# Efficient computation of equilibria for extensive games

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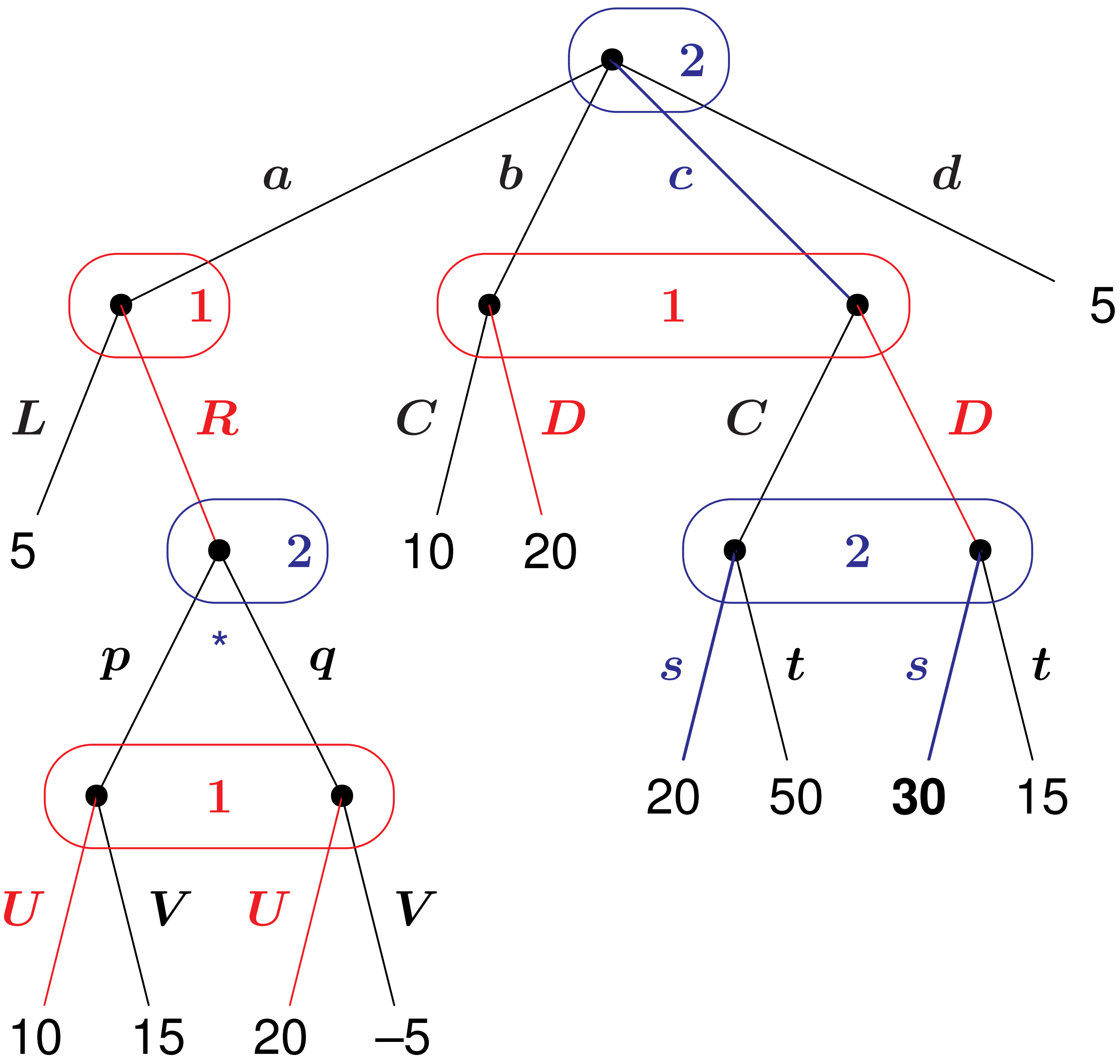
# Game tree (game in extensive form)



# Strategic (or normal) form

**Strategy** of a player:

specifies a move for **every** information set of that player.

| | $a$ | $a$ | $a$ | $a$ | $b$ | $b$ | $b$ | $b$ | $c$ | $c$ | $c$ | $c$ | $d$ | $d$ | $d$ | $d$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $p$ | $p$ | $q$ | $q$ | $p$ | $p$ | $q$ | $q$ | $p$ | $p$ | $q$ | $q$ | $p$ | $p$ | $q$ | $q$ |
| | $s$ | $t$ | $s$ | $t$ | $s$ | $t$ | $s$ | $t$ | $s$ | $t$ | $s$ | $t$ | $s$ | $t$ | $s$ | $t$ |
| $L, U, C$ | 5 | 5 | 5 | 5 | 10 | 10 | 10 | 10 | 20 | 50 | 20 | 50 | 5 | 5 | 5 | 5 |
| $L, V, C$ | 5 | 5 | 5 | 5 | 10 | 10 | 10 | 10 | 20 | 50 | 20 | 50 | 5 | 5 | 5 | 5 |
| $L, U, D$ | 5 | 5 | 5 | 5 | 20 | 20 | 20 | 20 | 30 | 15 | 30 | 15 | 5 | 5 | 5 | 5 |
| $L, V, D$ | 5 | 5 | 5 | 5 | 20 | 20 | 20 | 20 | 30 | 15 | 30 | 15 | 5 | 5 | 5 | 5 |
| $R, U, C$ | 10 | 10 | 20 | 20 | 10 | 10 | 10 | 10 | 20 | 50 | 20 | 50 | 5 | 5 | 5 | 5 |
| $R, U, D$ | 10 | 10 | 20 | 20 | 20 | 20 | 20 | 20 | 30 | 15 | 30 | 15 | 5 | 5 | 5 | 5 |
| $R, V, C$ | 20 | 20 | –5 | –5 | 10 | 10 | 10 | 10 | 20 | 50 | 20 | 50 | 5 | 5 | 5 | 5 |
| $R, V, D$ | 10 | 10 | 20 | 20 | 20 | 20 | 20 | 20 | 30 | 15 | 30 | 15 | 5 | 5 | 5 | 5 |

# Reduced strategic form

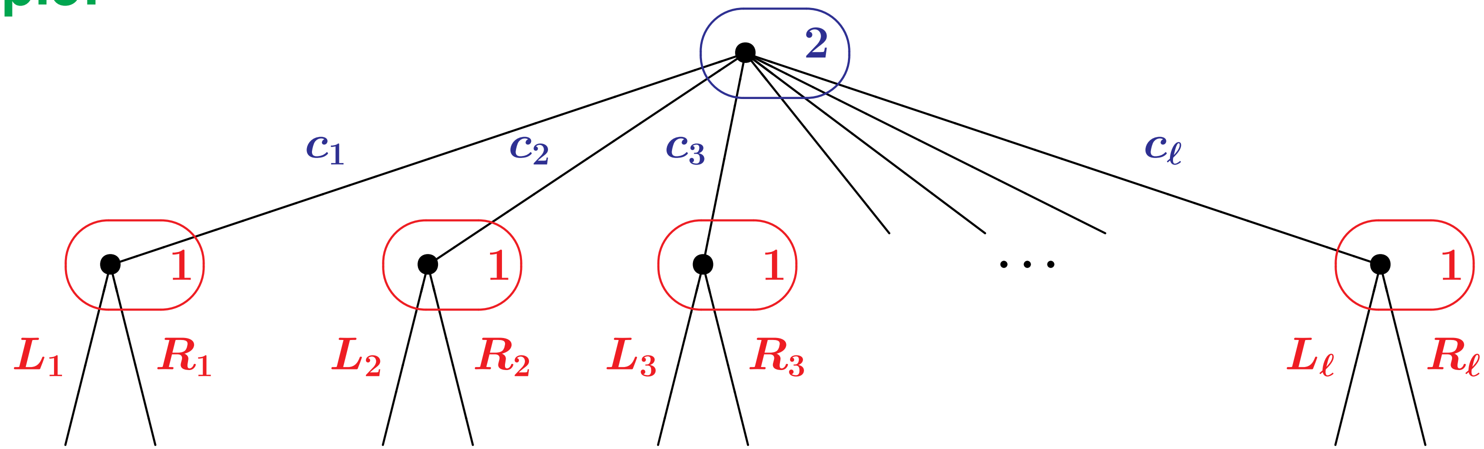
**Reduced strategy** of a player:
specifies a move for every information set of that player,
**except** for those information sets unreachable due to an **own** earlier move (where we write $*$ instead of a move).

| | $a, p, *$ | $a, q, *$ | $b, *, *$ | $c, *, s$ | $c, *, t$ | $d, *, *$ |
|---|---|---|---|---|---|---|
| $L, *, C$ | 5 | 5 | 10 | 20 | 50 | 5 |
| $L, *, D$ | 5 | 5 | 20 | 30 | 15 | 5 |
| $R, U, C$ | 10 | 20 | 10 | 20 | 50 | 5 |
| $R, U, D$ | 10 | 20 | 20 | **30** | 15 | 5 |
| $R, V, C$ | 15 | –5 | 10 | 20 | 50 | 5 |
| $R, V, D$ | 15 | –5 | 20 | 30 | 15 | 5 |

# Exponential blowup of strategic form

number of pure strategies typically
**exponential** in number of information sets.

**Example:**



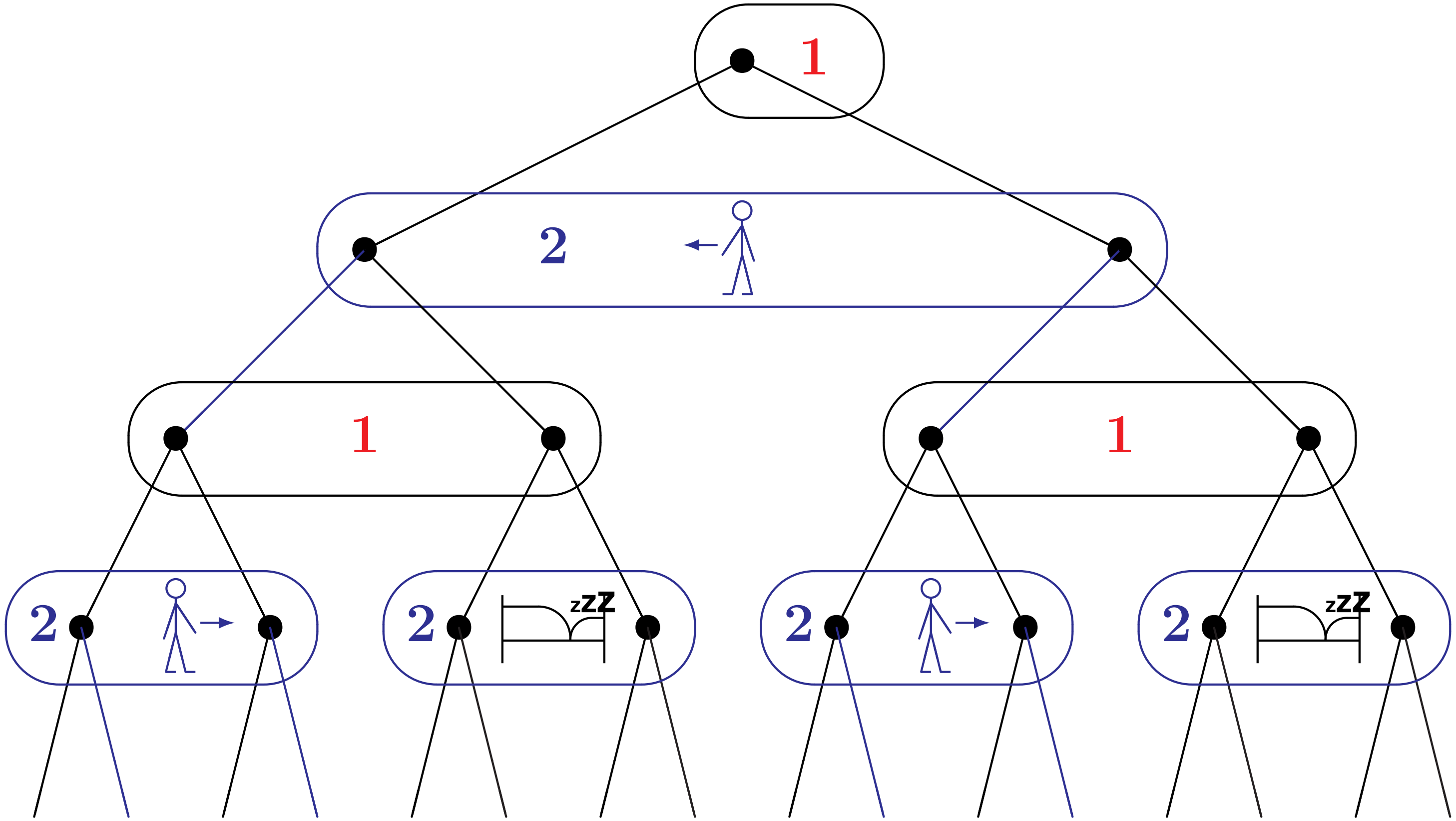

$$\begin{aligned}\text{number of information sets} &= \ell, \\ \text{number of pure strategies} &= 2^\ell.\end{aligned}$$

**Example** [Kuhn]: simplified poker game,

$$\begin{aligned}\text{number of information sets} &= 13, \\ \text{number of pure strategies} &= 8192.\end{aligned}$$

# Exponential blowup of reduced strategic form

**Example**: Game with (1) **bounded** number of moves per node, (2) no **subgames** (otherwise simplify by solving subgames first).




This tree with $n$ nodes: $\approx 2^{\sqrt{n}/2}$ strategies per player, reduced strategic form still (sub-)**exponential** in **tree** size.

# Our result (sneak preview)

The **sequence form** is a strategic description of an extensive game with perfect recall that has the **same** size as the game tree, as opposed to **exponential** size of reduced strategic form.

The same known strategic-form algorithms for **finding equilibria** can be applied to the sequence form:
linear programming (LP) for two-player zero-sum games,
linear complementarity (LCP) for general two-player games,

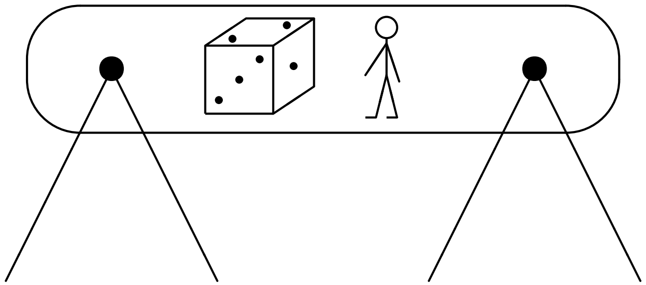
Game tree of size $n$:
sequence form size $n \times n$,
reduced strategic form: possibly size $2^{\sqrt{n}}$.

# Size of reduced strategic form versus sequence form

| tree depth | tree size (nodes) | number of reduced strategies: player 1 | number of reduced strategies: player 2 | Reduced Strategic Form size | indep. SF variables: pl. 1 | indep. SF variables: pl. 2 | SF size |
|---|---|---|---|---|---|---|---|
| 1 | 3 | 2 | 1 | 2 | 1 | | 2 |
| 2 | 7 | | 2 | 4 | | 1 | 2 |
| 3 | 15 | 4 | | 8 | 2 | | 4 |
| 4 | 31 | | 8 | 32 | | 4 | 16 |
| 5 | 63 | 16 | | 128 | 8 | | 64 |
| 6 | 127 | | 128 | 2048 | | 16 | 256 |
| 7 | 255 | 256 | | 32768 | 32 | | 1024 |
| 8 | 511 | | 32768 | 8388608 | | 64 | 4096 |
| 9 | 1023 | 65536 | | 2147483648 | 128 | | 16384 |
| 10 | 2047 | | 2147483648 | 140737488355328 | | 256 | 65536 |

# Use behavior strategies

**Behavior strategy** = **local** randomization



Mixed strategy too redundant, use behavior strategy instead:

- only one variable per **move**:
  player 1 chooses $L$ with probability $X_L$
  player 1 chooses $R$ with probability $X_R$ . . .
  player 2 chooses $a$ with probability $Y_a$ . . .
- expected payoff = 5 $Y_a X_L + 10\, Y_a X_R Y_p X_U + 15\, Y_a X_R Y_p X_V + \cdots$
- problem: **nonlinear**!

# Variable transformation

For each **sequence** $\sigma$ of moves of player 1
introduce new variable $x_\sigma$

- new variables replace products:
  if $\sigma = PQRS$ then $x_\sigma = X_P X_Q X_R X_S$
- Example:

$$
\begin{aligned}
x_L &= X_L \\
x_{RU} &= X_R X_U \\
\cdots \\
y_a &= Y_a \\
y_{ap} &= Y_a Y_p \\
\cdots
\end{aligned}
$$

- expected payoff = 5 $x_L\, y_a + 10\, x_{RU}\, y_{ap} + 15\, x_{RV}\, y_{ap} + \cdots$

  is **linear** in variables of one player.

# New paradigm: Sequences instead of pure strategies

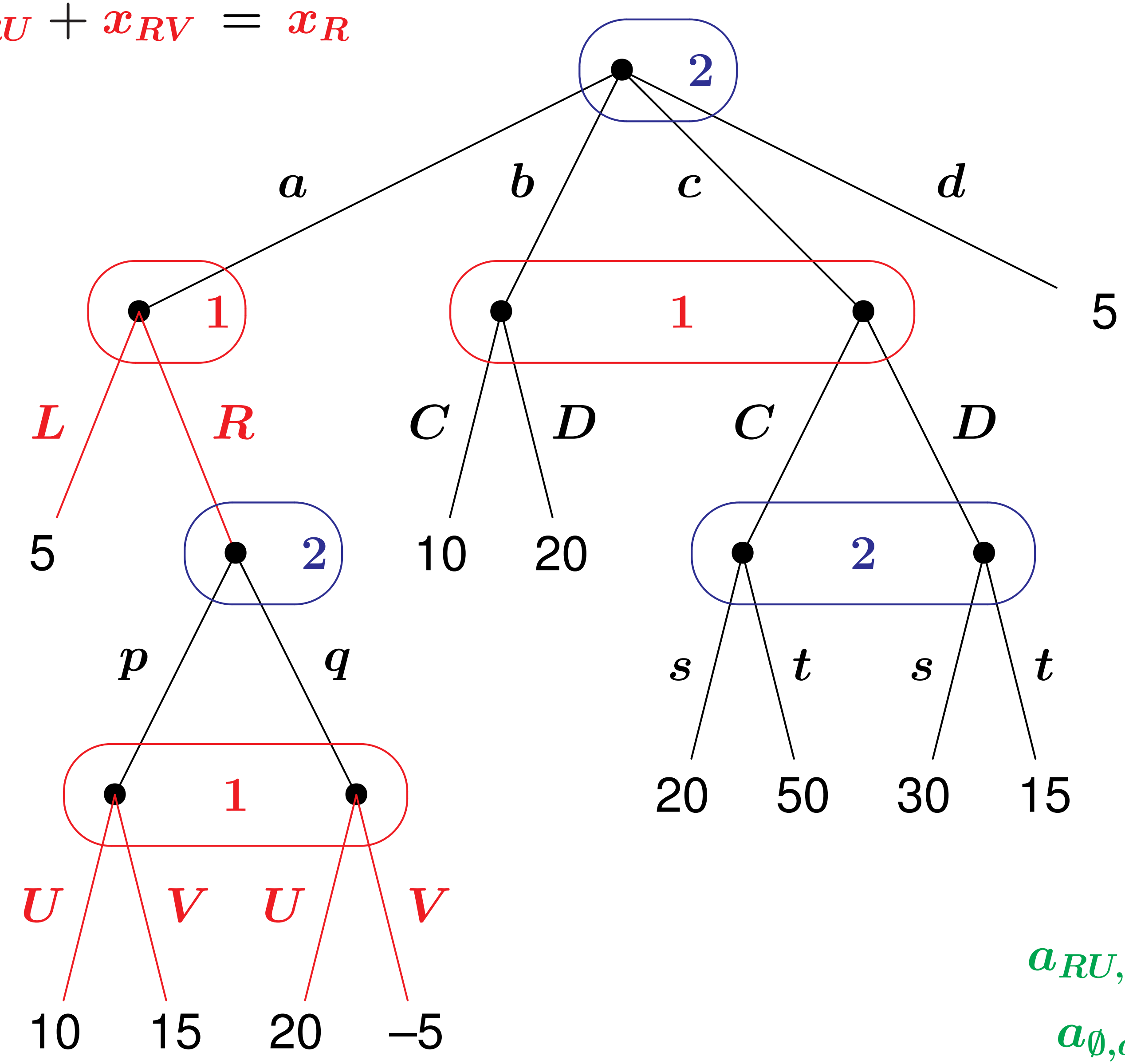
**Before:**

| | |
|---|---|
| pure strategy | $i$ |
| probability | $x_i$ |
| mixed strategy | $x$ |
| characterized by | $\mathbf{1}x = 1$ |
| expected payoff | $x^\top A y$ |

**After:**

| | |
|---|---|
| sequence | $\sigma$ |
| realization probability | $x_\sigma$ |
| **realization plan** | $x$ |
| characterized by | $Ex = e$ |
| expected payoff | $x^\top A y$ |

$$x_\emptyset = 1$$
$$x_L + x_R = x_\emptyset$$
$$x_{RU} + x_{RV} = x_R$$




$$a_{RU,ap} = 10$$
$$a_{\emptyset,d} = 5$$
$$a_{RU,b} = 0$$

# Realization plans

**Realization plan** $x = (x_\emptyset, x_L, x_R, x_C, x_D, x_{RU}, x_{RV})$

(= vector of realization probabilities)

characterized by $x \geq 0$ and **linear** equalities

$$\begin{aligned}
x_\emptyset &= 1 \\
x_\emptyset &= x_L + x_R \\
x_\emptyset &= x_C + x_D \\
x_R &= x_{RU} + x_{RV}
\end{aligned}$$

written as $Ex = e$ with

$$E = \begin{bmatrix} 1 & & & & & & \\ -1 & 1 & 1 & & & & \\ -1 & & & 1 & 1 & & \\ & & -1 & & & 1 & 1 \end{bmatrix}, \qquad e = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

# The sequence form

Payoff matrix $A$

| | $\emptyset$ | $a$ | $b$ | $c$ | $d$ | $ap$ | $aq$ | $bs$ | $bt$ |
|---|---|---|---|---|---|---|---|---|---|
| $\emptyset$ | | | | | 5 | | | | |
| $L$ | | 5 | | | | | | | |
| $R$ | | | | | | | | | |
| $RU$ | | | | | | 10 | 20 | | |
| $RV$ | | | | | | 15 | −5 | | |
| $C$ | | | 10 | | | | | 20 | 50 |
| $D$ | | | 20 | | | | | 30 | 15 |

expected payoff $x^\top A y$ ,

rows played with $x$ subject to $x \geq 0, \quad Ex = e$ ,

columns played with $y$ subject to $y \geq 0, \quad Fy = f$ .

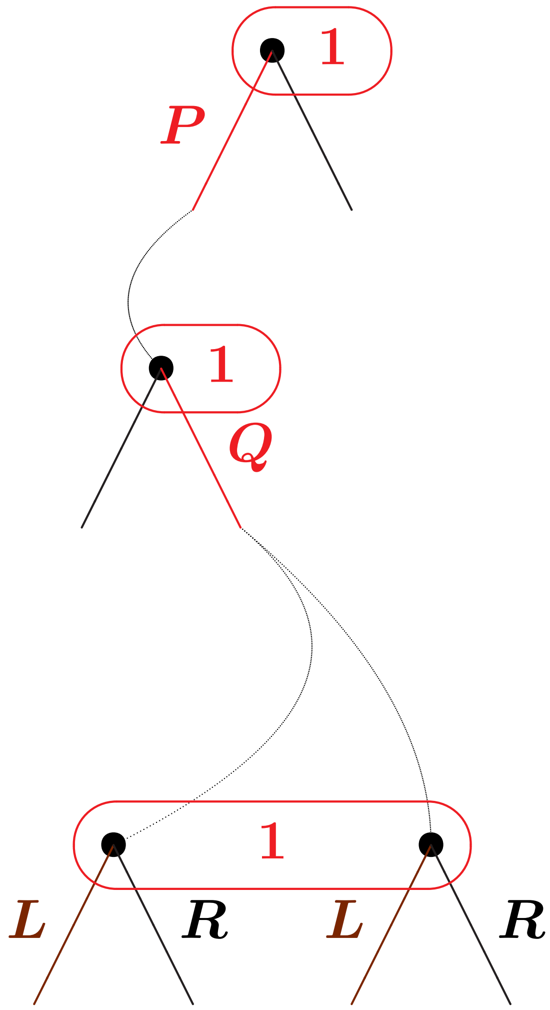
# How to play

**Given**: realization plan $x$ with $Ex = e$.

Move $L$ is last move of **unique** sequence,
say $PQL$, where $x_{PQL} + x_{PQR} = x_{PQ}$.

$$\Longrightarrow \quad \text{behavior-probability}(L) = \frac{x_{PQL}}{x_{PQ}}.$$

Required assumption of **perfect recall**
[Kuhn 1953, Selten 1975]:
Each node in an information set is
preceded by same sequence, here $PQ$,
of the player's **own** earlier moves.



# Best responses – LP duality

**1)** Best response $\boldsymbol{x}$ against fixed $\boldsymbol{y}$ solves LP:

$$\begin{array}{ll} \max\limits_{\boldsymbol{x}} & \boldsymbol{x}^\top(\boldsymbol{A}\boldsymbol{y}) \\ \text{subject to} & \boldsymbol{E}\boldsymbol{x} = \boldsymbol{e} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

**2)** Consider the **dual** of this LP:

$$\begin{array}{ll} \min\limits_{\boldsymbol{u}} & \boldsymbol{e}^\top \boldsymbol{u} \\ \text{subject to} & \boldsymbol{E}^\top \boldsymbol{u} \geq \boldsymbol{A}\boldsymbol{y} \end{array}$$

LP duality $\Longrightarrow$ same optimal value (payoff to player $1$).

# Best responses – LP duality

**2)** Consider the **dual** of this LP:

$$\min_{u} \quad e^\top u$$
$$\text{subject to} \quad E^\top u \geq Ay$$

LP duality $\Longrightarrow$ same optimal value (payoff to player $1$),

**3)** **minimized** by player $2$ if **zero-sum game**, $B = -A$:

$$\min_{u,\, y} \quad e^\top u$$
$$\text{subject to} \quad E^\top u \geq Ay$$
$$Fy = f$$
$$y \geq 0$$

# Example

**1)** Best response LP

$$\max_{x} \quad x^\top (Ay)$$

$$\text{subject to} \quad Ex = e$$

$$x \geq 0$$

$$\begin{array}{r|ccc||c|}
 & & & & \\
x_\emptyset & 1 & -1 & -1 & 0 \\
x_L & & 1 & & 2 \\
\geq 0 \quad x_R & & 1 & & 2 \\
x_C & & & 1 & 1 \\
x_D & & & 1 & 0 \\
 & & \| & & \downarrow \\
 & 1 & 0 & 0 & \text{max}
\end{array}$$

**2) dual LP**

$$\min_{u} \qquad e^\top u$$

$$\text{subject to} \quad E^\top u \geq Ay$$

$$\begin{array}{ccc|c|c}
u_0 & u_1 & u_2 & & \\
1 & -1 & -1 & & 0 \\
 & 1 & & & 2 \\
 & 1 & & \geq & 2 \\
 & & 1 & & 1 \\
 & & 1 & & 0 \\
\hline
1 & 0 & 0 & \longrightarrow & \text{min}
\end{array}$$

# Example

**2) dual LP**

$$\min_{u} \quad e^\top u$$

$$\text{subject to} \quad E^\top u \geq A y$$

$$\begin{array}{ccc} u_0 & u_1 & u_2 \end{array}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ & 1 & \\ & 1 & \\ & & 1 \\ & & 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \longrightarrow \text{min}$$

**3)** Treat $y$ as a variable:

$$\min_{u,\, y} \quad e^\top u$$

$$\text{subject to} \quad E^\top u \geq A y$$

$$F y = f$$

$$y \geq 0$$

$$\begin{array}{ccc} u_0 & u_1 & u_2 \end{array} \qquad \begin{array}{cccc} y_\emptyset & y_a & y_b & y_c \end{array} \geq 0$$

$$\begin{bmatrix} 1 & -1 & -1 \\ & 1 & \\ & 1 & \\ & & 1 \\ & & 1 \end{bmatrix} \geq \begin{bmatrix} & & & \\ & 6 & 0 & \\ & 2 & 4 & \\ & & & 3 \\ & & & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & & & & \end{bmatrix} \longrightarrow \text{min}$$

# Results

**Input:**

Two-person game tree with perfect recall.

**Theorem:**

A zero-sum game is solved via a Linear Program (LP) of **linear** size.

**Theorem:**

A non-zero-sum game is solved via a Linear Complementarity Problem (LCP) of **linear** size.

A sample equilibrium is found by Lemke's algorithm.

This algorithm mimicks the Harsanyi–Selten tracing procedure and finds a **normal form perfect** equilibrium.

# LCP – Lemke's algorithm

Consider a **prior** $(\overline{x}, \overline{y})$, and a new variable $z_0$ in the system

$$
\begin{array}{rcl}
 & & Ex + e\,z_0 = e \\
 & & Fy + f\,z_0 = f \\
r & = & E^\top u - Ay - A\overline{y}\,z_0 \geq 0 \\
s & = & F^\top v - B^\top x - B^\top \overline{x}\,z_0 \geq 0 \\
 & & x\,,\quad y,\quad z_0 \geq 0
\end{array}
$$

**Equilibrium condition** $x^\top r = 0, \quad y^\top s = 0, \quad [\, z_0 = 0\,].$

Initial solution $z_0 = 1,\ x = 0,\ y = 0.$

**Complementary pivoting:**

$x_\sigma \leftrightarrow r_\sigma,\ y_\tau \leftrightarrow s_\tau$, until $z_0$ leaves the basis.