \boldsymbol{v} \in \mathbb{R} \;:\; \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{1}\boldsymbol{v},\quad \boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{v}\boldsymbol{1}^\top$$

Gordan, Ville [1938], minimax theorem

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(via Ville by subtracting max-min value $\boldsymbol{v}$ from $\boldsymbol{A}$ giving $\boldsymbol{A}'$ with $\boldsymbol{y}^\top \boldsymbol{A}' \geq \mathbf{0}^\top$, shows min-max value of $\boldsymbol{A}'$ is $\mathbf{0}$).

From Gordan to Tucker

Let $\tilde{\boldsymbol{x}}$ with $\tilde{\boldsymbol{x}} \geq \mathbf{0}$, $\boldsymbol{A}\tilde{\boldsymbol{x}} = \mathbf{0}$ have maximum support

$\boldsymbol{S} = \{\, \boldsymbol{j} \mid \tilde{\boldsymbol{x}}_{\boldsymbol{j}} > \mathbf{0} \,\}$

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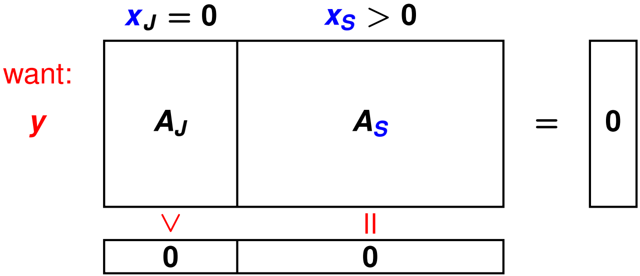
$\boldsymbol{S} = \{\, \boldsymbol{j} \mid \tilde{\boldsymbol{x}}_{\boldsymbol{j}} > \mathbf{0} \,\}$, write $\boldsymbol{x} = (\boldsymbol{x}_{\boldsymbol{J}}, \boldsymbol{x}_{\boldsymbol{S}})$, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}_{\boldsymbol{J}}\boldsymbol{x}_{\boldsymbol{J}} + \boldsymbol{A}_{\boldsymbol{S}}\boldsymbol{x}_{\boldsymbol{S}}$.

From Gordan to Tucker

Let $\tilde{\boldsymbol{x}}$ with $\tilde{\boldsymbol{x}} \geq \mathbf{0}$, $A\tilde{\boldsymbol{x}} = \mathbf{0}$ have maximum support

$S = \{\, j \mid \tilde{x}_j > 0 \,\}$, write $\boldsymbol{x} = (\boldsymbol{x}_J, \boldsymbol{x}_S)$, $A\boldsymbol{x} = A_J\boldsymbol{x}_J + A_S\boldsymbol{x}_S$.



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want: y

$$
\begin{array}{c|c}
x_J = 0 & x_S > 0 \\
\hline
D & \mathbf{0} \\
\hdashline
E & F \;\; \text{(basis of rows of } A_S\text{)} \\
\hline
\vee & \Vert \\
\mathbf{0} & \mathbf{0}
\end{array}
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 & \begin{array}{cc} x_J = 0 & x_S > 0 \end{array} \\
\text{want:} & \left[\begin{array}{c|c} D & 0 \\ \hdashline E & F \;\text{(basis of rows of } A_S\text{)} \end{array}\right] = \left[\begin{array}{c} 0 \end{array}\right] \\
y & \begin{array}{cc} \vee & \parallel \end{array} \\
 & \left[\begin{array}{c|c} 0 & 0 \end{array}\right]
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$$

$$
\begin{aligned}
Ax = \mathbf{0} \quad &\Leftrightarrow \quad CAx = CA_J x_J + CA_S x_S = \mathbf{0} \\
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\begin{array}{ccc|cc}
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\mathbf{w} & D & & \mathbf{0} & \\
\mathbf{0} & E & & F \ \text{(basis of rows of } A_S\text{)} & \\
\end{array}
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$A\tilde{x} = 0\,,\ \tilde{x} \geq 0\,,\ \tilde{x}_S > 0$ where $\tilde{x}$ has maximum support S.

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F has full rank $\Rightarrow\; \exists\, x_S \,:\, Ex_J + Fx_S = 0\,.$

$\Rightarrow\; C(A_J x_J + A_S \underbrace{(x_S + \tilde{x}_S \alpha)}_{>0 \text{ as } \alpha \to \infty}) = 0\,,\quad S$ not maximal. ↯

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Gordan $\Rightarrow$

$\exists\, w\ :\ w^\top D > 0^\top,\ \ (\binom{w}{0}^\top C)A_J > 0\,,\ \ (\binom{w}{0}^\top C)A_S = 0\,.$ □

Summary: minimax theorem $\Rightarrow$ LP duality

Recall: Using Dantzig's game $\quad \boldsymbol{B} = \begin{bmatrix} \mathbf{0} & \boldsymbol{A} & -\boldsymbol{b} \\ -\boldsymbol{A}^\top & \mathbf{0} & \boldsymbol{c} \\ \boldsymbol{b}^\top & -\boldsymbol{c}^\top & 0 \end{bmatrix}$

with $\boldsymbol{B} = -\boldsymbol{B}^\top$ **assumes** Tucker's Lemma

$$\exists\, \boldsymbol{z} \geq \mathbf{0},\ \ \boldsymbol{B}\boldsymbol{z} \leq \mathbf{0},\ \ \boldsymbol{z}_k - (\boldsymbol{B}\boldsymbol{z})_k > 0\,.$$

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minimax theorem $\Rightarrow$ Gordan's Theorem, $\Rightarrow$ **Tucker's Theorem**

$$\exists\, \boldsymbol{z} \geq \mathbf{0},\ \boldsymbol{B}\boldsymbol{z} \leq \mathbf{0},\ \boldsymbol{z} - \boldsymbol{B}\boldsymbol{z} > \mathbf{0}$$

$\Rightarrow$ LP duality with **strict complementarity**: for feasible LPs

$$\exists\, \boldsymbol{x}, \boldsymbol{y}\ :\ (\boldsymbol{y}^\top \boldsymbol{A} - \boldsymbol{c}^\top)\boldsymbol{x} = 0\,, \qquad \boldsymbol{y}^\top(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) = 0\,,$$
$$(\boldsymbol{y}^\top \boldsymbol{A} - \boldsymbol{c}^\top) + \boldsymbol{x}^\top > \mathbf{0}^\top, \quad \boldsymbol{y} + (\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) > \mathbf{0}\,.$$

Karp-type reduction from LP to Minimax [motivated by Brooks & Reny, 2023]

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Theorem Consider **max-min** strategy $(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{s}, \boldsymbol{v})$ for the game

$$\boldsymbol{B_M} = \begin{bmatrix} \mathbf{0} & \boldsymbol{A} & -\boldsymbol{b} \\ -\boldsymbol{A}^\top & \mathbf{0} & \boldsymbol{c} \\ \boldsymbol{b}^\top & -\boldsymbol{c}^\top & \mathbf{0} \\ \mathbf{1}^\top & \mathbf{1}^\top & -\boldsymbol{M} \end{bmatrix}$$

for sufficiently large $\boldsymbol{M}$ (polynomial bit-size for rational $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}$).

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$\boldsymbol{v} = \mathbf{0} \Rightarrow \boldsymbol{s} > \mathbf{0}, \quad \boldsymbol{A}\boldsymbol{x}\frac{\mathbf{1}}{\boldsymbol{s}} \leq \boldsymbol{b}, \quad \boldsymbol{A}^\top\boldsymbol{y}\frac{\mathbf{1}}{\boldsymbol{s}} \geq \boldsymbol{c}, \quad \boldsymbol{b}^\top\boldsymbol{y}\frac{\mathbf{1}}{\boldsymbol{s}} = \boldsymbol{c}^\top\boldsymbol{x}\frac{\mathbf{1}}{\boldsymbol{s}}$ (opt.)

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$$B_M = \begin{bmatrix} 0 & A & -b \\ -A^\top & 0 & c \\ b^\top & -c^\top & 0 \\ 1^\top & 1^\top & -M \end{bmatrix}$$

for sufficiently large M (polynomial bit-size for rational A, b, c).

Then v is the value of B_M, $v \geq 0$, and

$v = 0 \;\Rightarrow\; s > 0,\;\; Ax\frac{1}{s} \leq b,\;\; A^\top y\frac{1}{s} \geq c,\;\; b^\top y\frac{1}{s} = c^\top x\frac{1}{s}$ (opt.),

$v > 0 \;\Rightarrow\; s = 0,\;\; Ax \leq 0,\;\; A^\top y \geq 0,\;\; b^\top y < c^\top x$ (infeasible).

Minimax theorem: Proof by Loomis [1946]

min-max strategy $\boldsymbol{x} \in \boldsymbol{X}$: minimize $\boldsymbol{v}$ s.t. $\boldsymbol{A}\boldsymbol{x} \leq \mathbf{1}\boldsymbol{v}$,

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Assume $(\boldsymbol{Ax})_k < \boldsymbol{v}$ for some row $\boldsymbol{k}$, let $\overline{\boldsymbol{A}}$ be $\boldsymbol{A}$ without row $\boldsymbol{k}$.

By **inductive hypothesis**, $\overline{\boldsymbol{A}}$ has game value $\overline{\boldsymbol{v}}$, $\overline{\boldsymbol{A}}\overline{\boldsymbol{x}} \leq \mathbf{1}\overline{\boldsymbol{v}}$.

$$\overline{\boldsymbol{v}} \leq \boldsymbol{u}, \quad \overline{\boldsymbol{v}} \leq \boldsymbol{v}, \quad (\overline{\boldsymbol{A}} \text{ better than } \boldsymbol{A} \text{ for minimizer}).$$

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Claim : $\overline{\boldsymbol{v}} = \boldsymbol{v}$. Intuition: maximizer avoids row $\boldsymbol{k}$ of $\boldsymbol{A}$ anyhow.

Proof that $\overline{v} = v$

minimal v s.t. $Ax \leq \mathbf{1}v$, maximal u s.t. $y^\top A \geq u\mathbf{1}^\top$, $\quad u \leq v$.

$(Ax)_k < v$, matrix $\overline{A}$ is A without row k, value $\overline{v} \leq u$, $\overline{v} \leq v$.

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Suppose $\overline{v} < v$. For $0 < \varepsilon \le 1$,

$$\overline{A}(\underbrace{x(1-\varepsilon) + \overline{x}\varepsilon}_{x(\varepsilon) \in X \text{ (convex)}}) \le \mathbf{1}(v(1-\varepsilon) + \overline{v}\varepsilon) = \mathbf{1}(v - \varepsilon(v - \overline{v})) < \mathbf{1}v$$

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For missing row k of A and sufficiently small $\varepsilon > 0$:

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overall $Ax(\varepsilon) < \mathbf{1}v$, contradicting minimality of v.
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$\Rightarrow \quad \overline{v} \le u \le v = \overline{v}$, $\boxed{u = v}$. Induction complete . □

“On a theorem of von Neumann”

Theorem Loomis [1946]

Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m \times n}$, $\boldsymbol{B} > \mathbf{0}$.

Then there exist $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{y} \in \boldsymbol{Y}$, $\boldsymbol{v} \in \mathbb{R}$:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{B}\boldsymbol{x}\boldsymbol{v}, \qquad \boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{v}\boldsymbol{y}^\top \boldsymbol{B}.$$

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$\boldsymbol{B} = \mathbf{1}\,\mathbf{1}^\top$: minimax theorem, $\quad \boldsymbol{A}\boldsymbol{x} \leq \mathbf{1}\boldsymbol{v}, \qquad \boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{v}\mathbf{1}^\top.$

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Conversely, theorem is **implied** by the minimax theorem:

value$(\boldsymbol{A} - \alpha\boldsymbol{B}) < \mathbf{0}$ for $\alpha \to \infty$,

value$(\boldsymbol{A} - \alpha\boldsymbol{B}) > \mathbf{0}$ for $\alpha \to -\infty$, continuous in α, hence

value$(\boldsymbol{A} - \alpha\boldsymbol{B}) = \mathbf{0}$ for some $\boldsymbol{v} = \alpha$. $\square$

Conforti, Di Summa, Zambelli [2007]

Theorem

$Ax \leq b$ minimally infeasible $\Rightarrow$ $Ax = b$ minimally infeasible.

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$\Rightarrow$ reversing any inequality $\boldsymbol{a_i x} \leq \boldsymbol{b_i}$ creates feasible system:

$$\forall \text{ row } \boldsymbol{i} \quad \exists\, \boldsymbol{x} \;:\; \boldsymbol{a_i x} > \boldsymbol{b_i}\,, \quad \forall\, \boldsymbol{k} \neq \boldsymbol{i} \;:\; \boldsymbol{a_k x} = \boldsymbol{b_k}\,.$$

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Then apply **linear algebra** (get $0 = -1$ from infeasible $Ax = b$):

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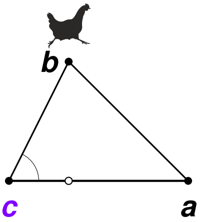
to prove inequality-Farkas (get $0 \leq -1$ from infeasible $Ax \leq b$):

$$\nexists\, x \;:\; Ax \leq b \quad \Leftrightarrow \quad \exists\, y \geq 0 \;:\; y^\top A = 0^\top, \;\; y^\top b < 0\,.$$

How did the chicken cross the triangle?



Consider a triangle with corners ***a***, ***b***, ***c*** and a chicken at ***b*** that wants ???

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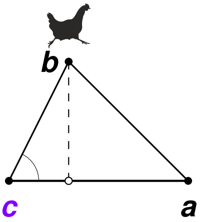
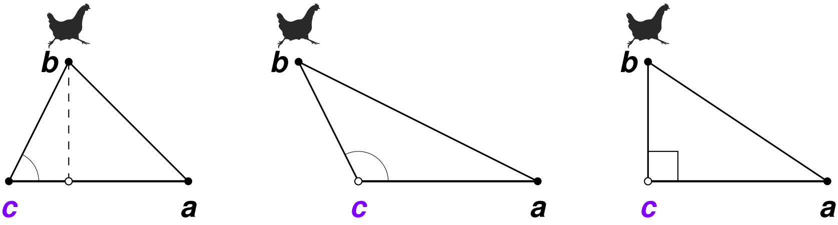


Consider a triangle with corners ***a***, ***b***, ***c*** and a chicken at ***b*** that wants to get to the other side.[citation needed]

How did the chicken cross the triangle?



Consider a triangle with corners $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and a chicken at $\boldsymbol{b}$ that wants to get to the other side.

Then the closest point to get there is $\boldsymbol{c}$ if and only if the angle at $\boldsymbol{c}$ is not acute, that is,

$$(\boldsymbol{b} - \boldsymbol{c})^\top (\boldsymbol{a} - \boldsymbol{c}) \leq \mathbf{0}\,.$$

Supporting hyperplane theorem

Theorem

Let $\emptyset \neq \boldsymbol{C} \subset \mathbb{R}^{\boldsymbol{m}}$, closed, convex, $\boldsymbol{b} \notin \boldsymbol{C}$.

Let $\boldsymbol{c} \in \boldsymbol{C}$ with smallest $\|\boldsymbol{b} - \boldsymbol{c}\|$.

Consider hyperplane $\boldsymbol{H}$ with normal vector $\boldsymbol{b} - \boldsymbol{c}$ through $\boldsymbol{c}$:
then all of $\boldsymbol{C}$ on one side, $\boldsymbol{b}$ strictly on the other side of $\boldsymbol{H}$,

$$(\boldsymbol{b} - \boldsymbol{c})^\top(\boldsymbol{b} - \boldsymbol{c}) > \mathbf{0}\,, \quad \forall\, \boldsymbol{a} \in \boldsymbol{C} \;:\; (\boldsymbol{b} - \boldsymbol{c})^\top(\boldsymbol{a} - \boldsymbol{c}) \leq \mathbf{0}\,.$$

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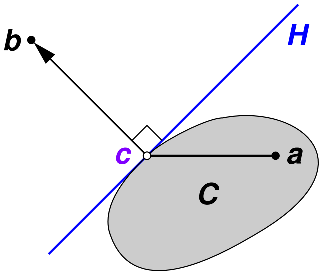

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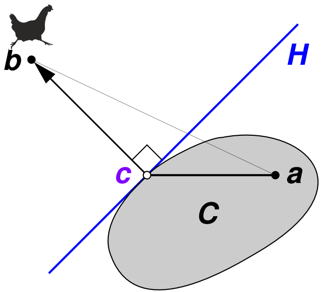
Theorem

Let $\emptyset \neq \boldsymbol{C} \subset \mathbb{R}^{\boldsymbol{m}}$, closed, convex, $\boldsymbol{b} \notin \boldsymbol{C}$.

Let $\boldsymbol{c} \in \boldsymbol{C}$ with smallest $||\boldsymbol{b} - \boldsymbol{c}||$.

Consider hyperplane $\boldsymbol{H}$ with normal vector $\boldsymbol{b} - \boldsymbol{c}$ through $\boldsymbol{c}$:
then all of $\boldsymbol{C}$ on one side, $\boldsymbol{b}$ strictly on the other side of $\boldsymbol{H}$,

$$(\boldsymbol{b} - \boldsymbol{c})^{\top}(\boldsymbol{b} - \boldsymbol{c}) > \mathbf{0}, \quad \forall\, \boldsymbol{a} \in \boldsymbol{C} \;:\; (\boldsymbol{b} - \boldsymbol{c})^{\top}(\boldsymbol{a} - \boldsymbol{c}) \leq \mathbf{0}.$$

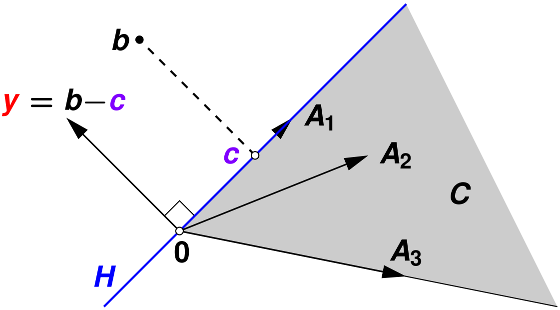


Lemma of Farkas

Cone $\boldsymbol{C} = \{\boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{x} \geq \mathbf{0}\}$ and $\boldsymbol{b} \notin \boldsymbol{C}$.

Consider $\boldsymbol{c} \in \boldsymbol{C}$ with smallest $\|\boldsymbol{b} - \boldsymbol{c}\|$, and $\boldsymbol{y} = \boldsymbol{b} - \boldsymbol{c}$. Then

$$\boldsymbol{y}^\top \boldsymbol{b} > \mathbf{0}, \quad (\forall\, \boldsymbol{a} \in \boldsymbol{C} \;:\; \boldsymbol{y}^\top \boldsymbol{a} \leq \mathbf{0}) \quad \boldsymbol{y}^\top \boldsymbol{A} \leq \mathbf{0}^\top.$$



Why is the cone $C = \{Ax \mid x \geq 0\}$ closed?

- show: limit a of any sequence of points $a^{(k)}$ in C is in C
- $\forall k \; \exists$ basis B, $x_B \geq 0$: $a^{(k)} = A_B x_B$
- only finitely many bases B
- restrict to subsequence with one B that occurs infinitely often
- $a = \lim_{k \to \infty} a^{(k)} = A_B \lim_{k \to \infty} \underbrace{A_B^{-1} a^{(k)}}_{\geq 0} \in C$
- need theorem of Carathéodory (and Weierstrass).

Fourier–Motzkin elimination = projection

Lemma (ineq-Farkas, get $\mathbf{0} \leq -\mathbf{1}$ from infeasible $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$):

$$\nexists\, \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}} \;:\; \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \quad \Leftrightarrow \quad \exists\, \boldsymbol{y} \geq \mathbf{0} \;:\; \boldsymbol{y}^\top \boldsymbol{A} = \mathbf{0}^\top, \;\; \boldsymbol{y}^\top \boldsymbol{b} < \mathbf{0}\,.$$

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Proof By induction on $\boldsymbol{n}$.

Scale rows of $\boldsymbol{Ax} \leq \boldsymbol{b}$ with **affine** $\boldsymbol{a_i}$, $\boldsymbol{b_j}$, $\boldsymbol{c_k}$ as

$$\boldsymbol{a_i}(\boldsymbol{x_2}, \ldots, \boldsymbol{x_n}) \leq \boldsymbol{x_1}\,, \quad \boldsymbol{x_1} \leq \boldsymbol{b_j}(\boldsymbol{x_2}, \ldots, \boldsymbol{x_n})\,,$$
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Eliminate $\boldsymbol{x_1}$ by writing $\boldsymbol{a_i} \leq \boldsymbol{b_j}$ for all pairs $\boldsymbol{i}, \boldsymbol{j}$.

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By inductive hypothesis: Either solve in $\boldsymbol{x_2}, \ldots, \boldsymbol{x_n} \ge \mathbf{0}$ and choose any $\boldsymbol{x_1}$ with $\boldsymbol{a_i} \le \boldsymbol{x_1} \le \boldsymbol{b_j}$ for all $\boldsymbol{i}, \boldsymbol{j}$, or linearly combine (then also in terms of rows of $\boldsymbol{Ax} \le \boldsymbol{b}$) to get $\mathbf{0} \le -\mathbf{1}$. □

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