Constructing and computing equilibria for two-player games

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Nash equilibria of bimatrix games

$$A = \begin{bmatrix} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 3 \end{bmatrix}$$

Nash equilibrium =

pair of strategies x , y with

x best response to y and
y best response to x.

Mixed equilibria

$$A = \begin{bmatrix} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 3 \end{bmatrix}$$

$$x = \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \end{bmatrix} \qquad x^T B = \begin{bmatrix} 5/3 & 5/3 \end{bmatrix}$$

$$Ay = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} \qquad y^T = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}$$

only **pure best responses** can have probability > 0

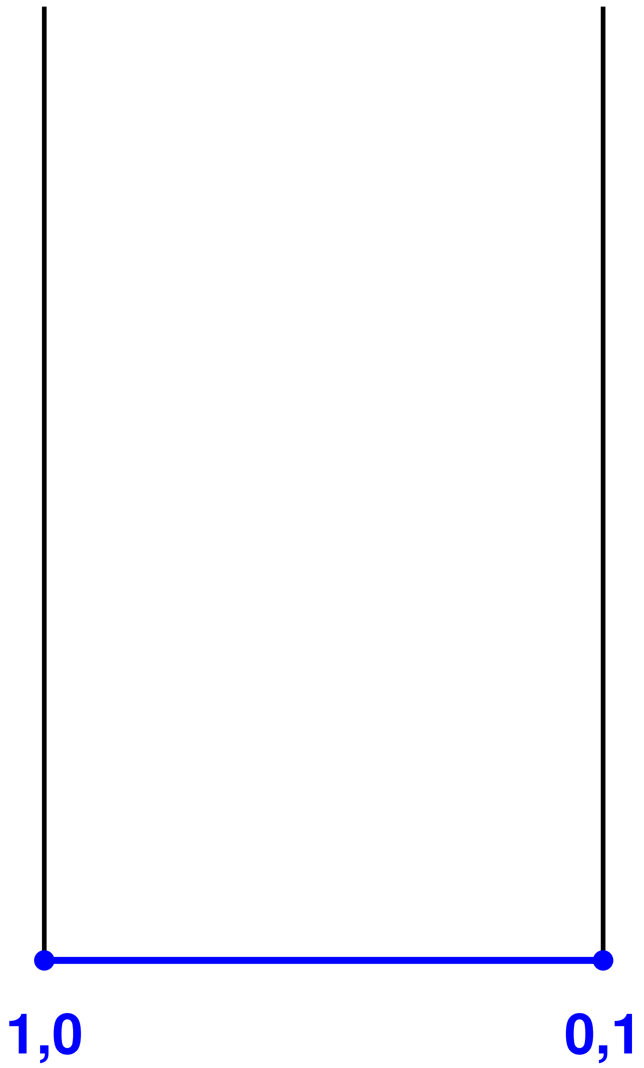
Best responses to mixed strategy of player 2

	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to
player I

Best responses to mixed strategy of player 2



	4	5
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= A

payoffs to
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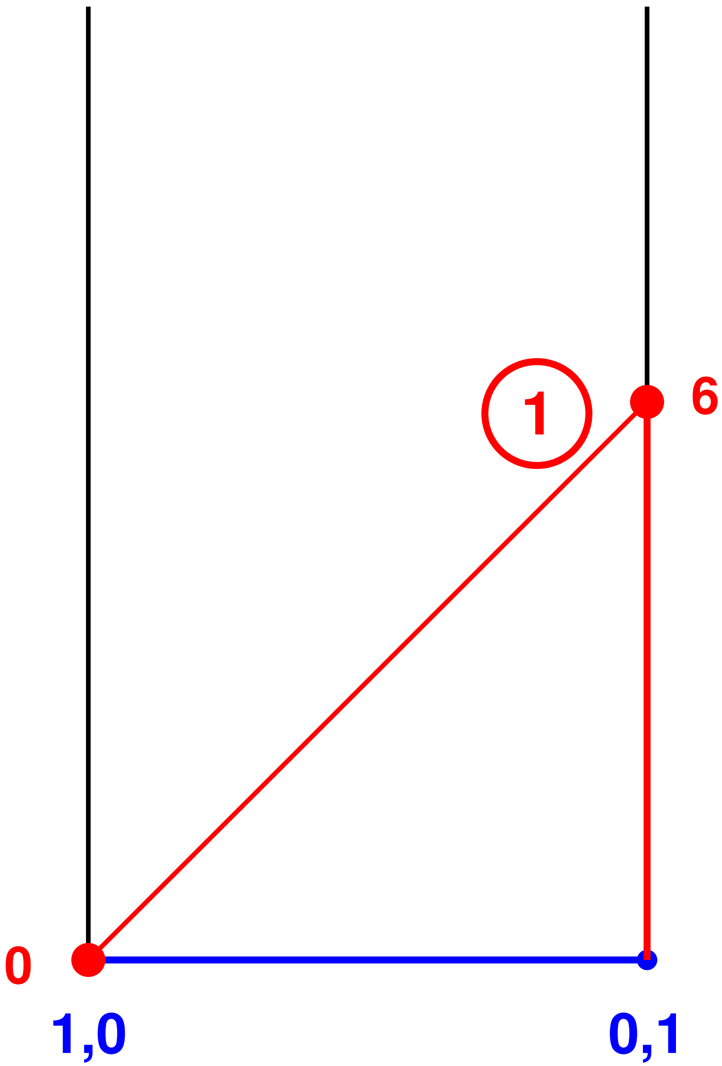

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= A

payoffs to
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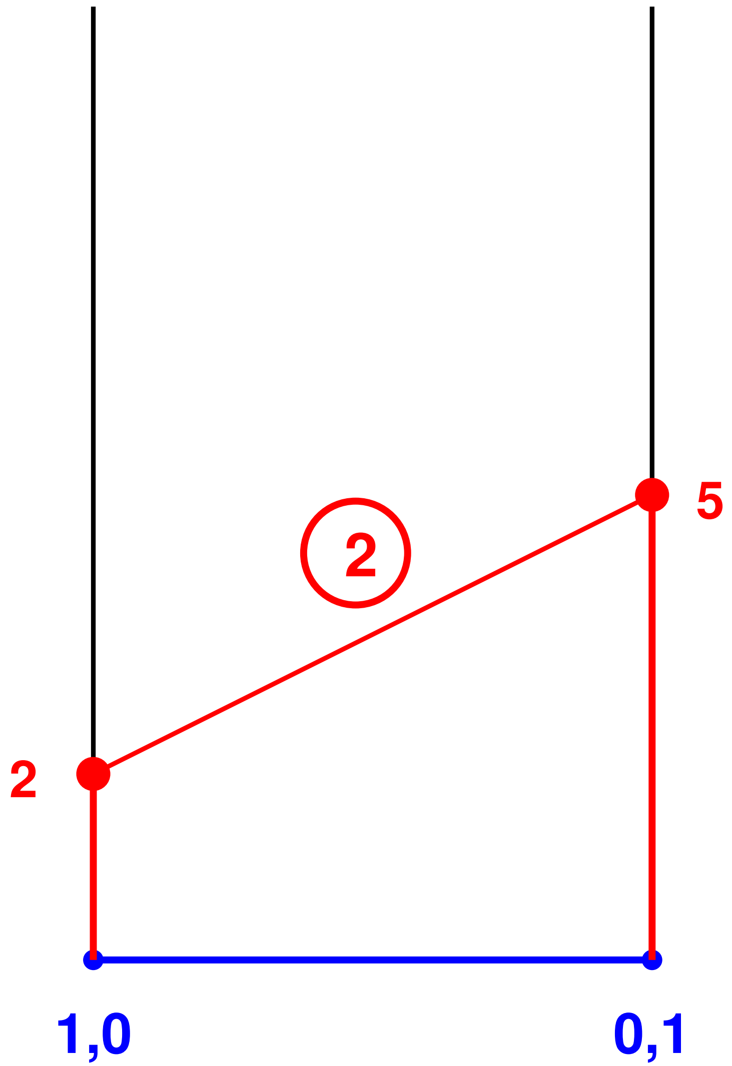
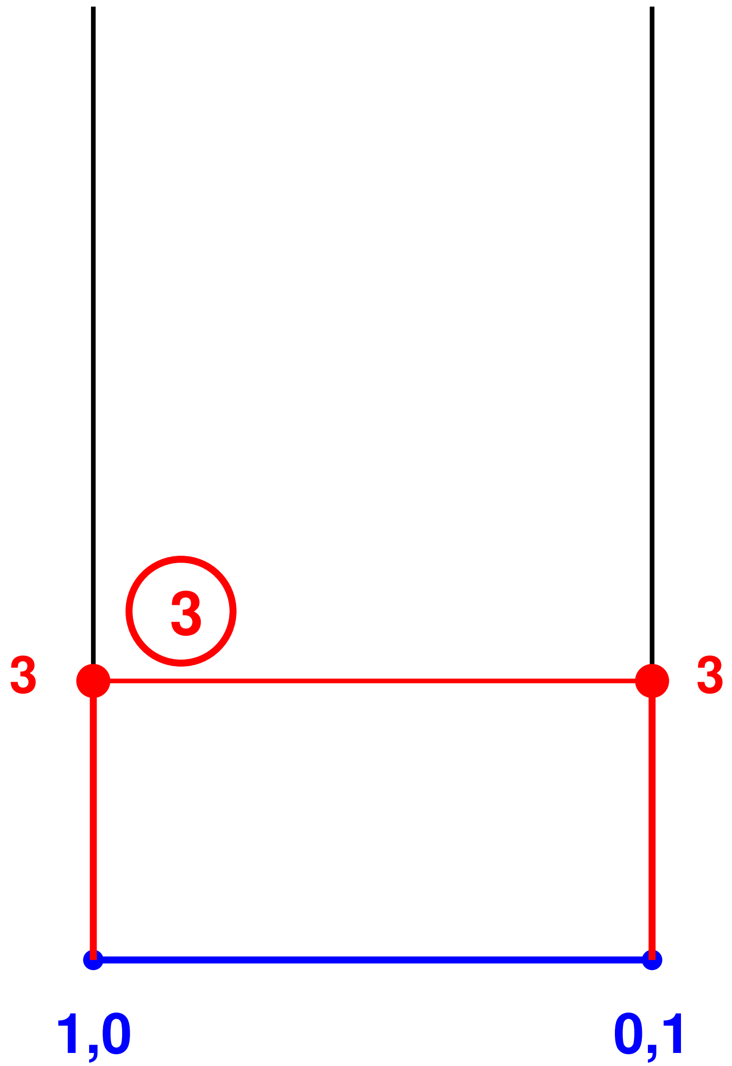


	4	5	
1	0	6	
2	2	5	= A
3	3	3	

payoffs to
player I

Best responses to mixed strategy of player 2

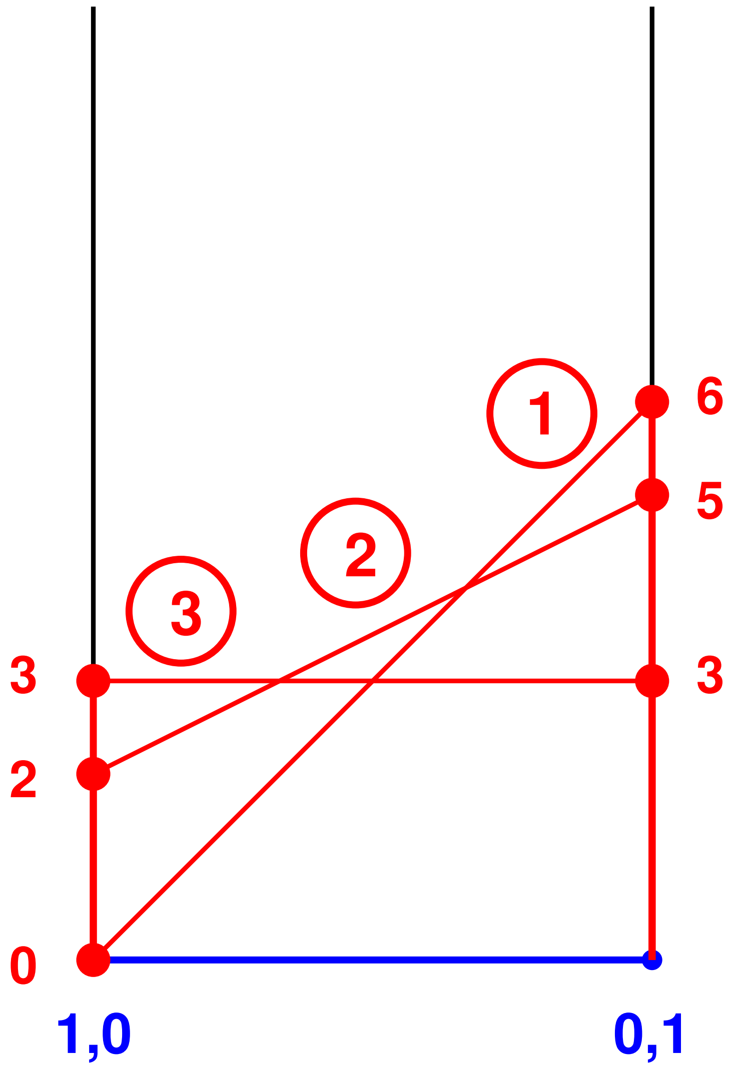


	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to player I

Best responses to mixed strategy of player 2

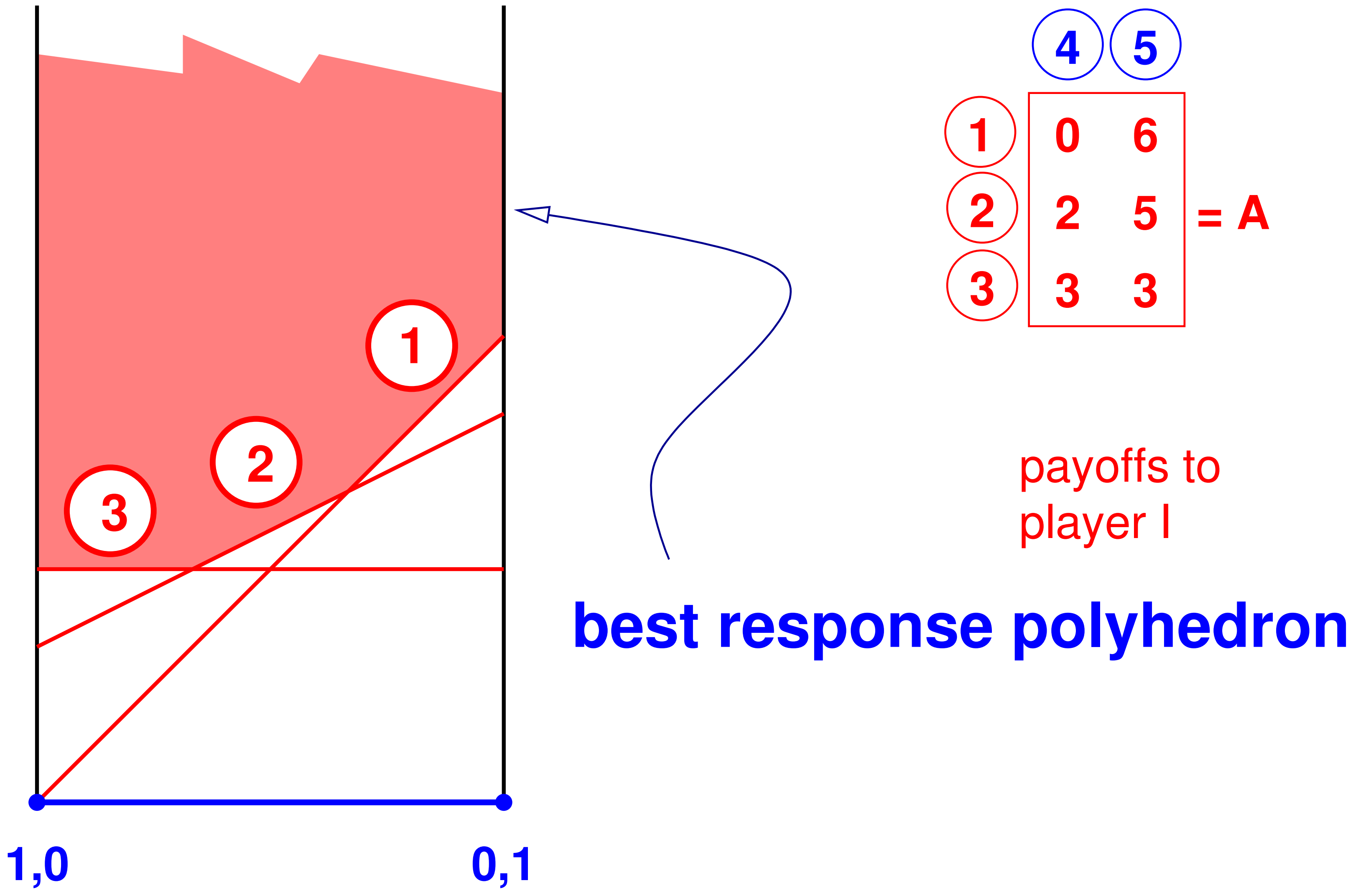


	4	5
1	0	6
2	2	5
3	3	3

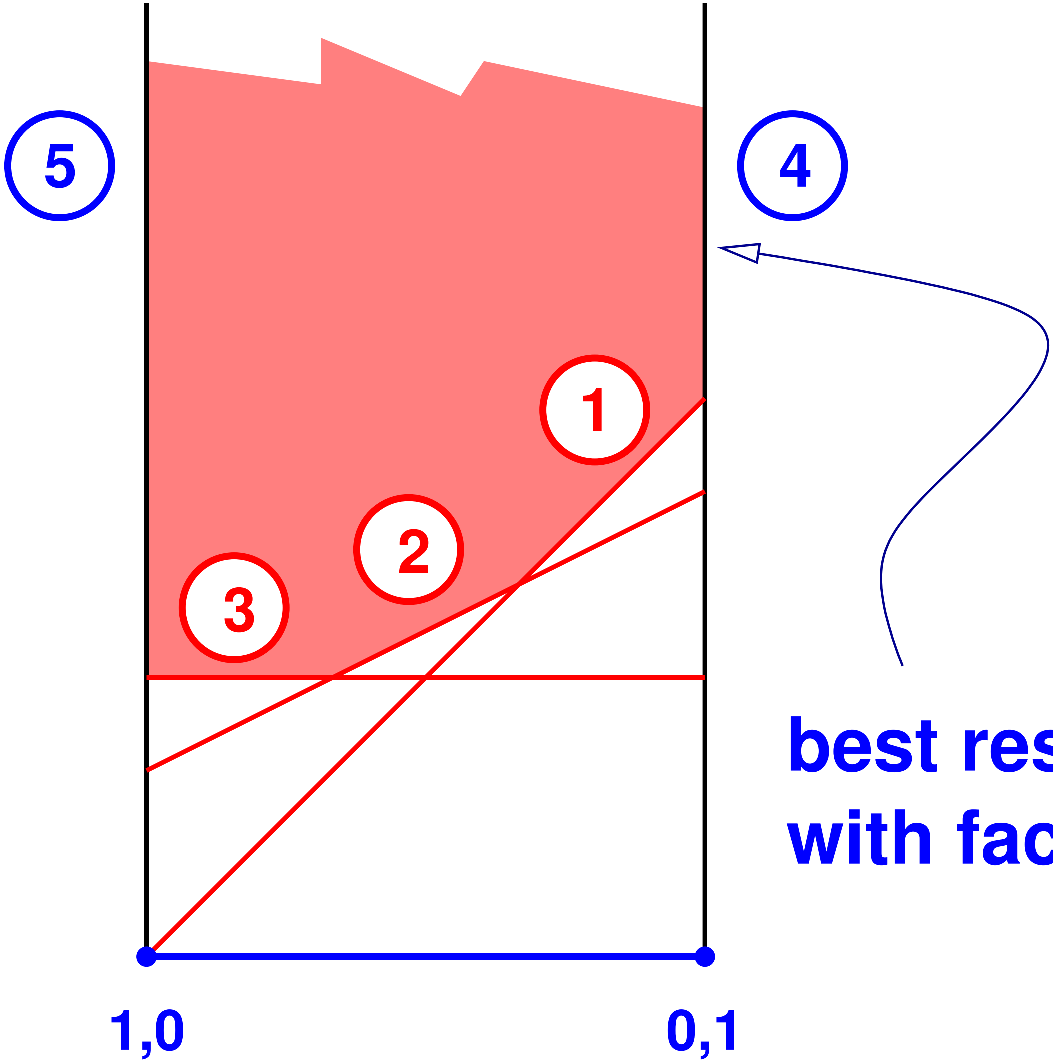
= A

payoffs to player I

Best responses to mixed strategy of player 2



Best responses to mixed strategy of player 2

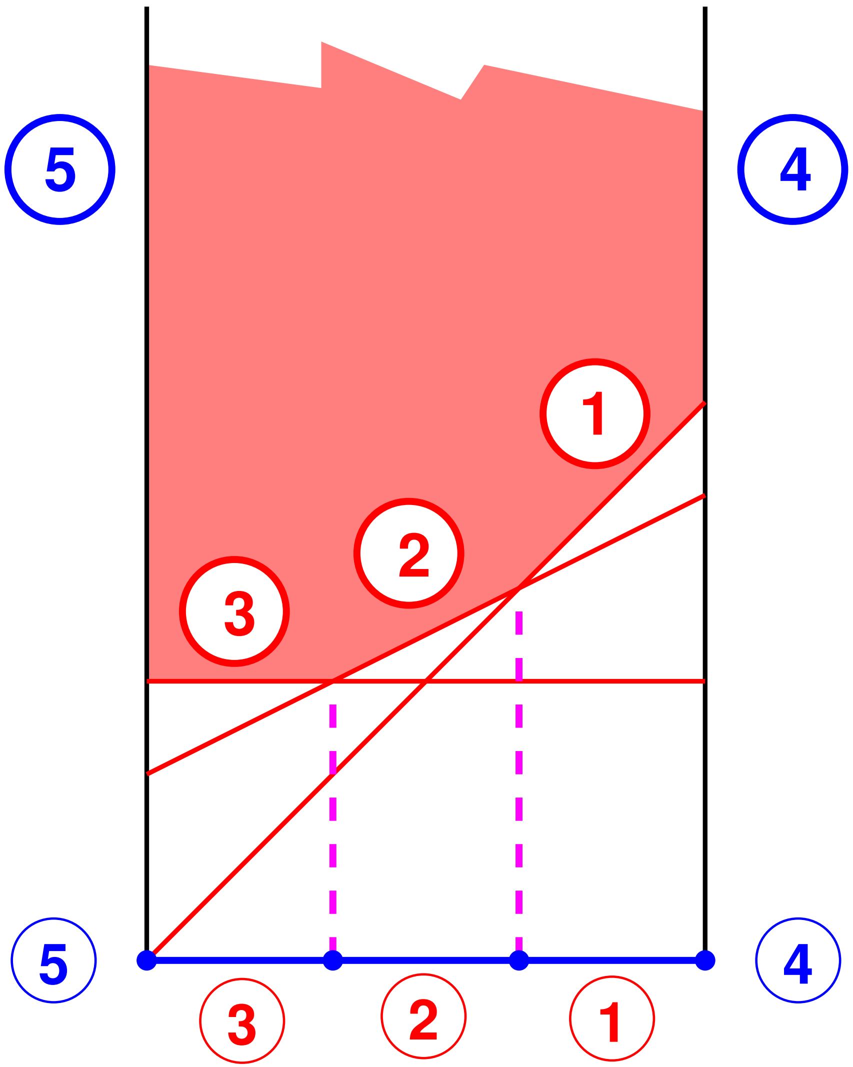


	4	5
1	0	6
2	2	5
3	3	3

= A

payoffs to
player I

Best responses to mixed strategy of player 2



	4	5
1	0	6
2	2	5
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= A

payoffs to player I

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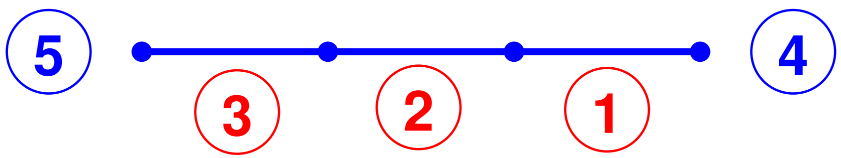
1	0	6
2	2	5
3	3	3

= A

payoffs to
player I

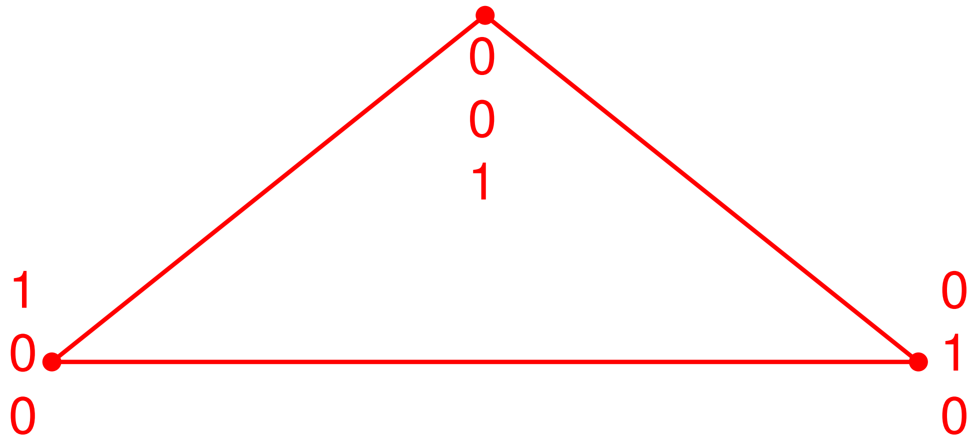


Best responses to mixed strategy of player 1

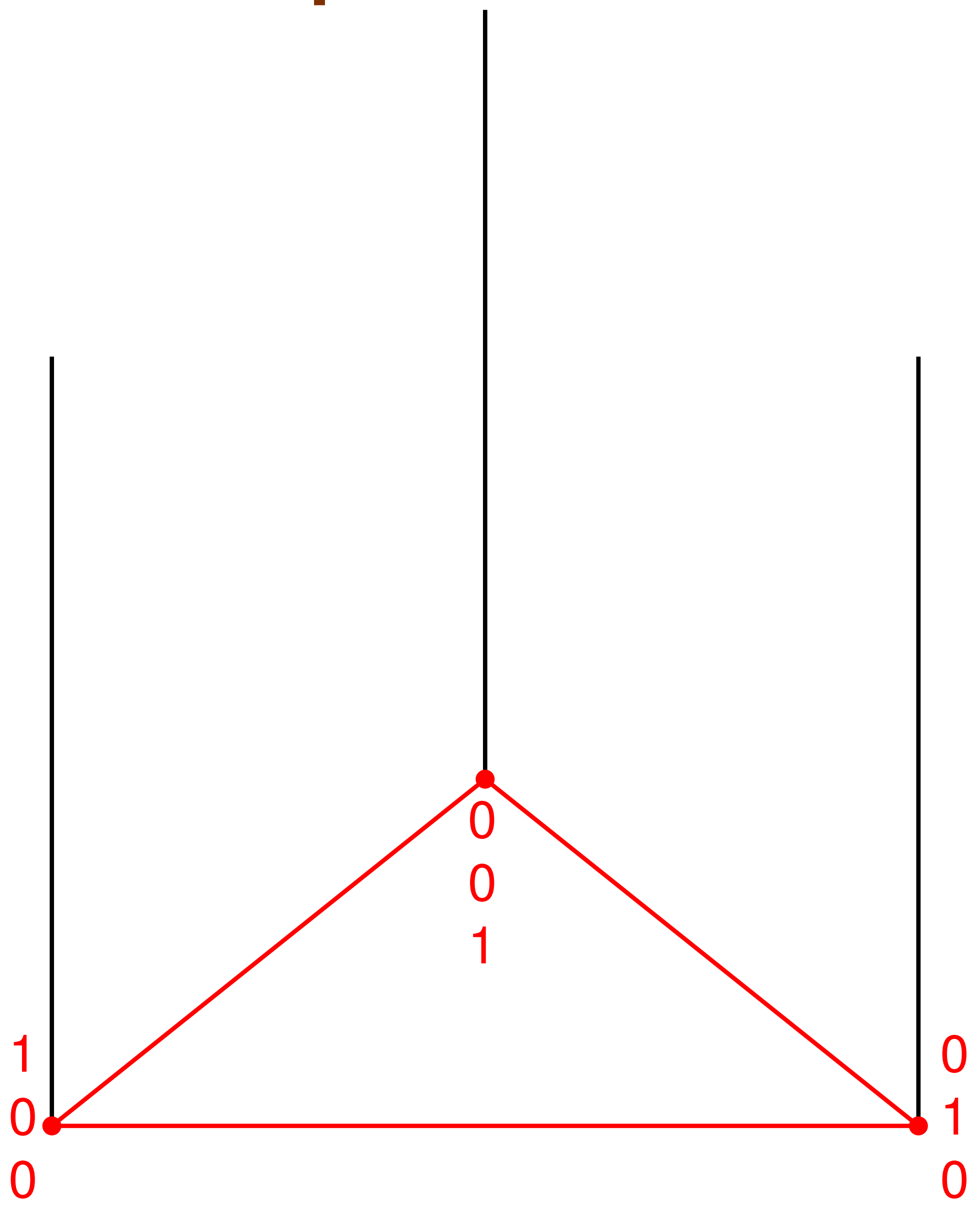
	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II



Best responses to mixed strategy of player 1

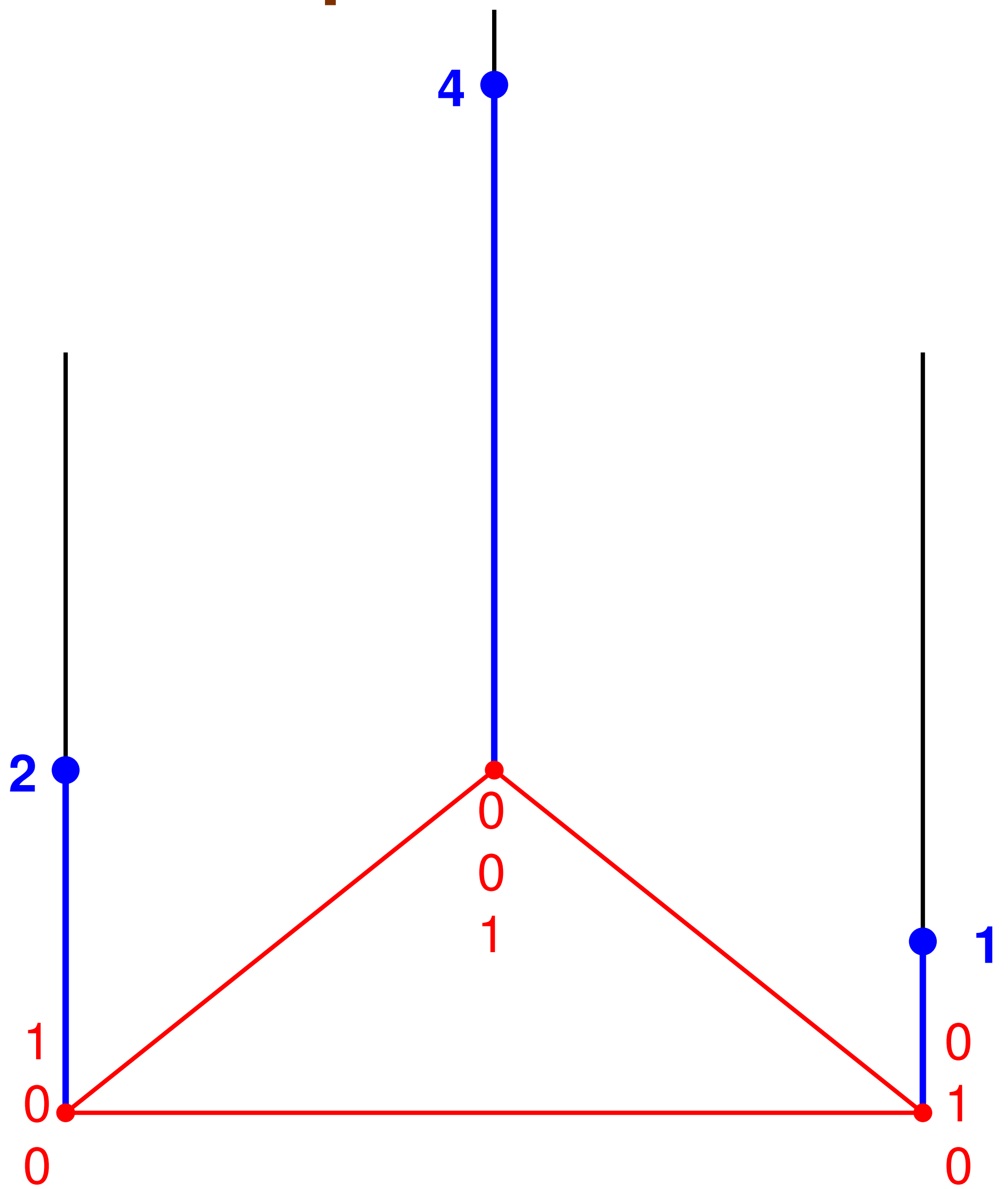


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Best responses to mixed strategy of player 1



	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

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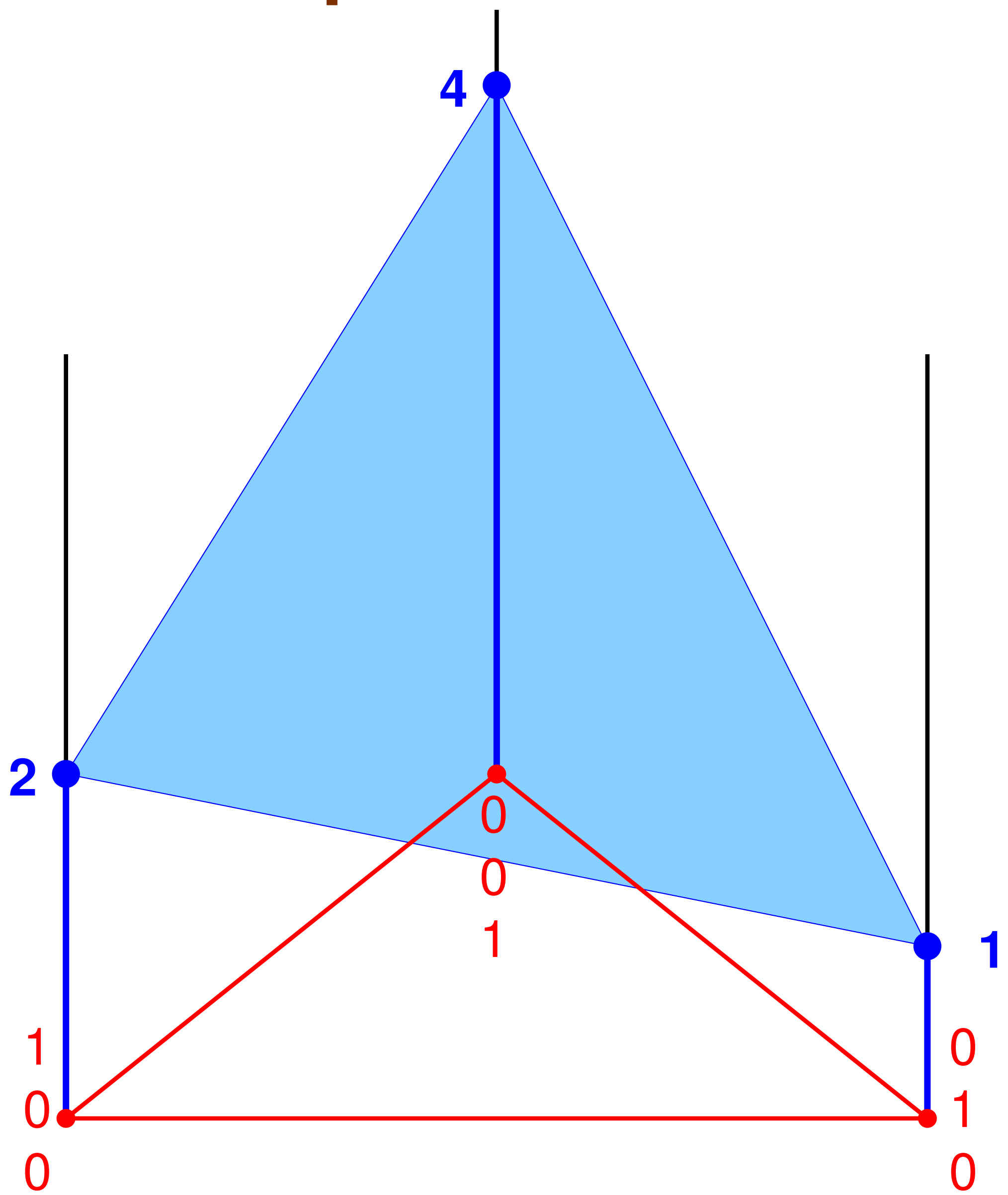

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1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

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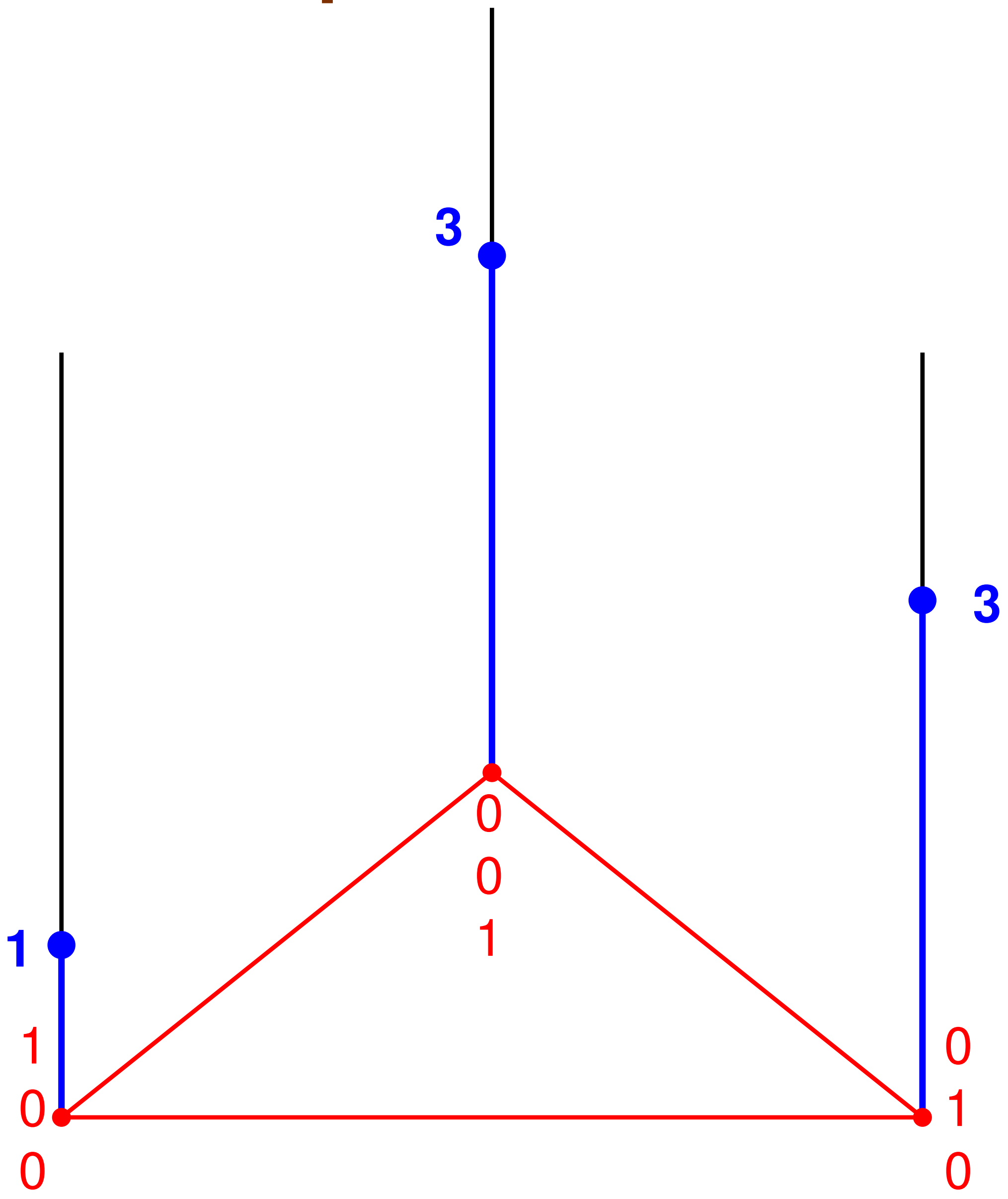
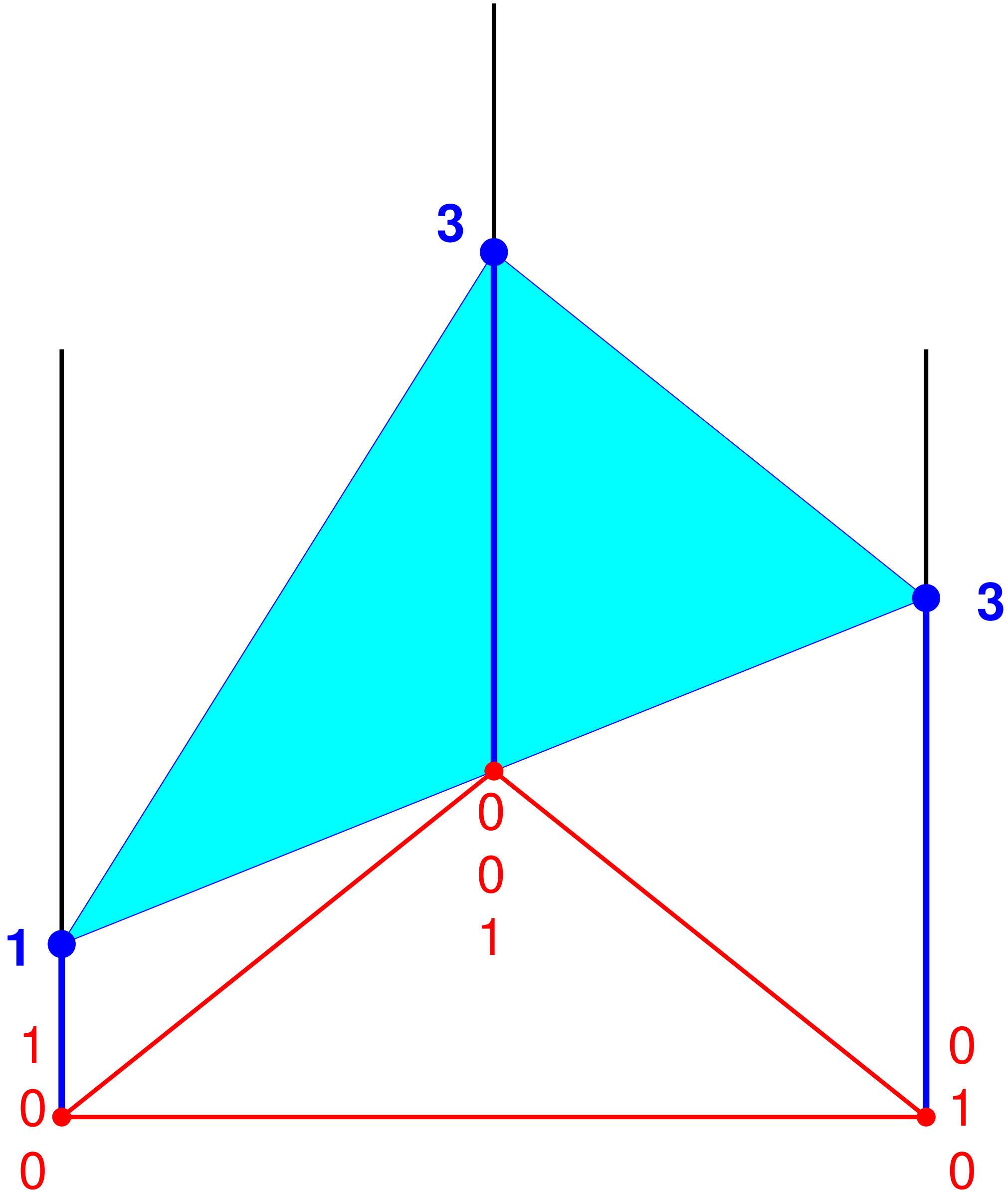


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to player II

Best responses to mixed strategy of player 1



	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

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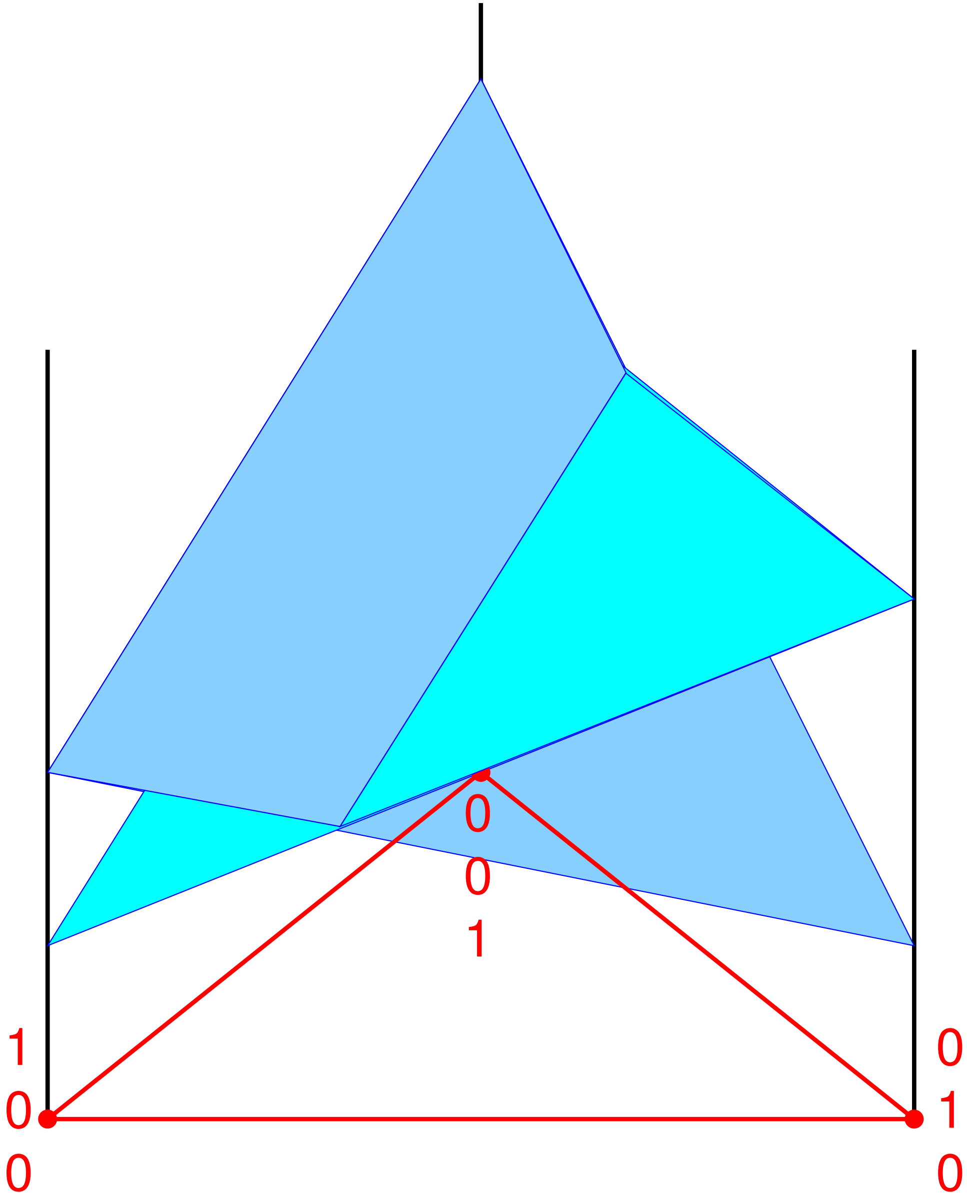
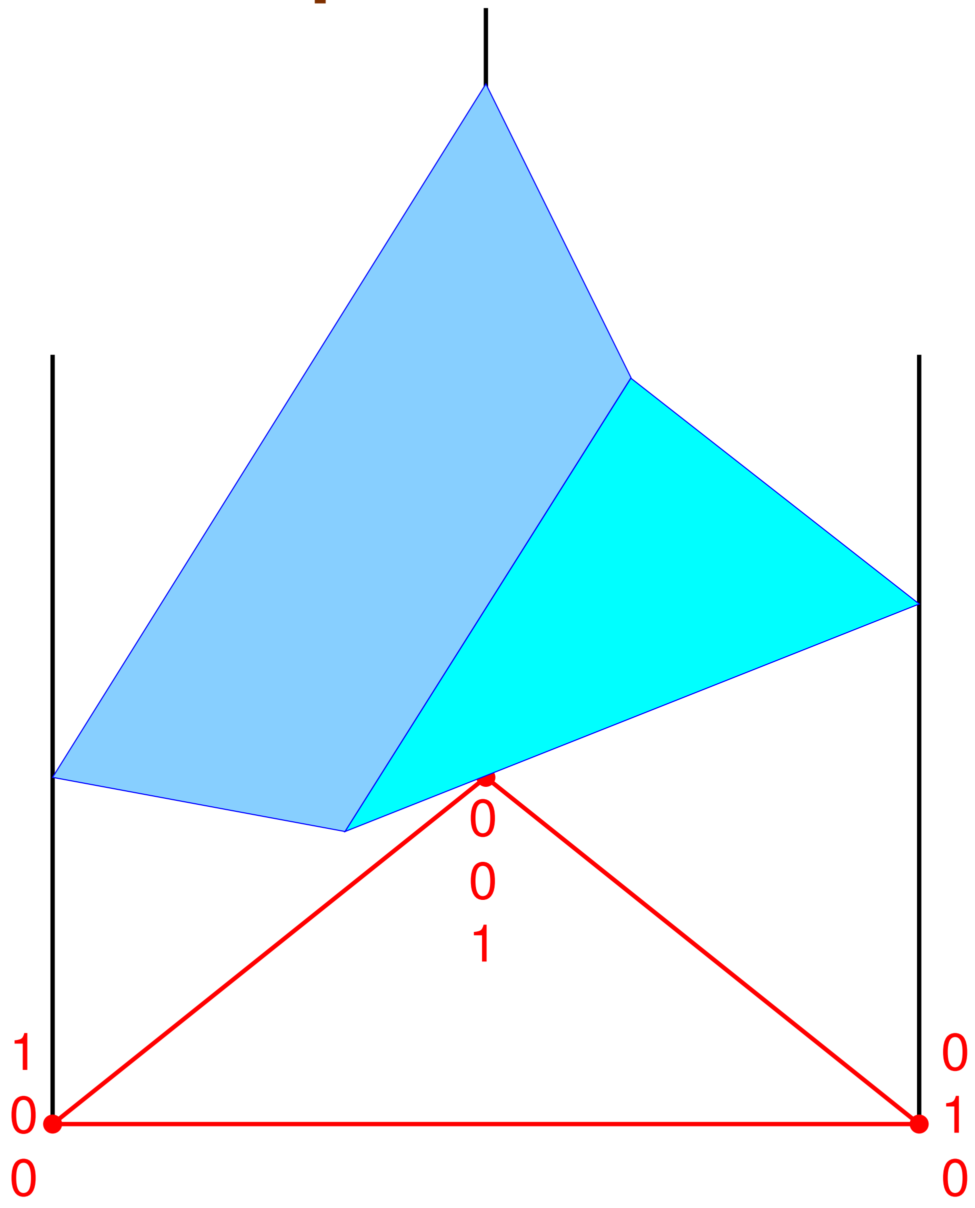


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Best responses to mixed strategy of player 1

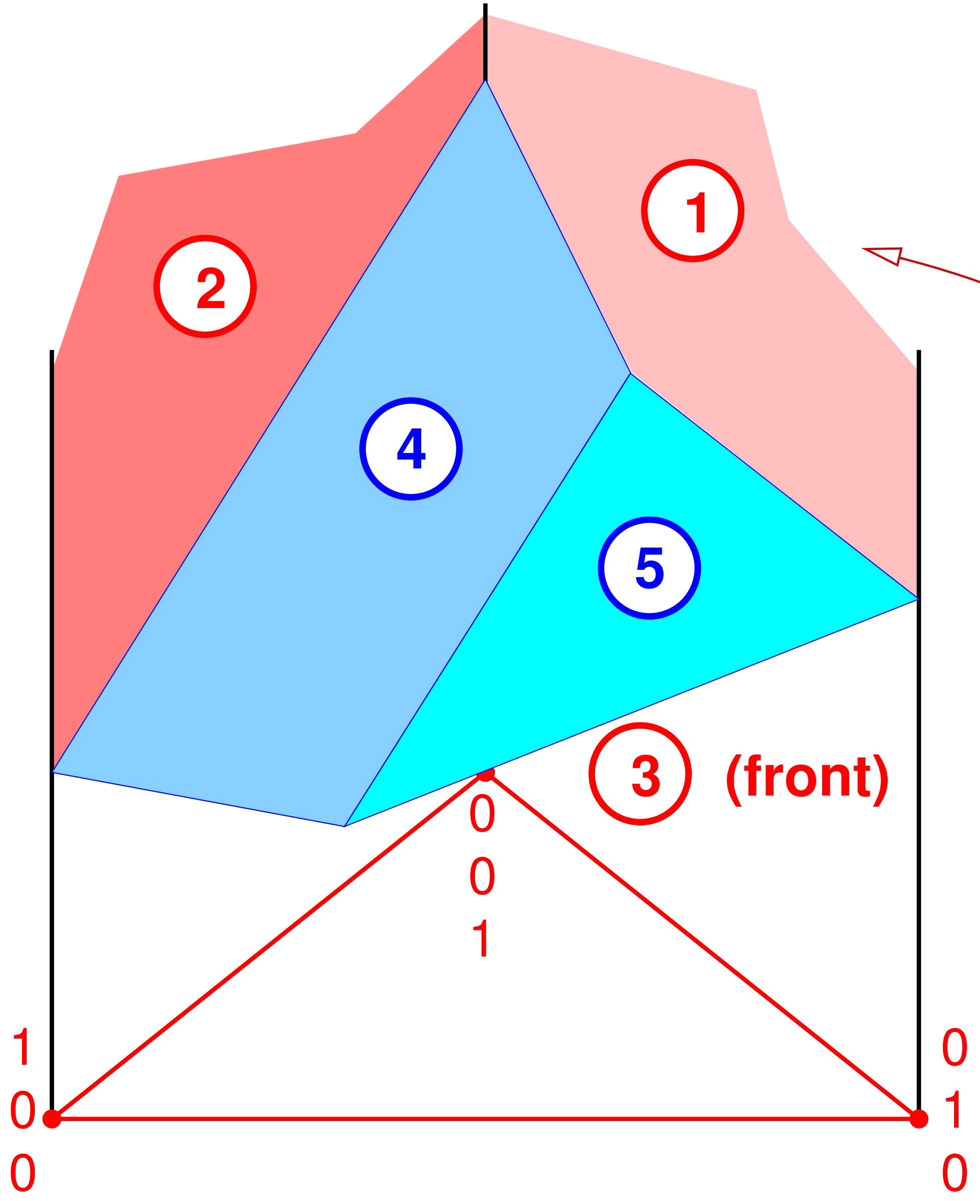


	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
player II

Best responses to mixed strategy of player 1



	4	5
1	2	1
2	1	3
3	4	3

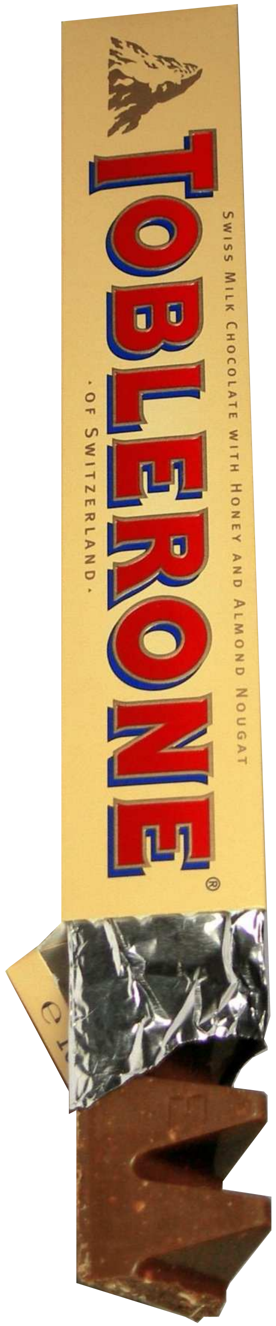
= B

payoffs to player II

best response polyhedron with facet labels

Alternative view

Chop off Toblerone prism



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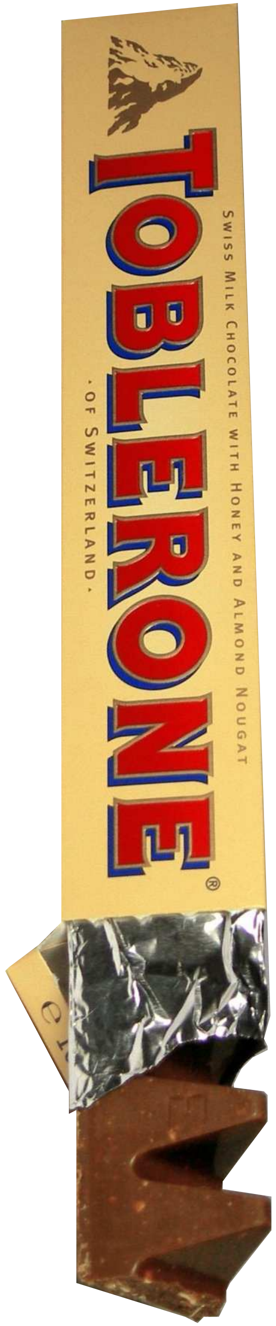

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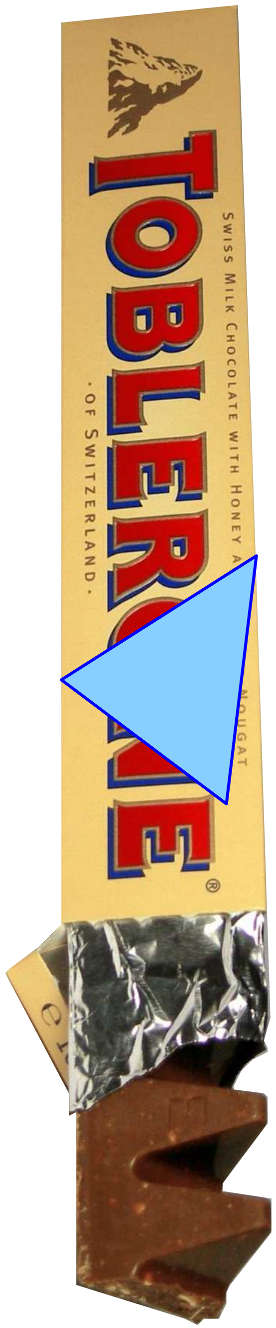

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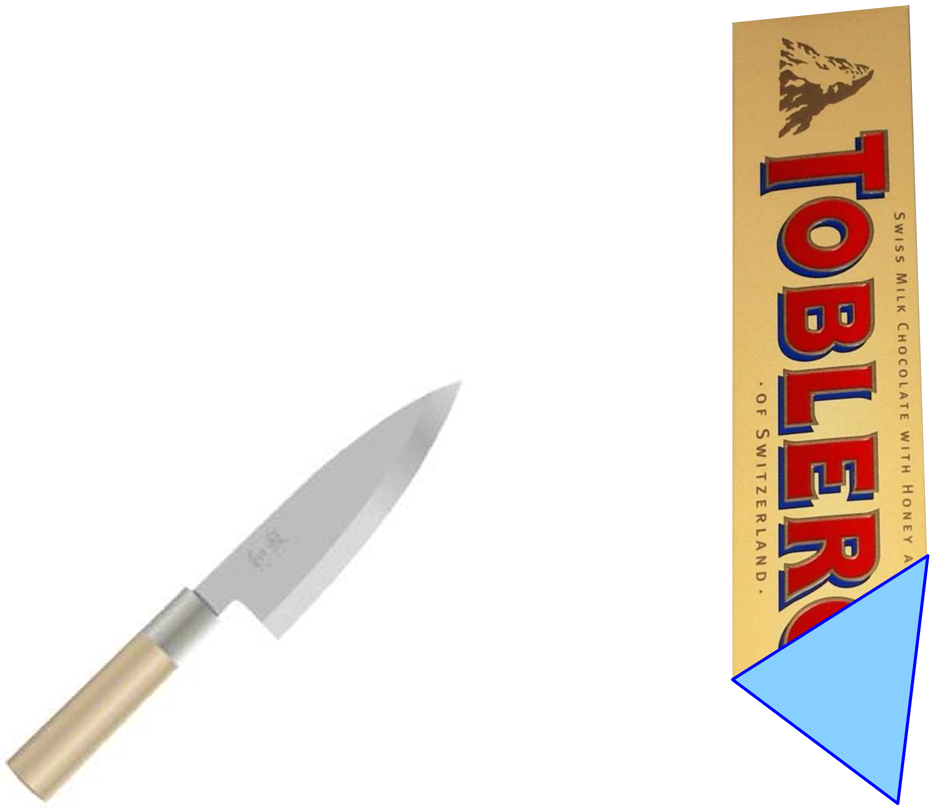
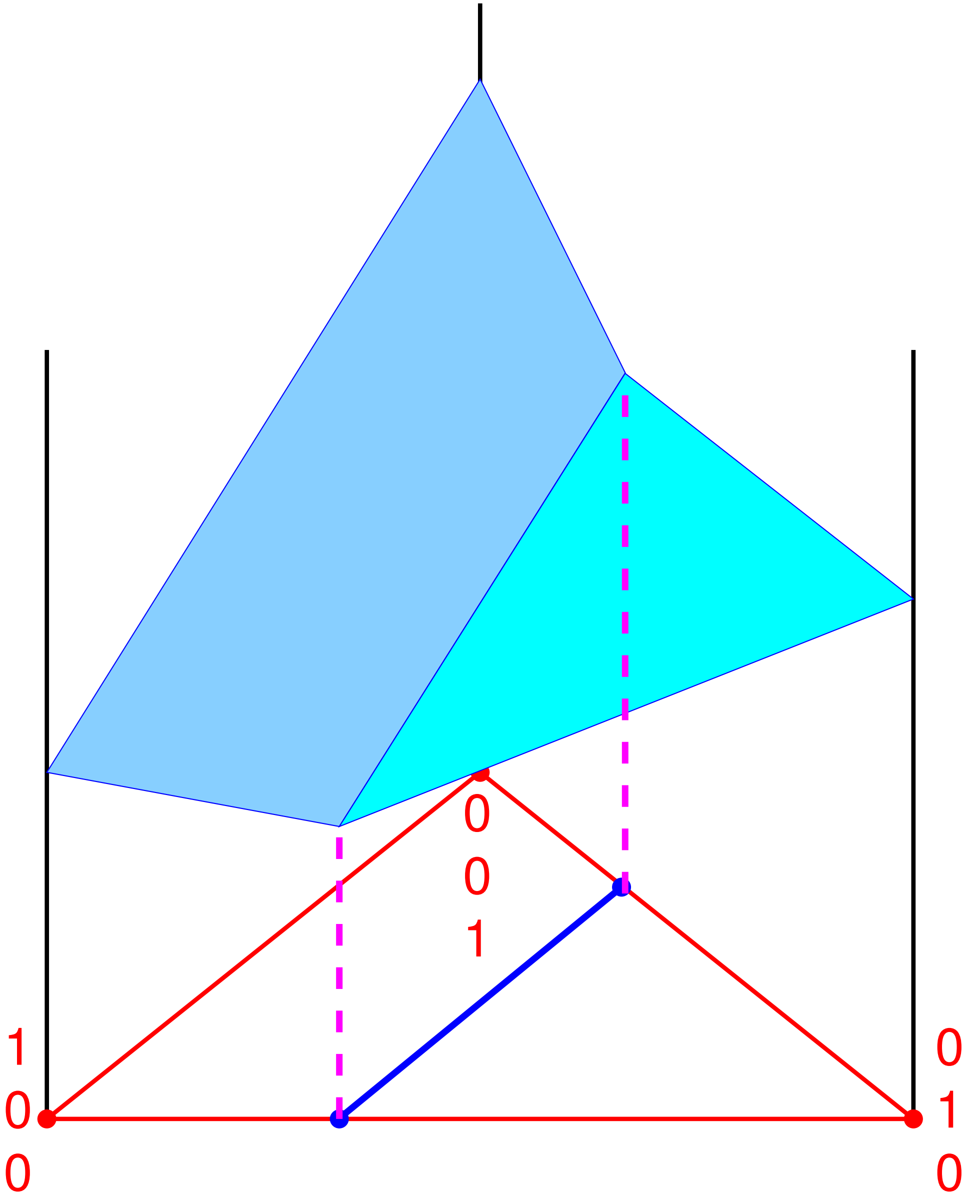


Chop off Toblerone prism

Best responses to mixed strategy of player 1



	4	5
1	2	1
2	1	3
3	4	3

= B

payoffs to
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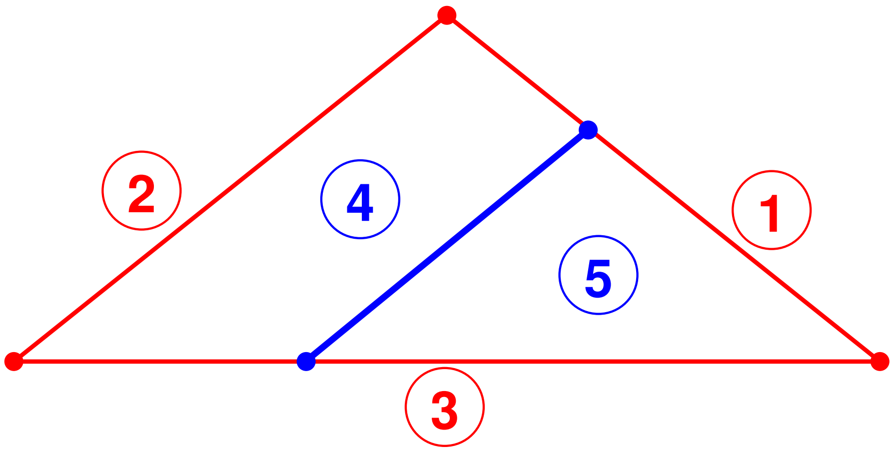
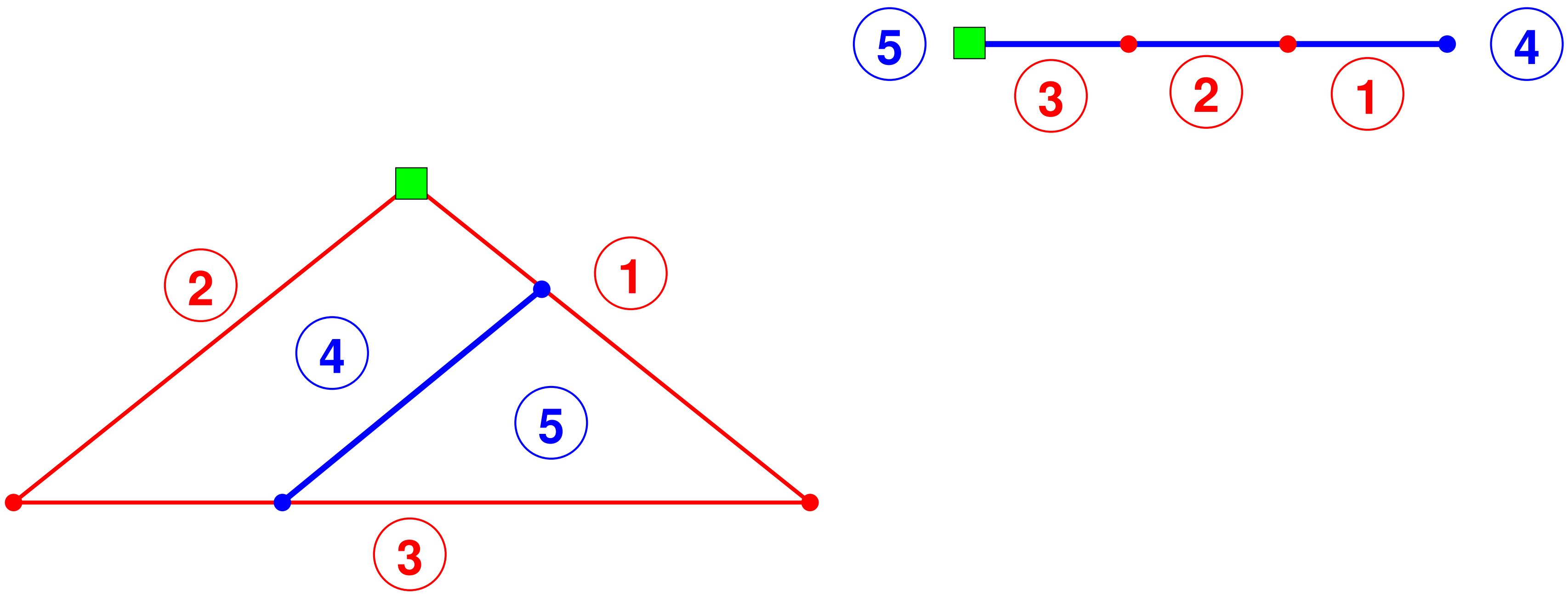
= B

payoffs to
player II



Equilibrium = completely labeled strategy pair



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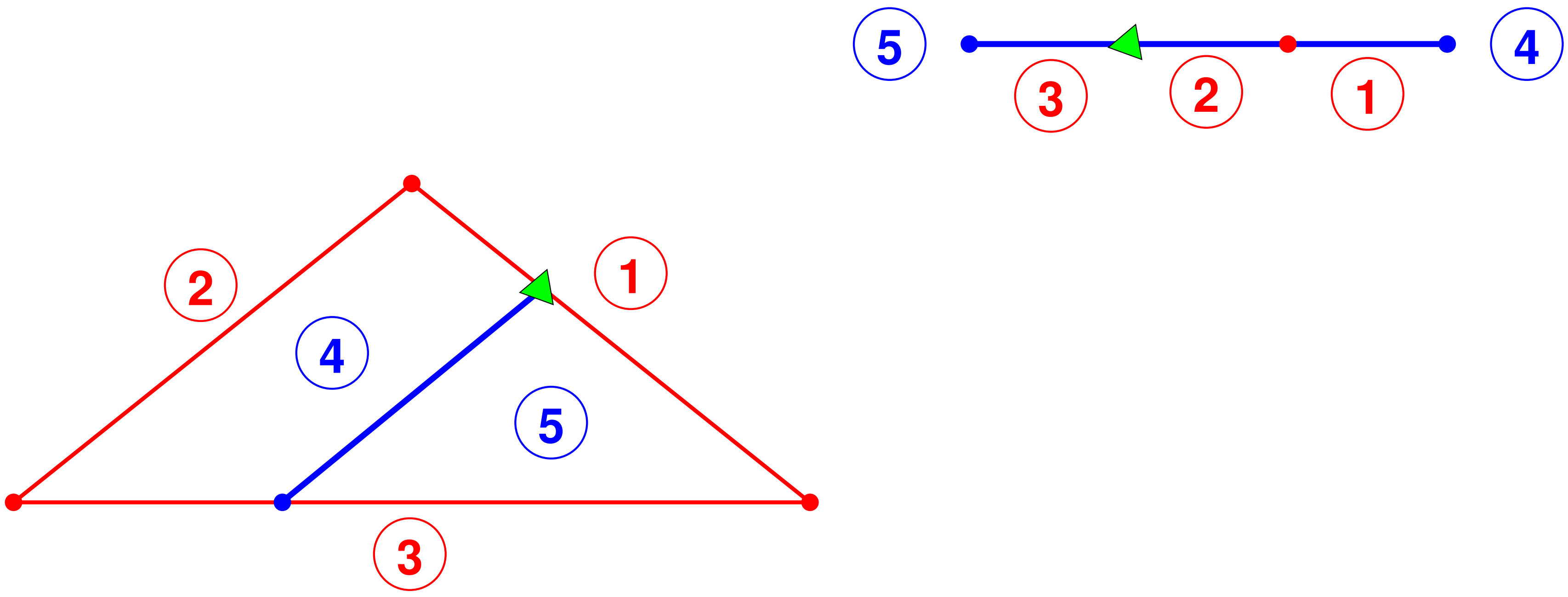
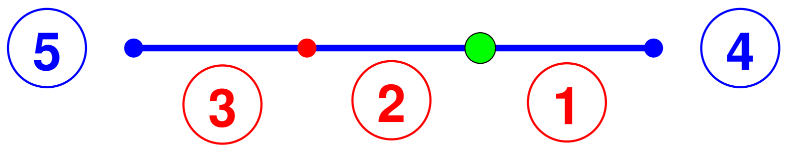
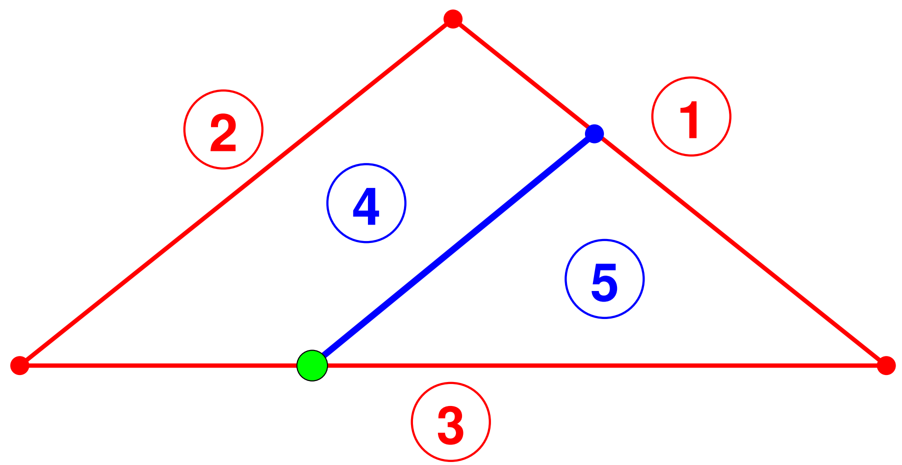


Equilibrium = completely labeled strategy pair





Constructing games using geometry

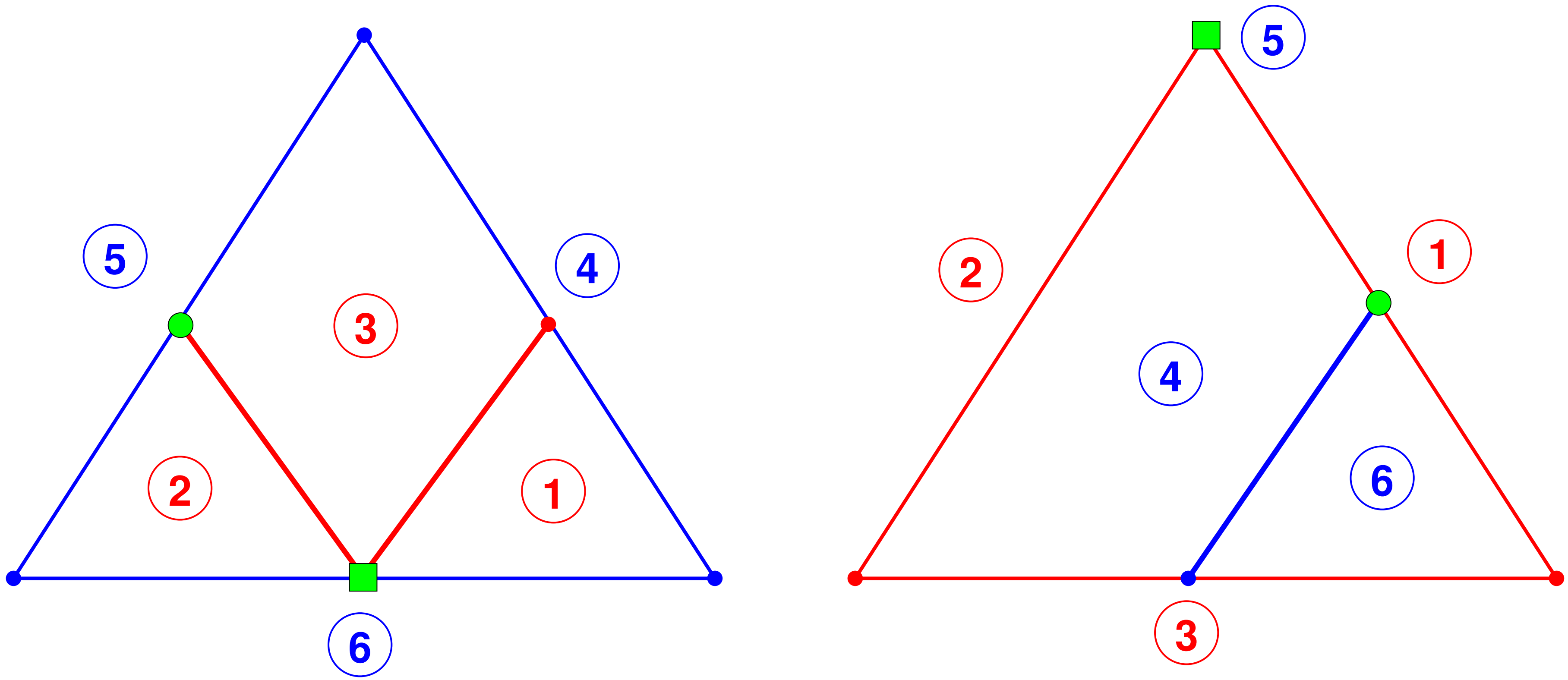
low dimension: 2, 3, (4) pure strategies:

subdivide mixed strategy simplex into
response regions, label suitably

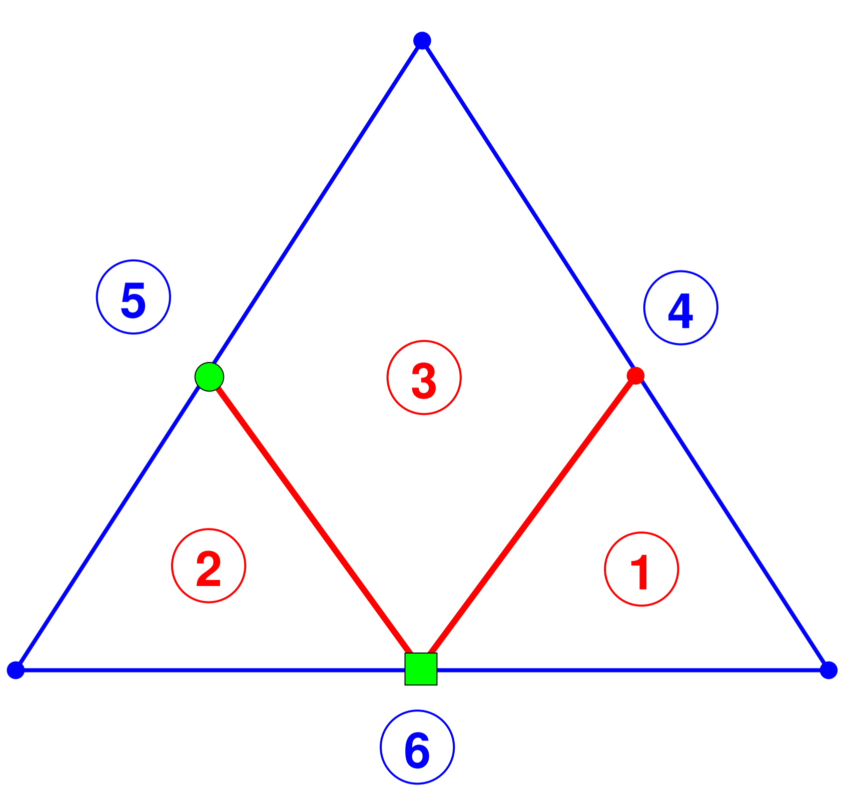
high dimension:

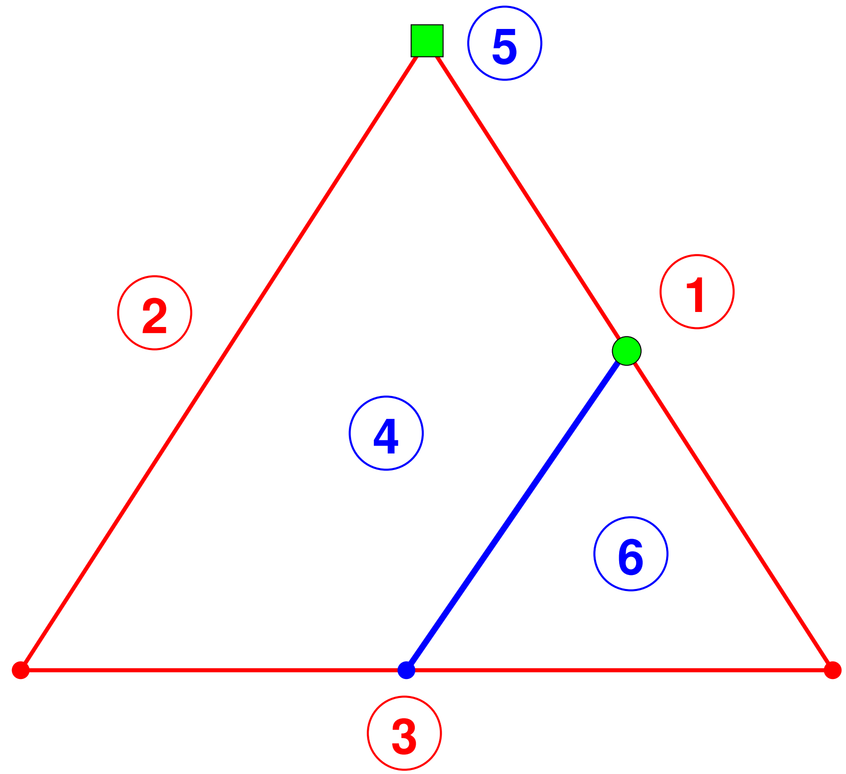
use polytopes with known combinatorial structure
e.g. for constructing games with many equilibria,
or long Lemke-Howson computations
[Savani & von Stengel, *FOCS 2004*,
Econometrica 2006]

Construct isolated non–quasi–strict equilibrium



Construct isolated non–quasi–strict equilibrium





$$A = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{matrix} ① \\ ② \\ ③ \end{matrix}$$

$$\begin{matrix} & ④ & ⑤ & ⑥ \end{matrix}$$
$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

Best response polyhedron H_2 for player 2

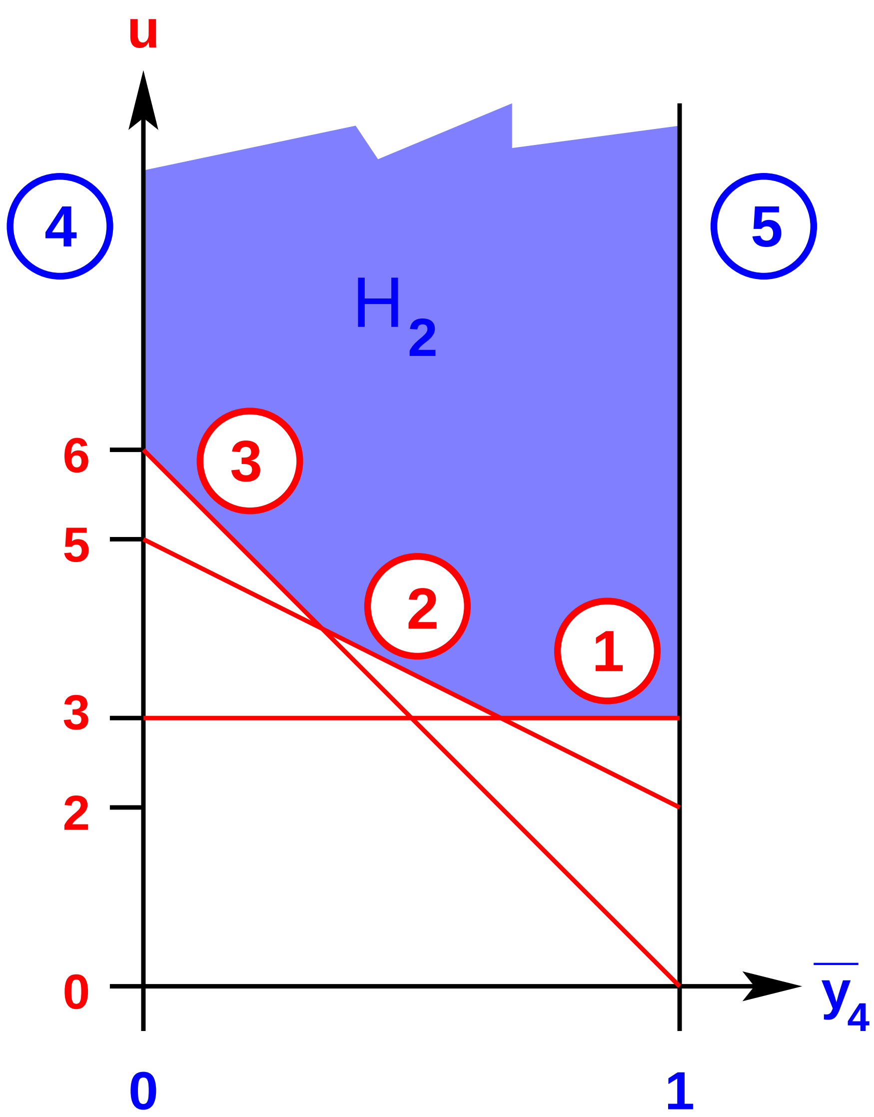
$$\begin{array}{c|cc|} & \bar{y}_4 & \bar{y}_5 \\ \hline ① & 3 & 3 \\ ② & 2 & 5 \\ ③ & 0 & 6 \\ \hline \end{array} = A$$

$$H_2 = \{ \; (\bar{y}_4, \bar{y}_5, u) \; | $$

$$\begin{array}{rl} ①: & 3\bar{y}_4 + 3\bar{y}_5 \leq u \\ ②: & 2\bar{y}_4 + 5\bar{y}_5 \leq u \\ ③: & 6\bar{y}_5 \leq u \end{array}$$

$$\bar{y}_4 + \bar{y}_5 = 1$$

$$\begin{array}{rl} ④: & \bar{y}_4 \geq 0 \\ ⑤: & \bar{y}_5 \geq 0 \;\; \} \end{array}$$



Best response polytope Q for player 2

	y_4	y_5
①	3	3
②	2	5
③	0	6

$= A$

$$Q = \{ y \mid Ay \leq 1,\ y \geq 0 \}$$

$$Q = \{ (y_4, y_5) \mid$$

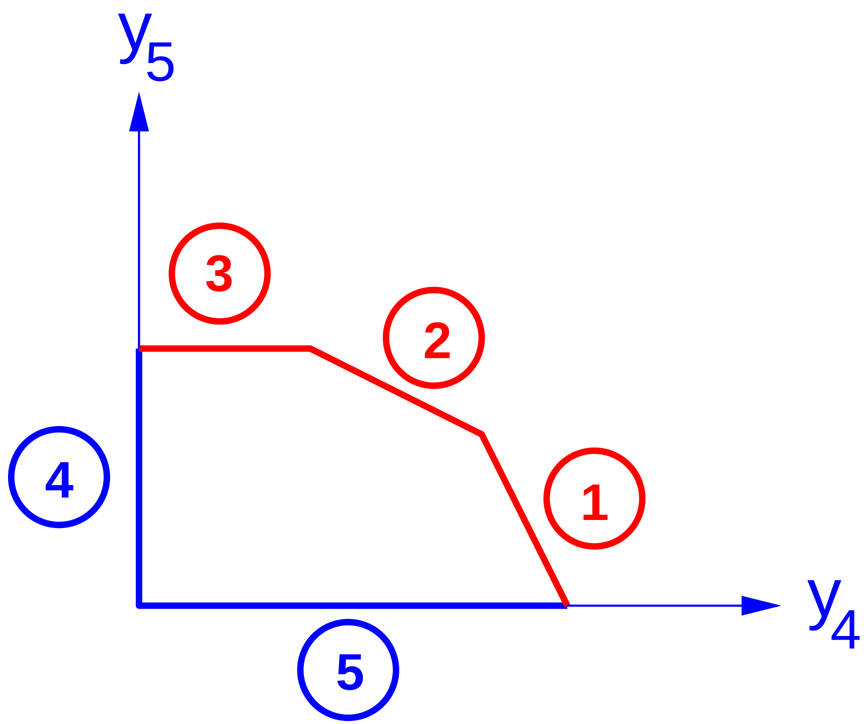
① : $3y_4 + 3y_5 \leq 1$

② : $2y_4 + 5y_5 \leq 1$

③ : $6y_5 \leq 1$

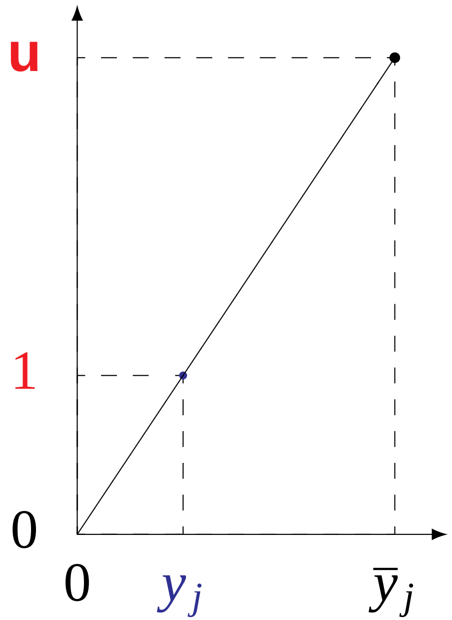
④ : $y_4 \geq 0$

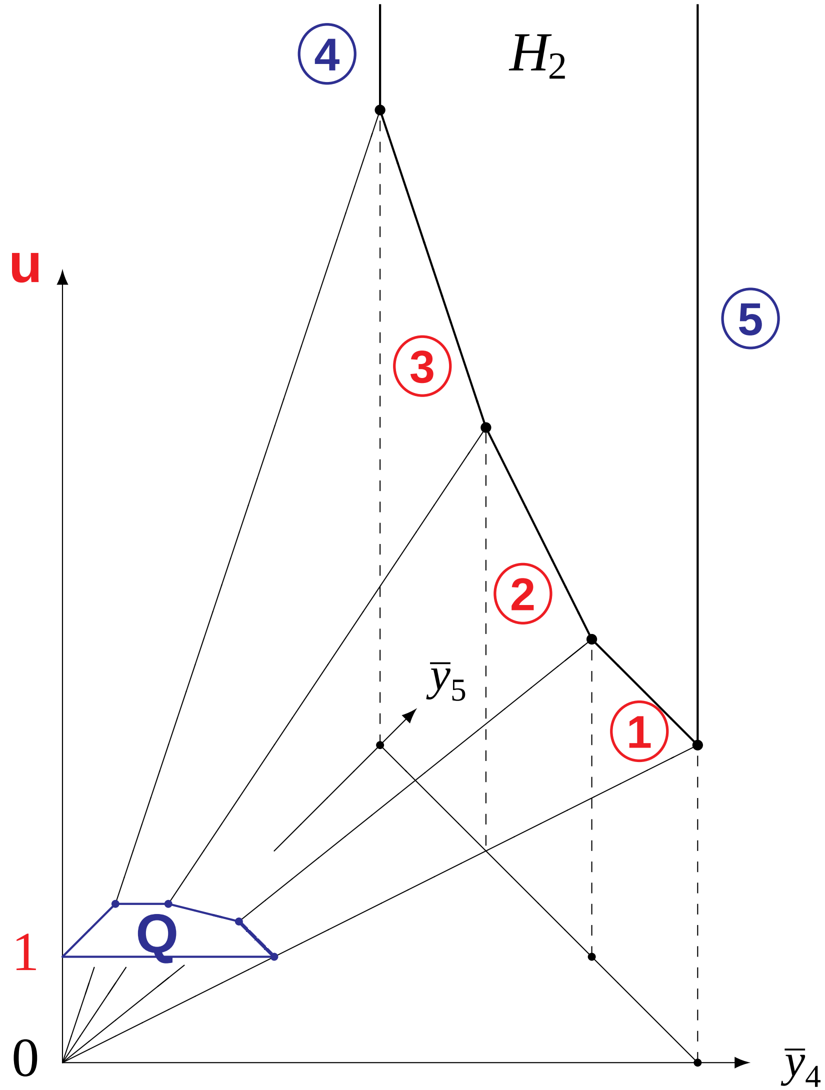
⑤ : $y_5 \geq 0 \}$



Projective transformation

H_2, **Q** same face incidences





Best response polytope Q for player 2

	y_4	y_5
①	3	3
②	2	5
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$= A$

$$Q = \{\, y \mid Ay \le 1,\ y \ge 0 \,\}$$

$Q = \{\ (y_4, y_5)\ |$

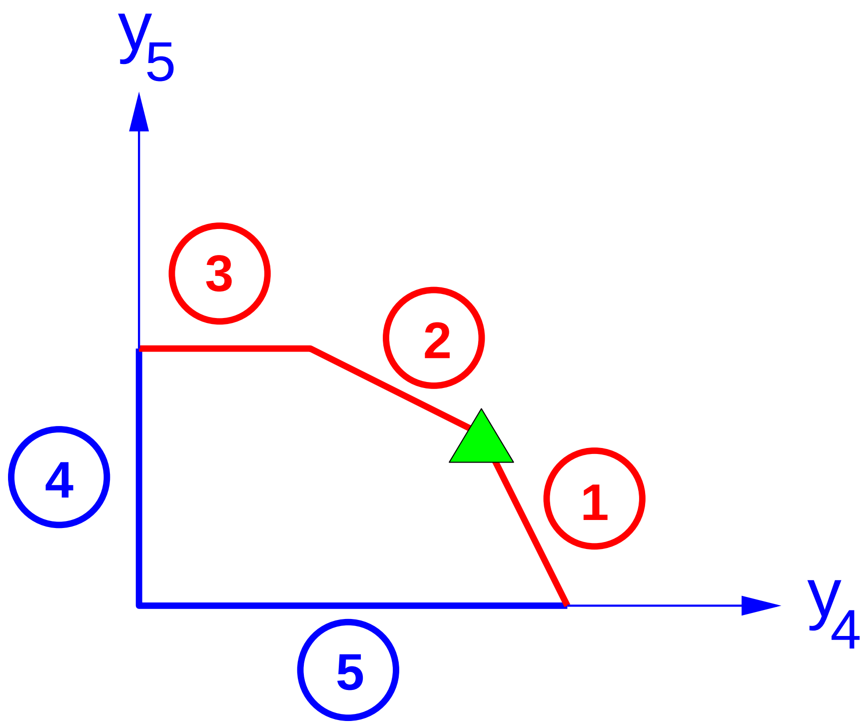
① : $3y_4 + 3y_5 \le 1$

② : $2y_4 + 5y_5 \le 1$

③ : $6y_5 \le 1$

④ : $y_4 \ge 0$

⑤ : $y_5 \ge 0\ \}$

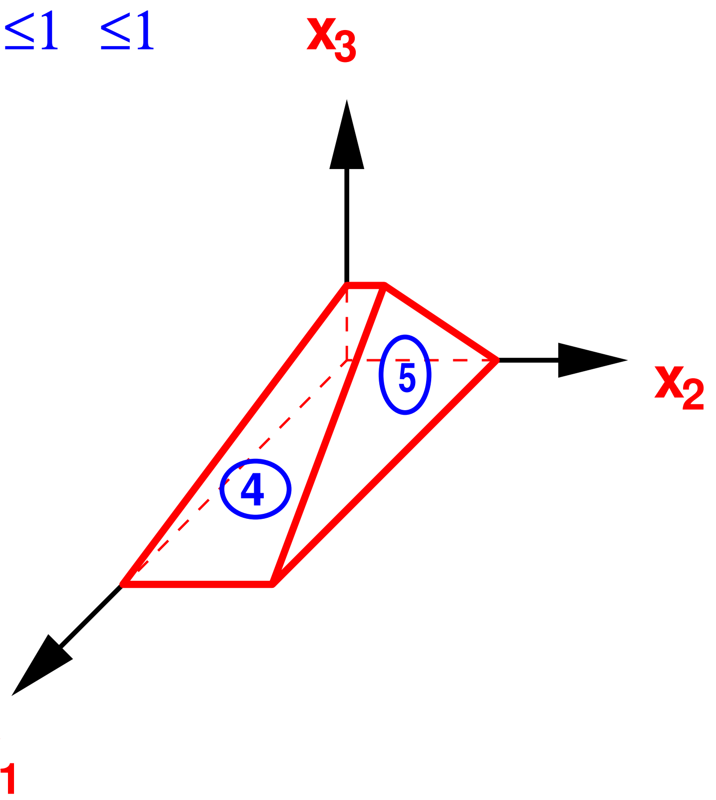


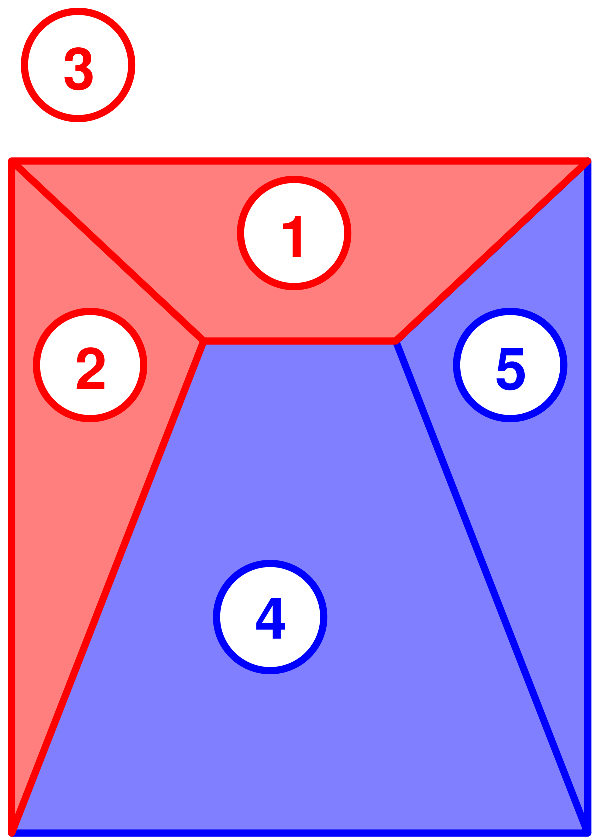
Best response polytope P for player 1

x_1	1	0
x_2	0	2
x_3	4	3
	≤ 1	≤ 1

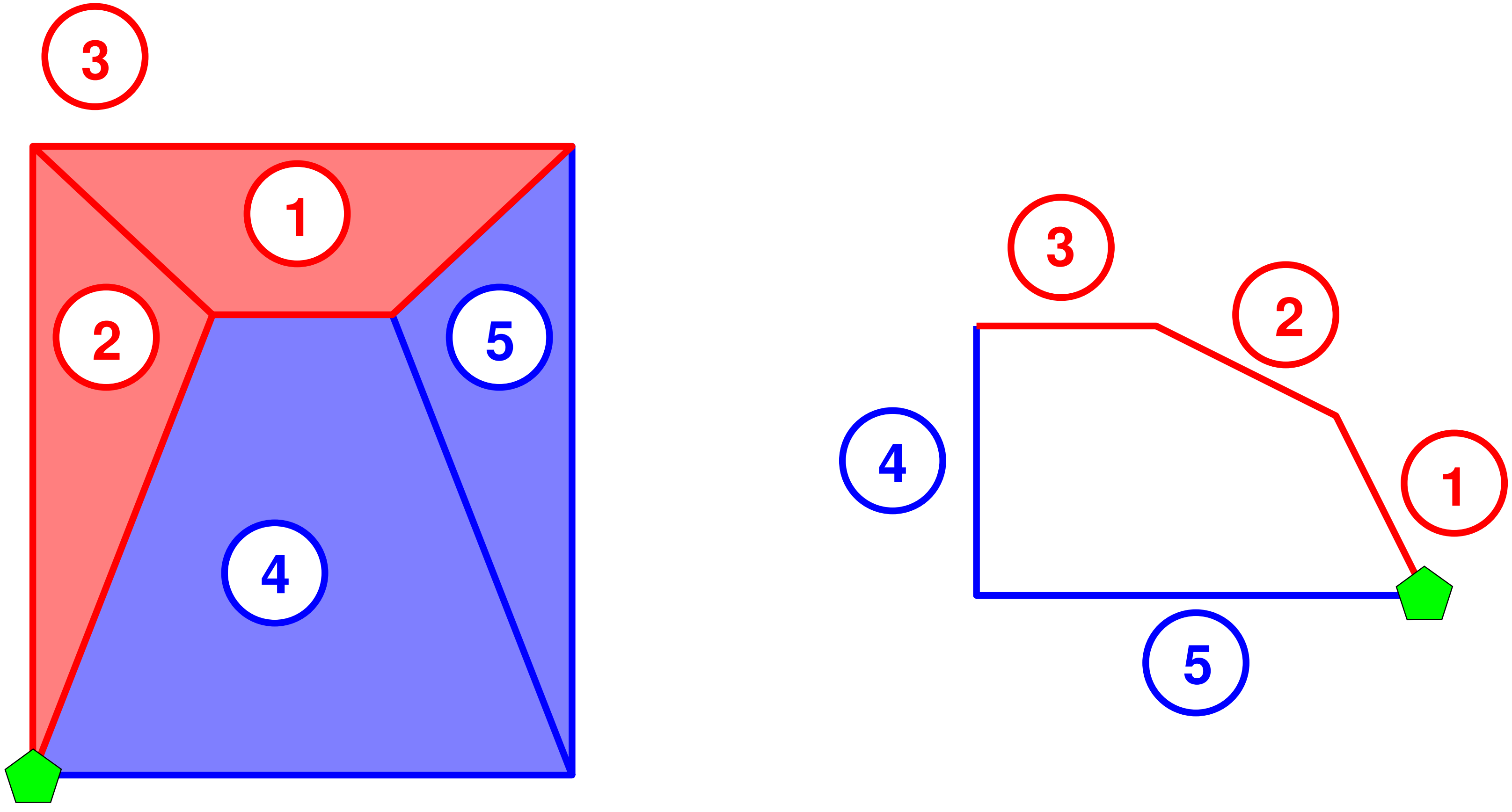
$= B$

$$P = \{ x \mid x \ge 0, \; x^\top B \le 1 \}$$



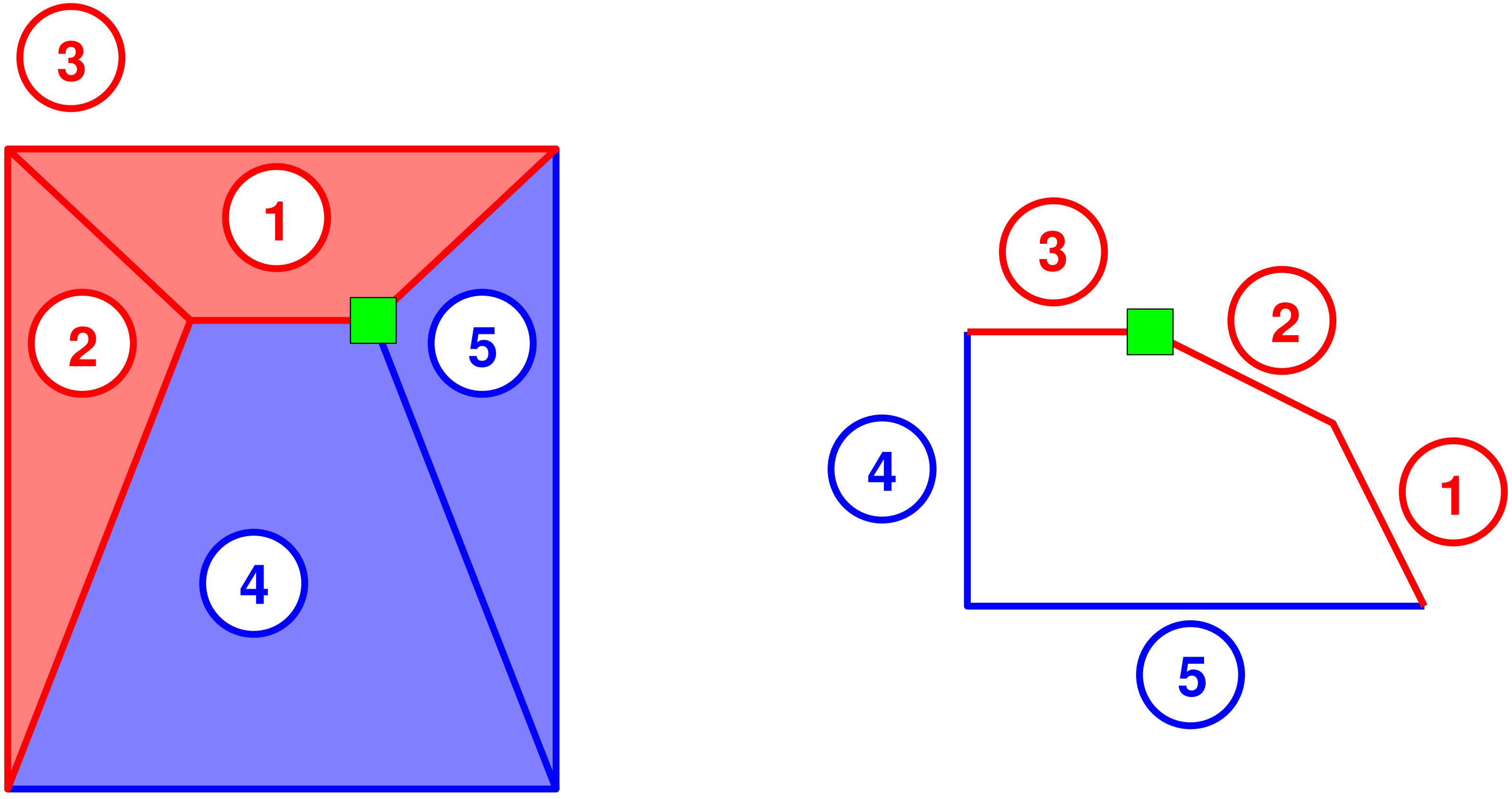


Equilibrium = completely labeled pair



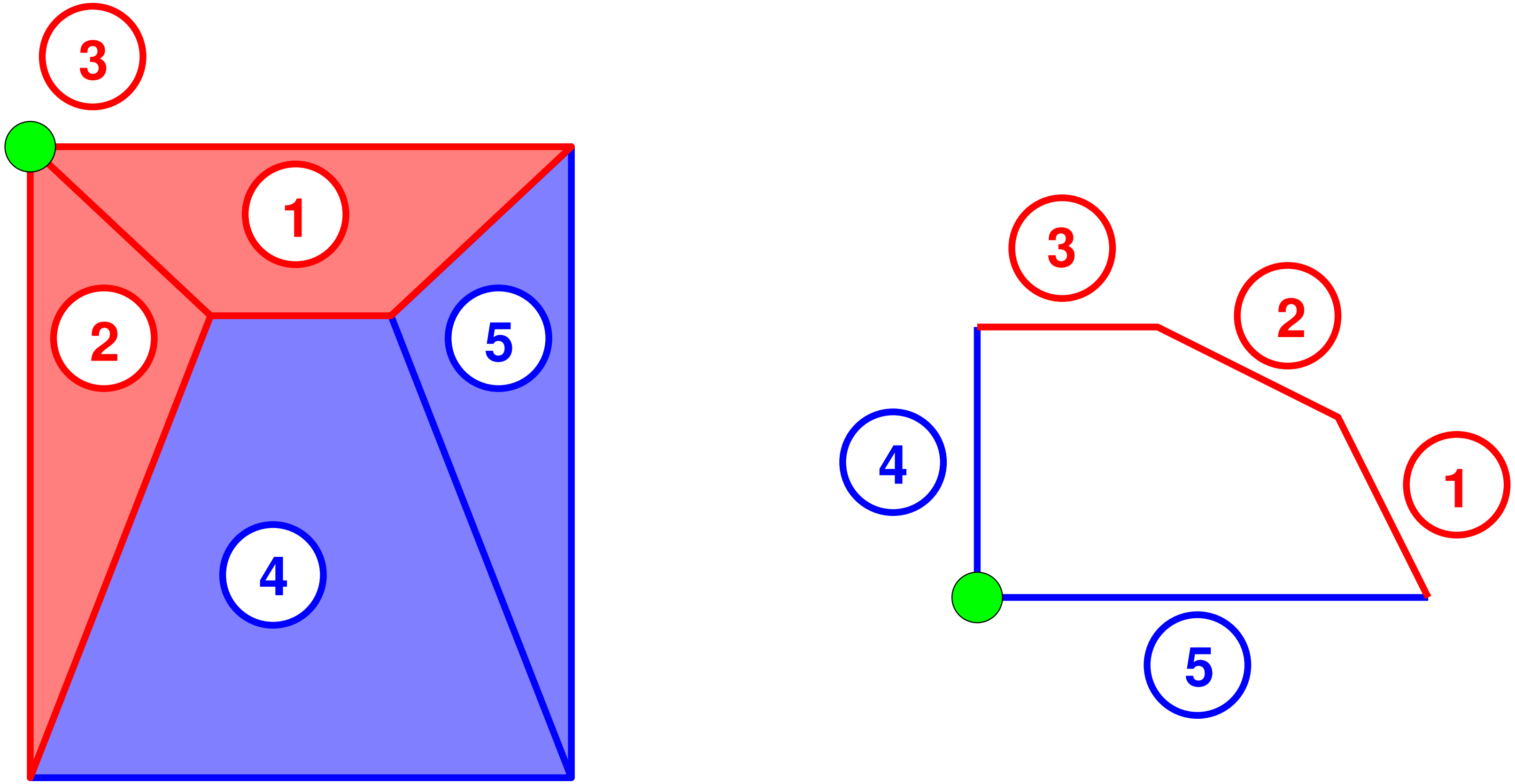
pure equilibrium

Equilibrium = completely labeled pair

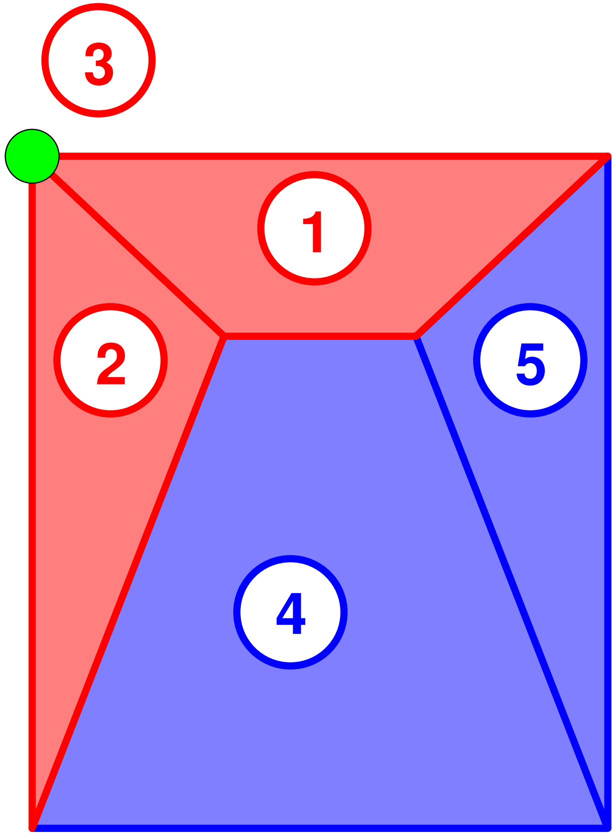


mixed equilibrium

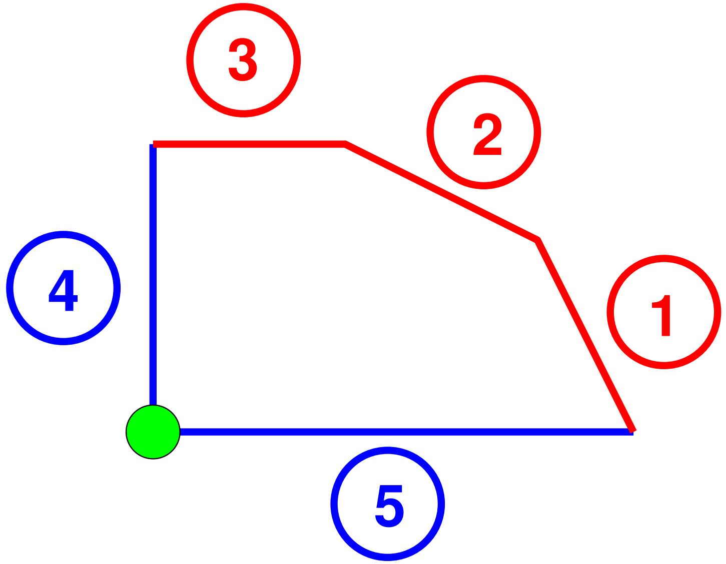
The Lemke–Howson algorithm



The Lemke–Howson algorithm



Drop label 3



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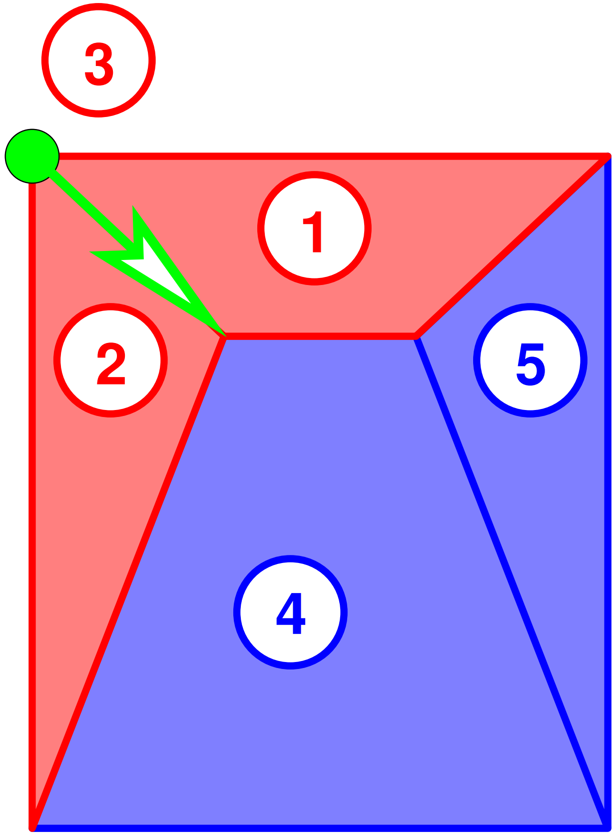

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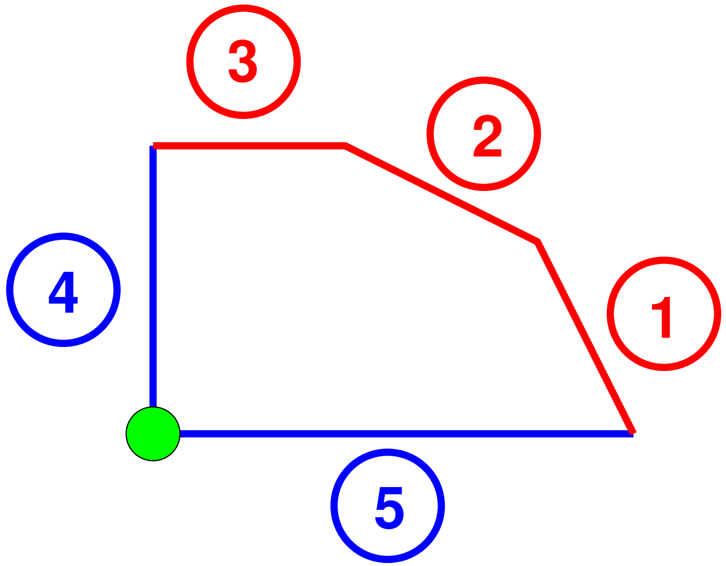

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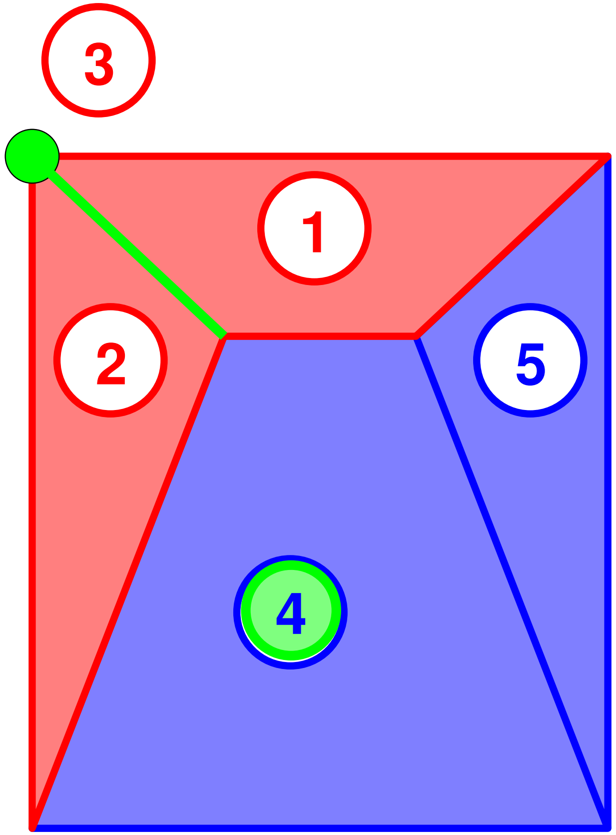

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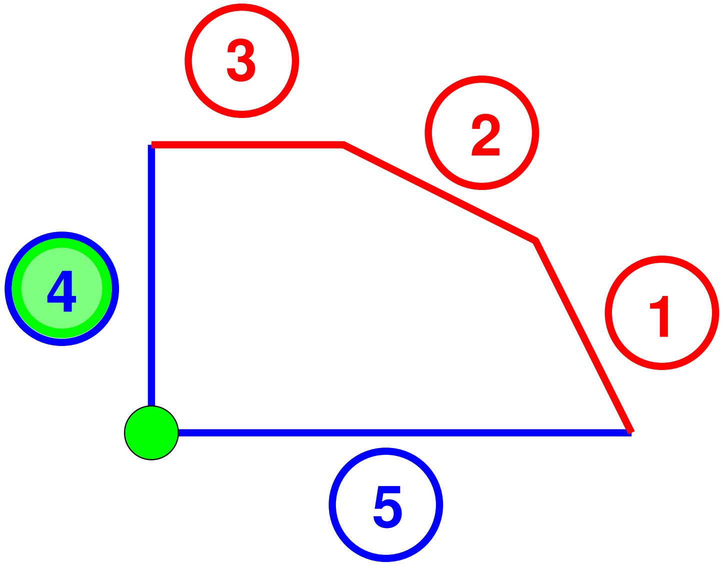

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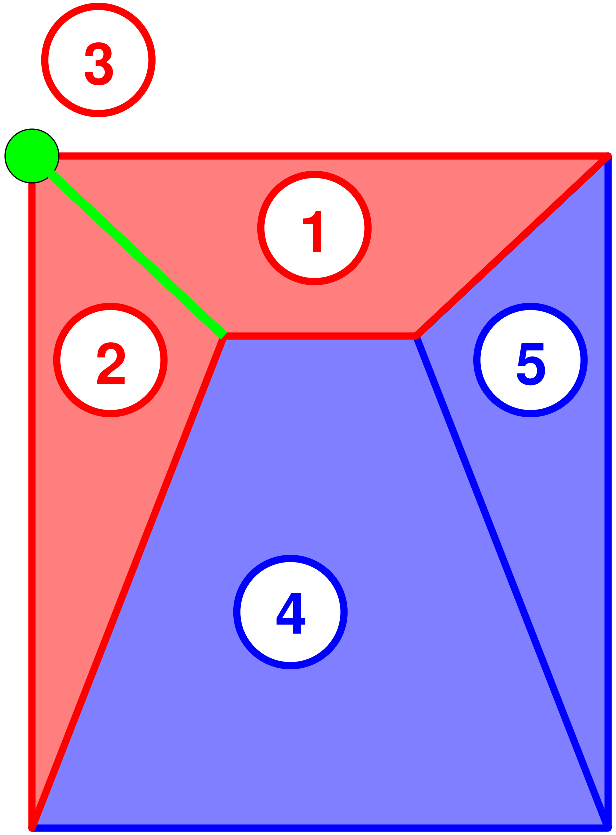

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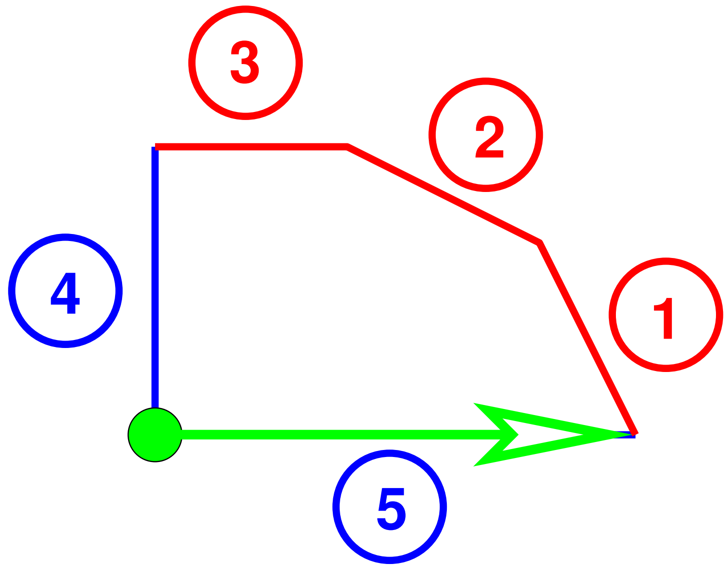

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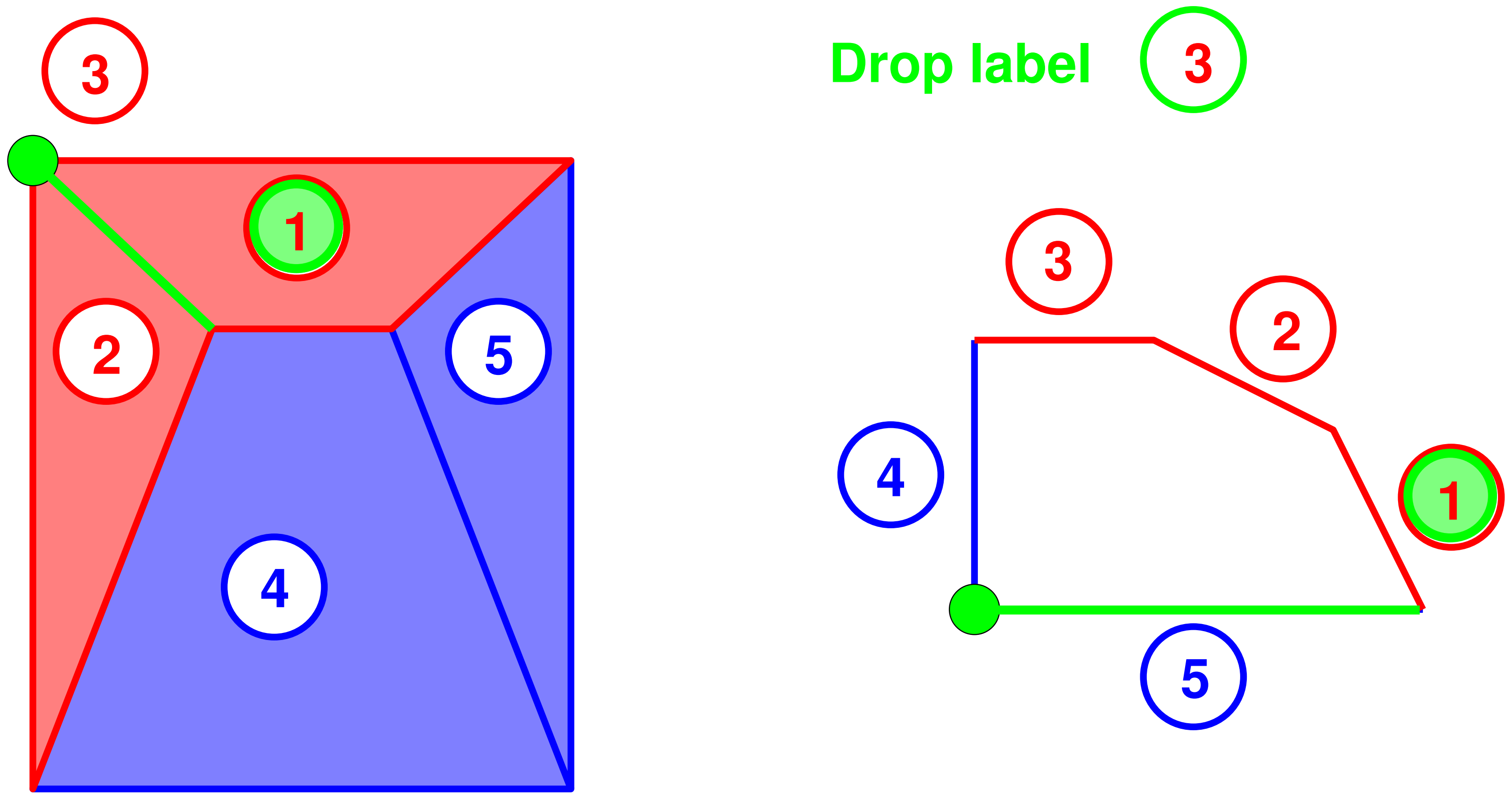

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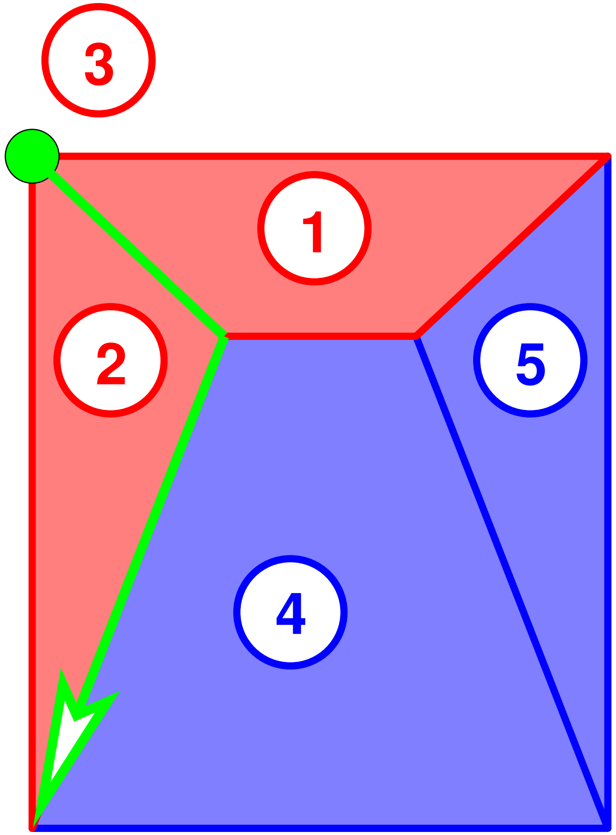

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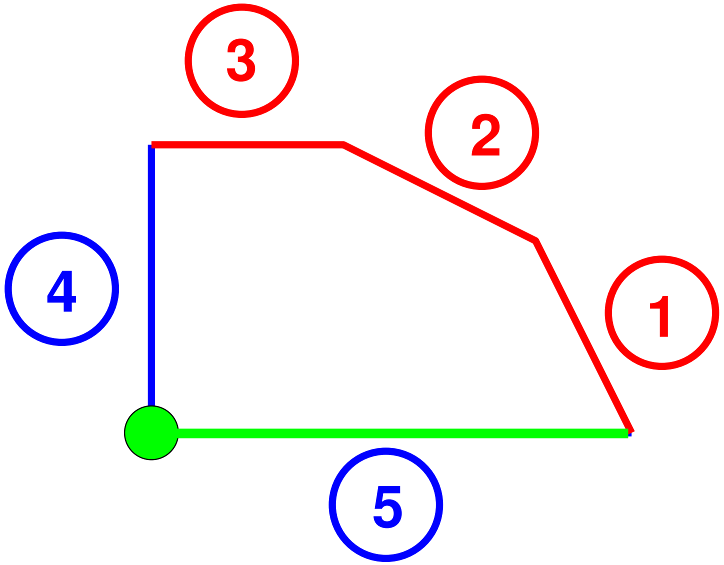

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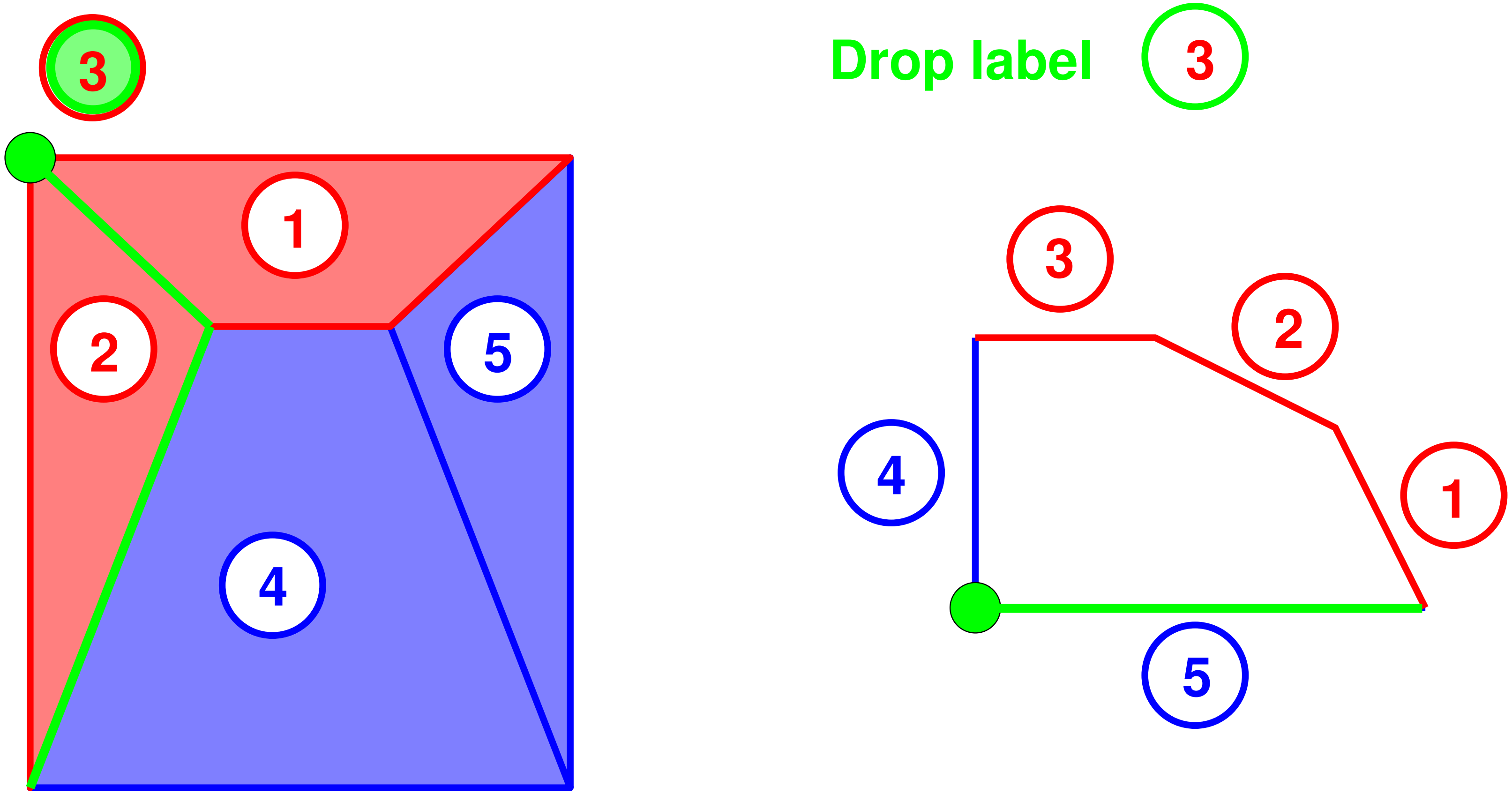

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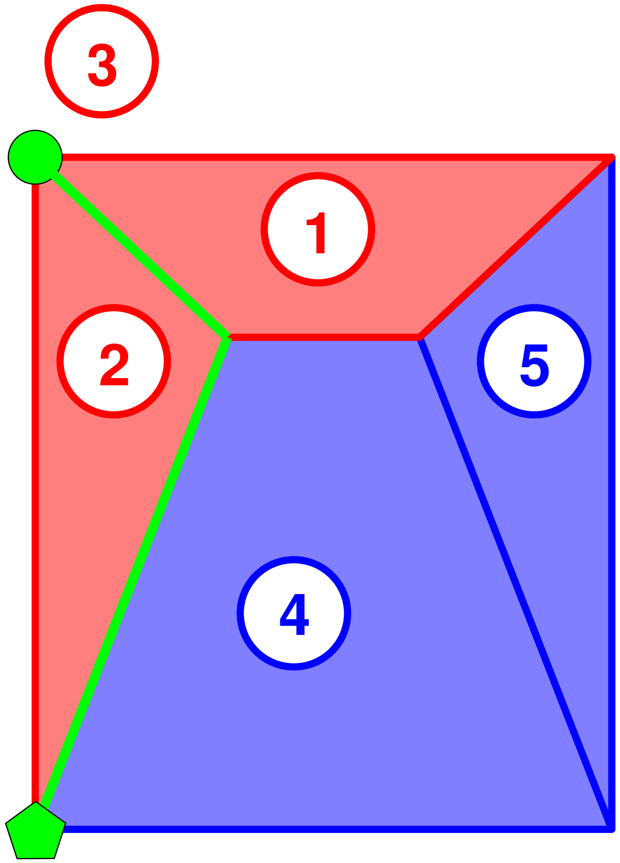

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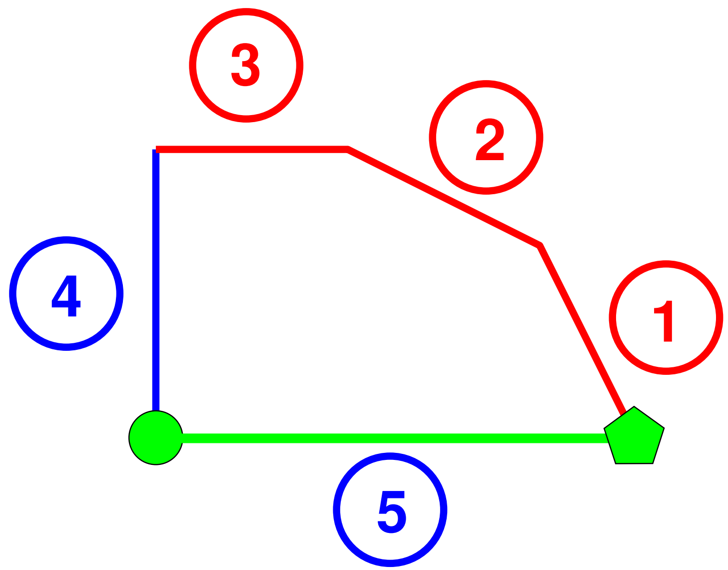


Why Lemke-Howson works

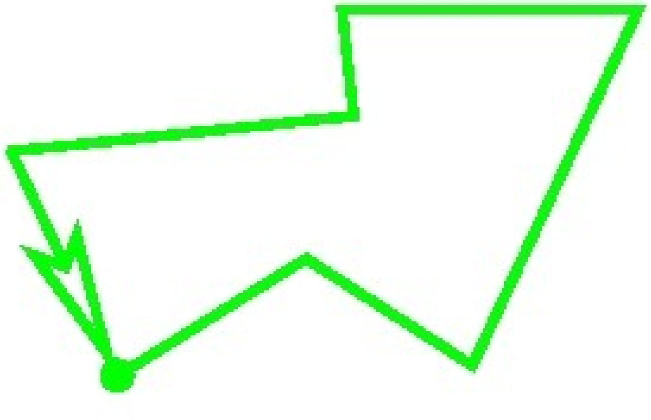
LH finds at least one Nash equilibrium because

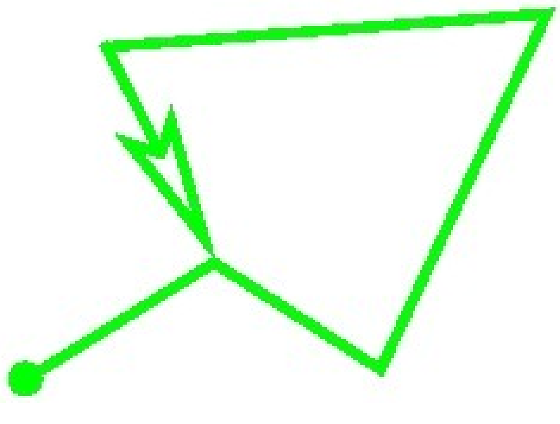
- finitely many "vertices"

for nondegenerate (generic) games:

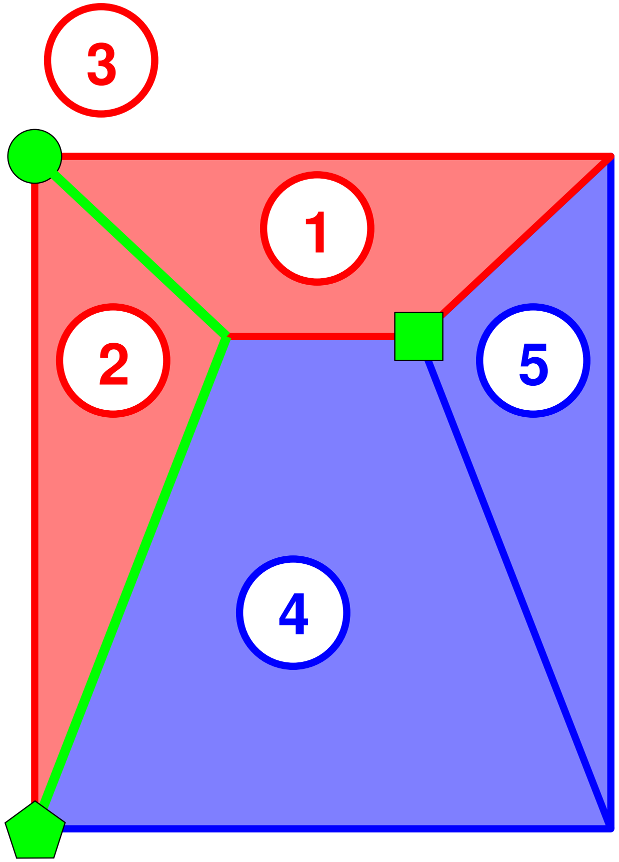
- **unique** starting edge given missing label
- **unique** continuation

$\Rightarrow$ precludes "coming back" like here:

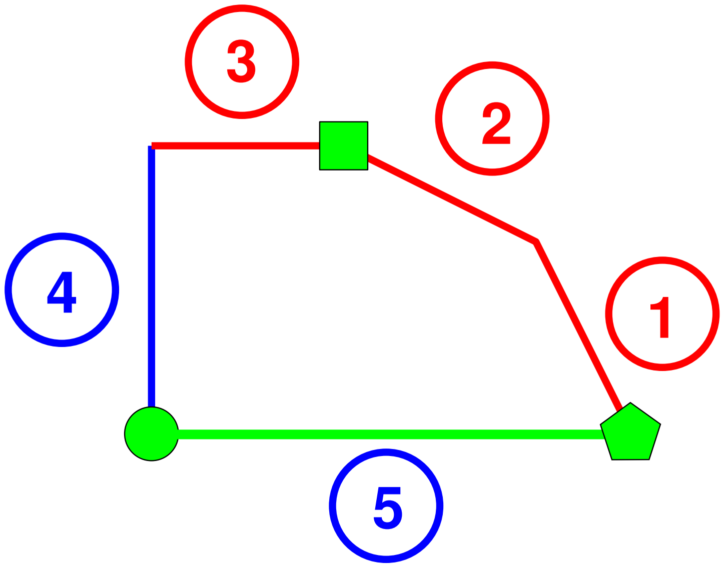




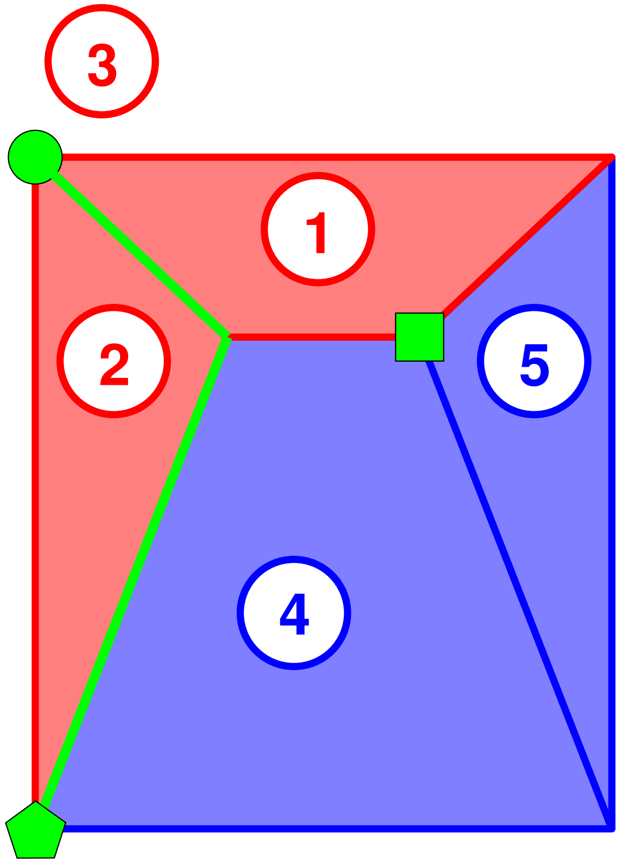
The Lemke–Howson algorithm



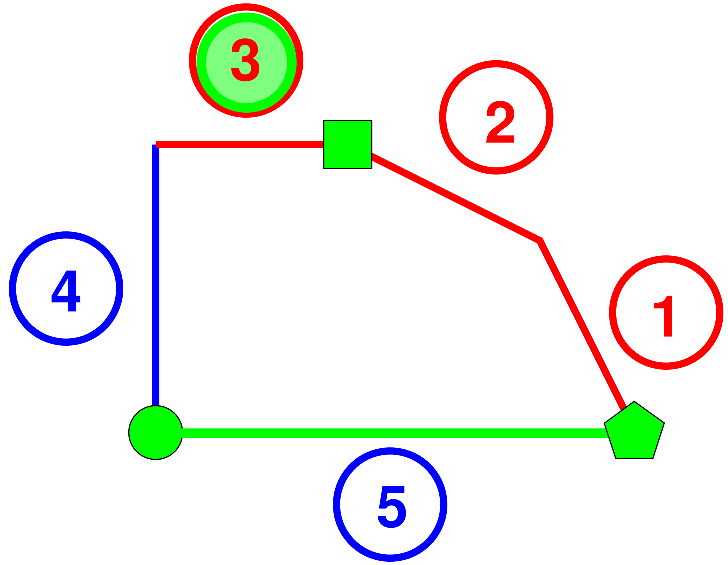
Drop label 3



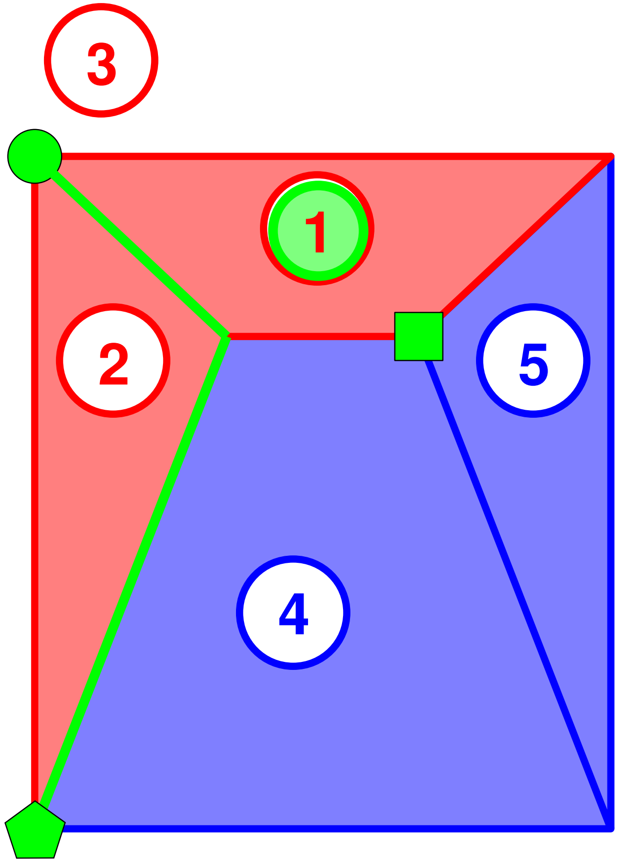
The Lemke–Howson algorithm



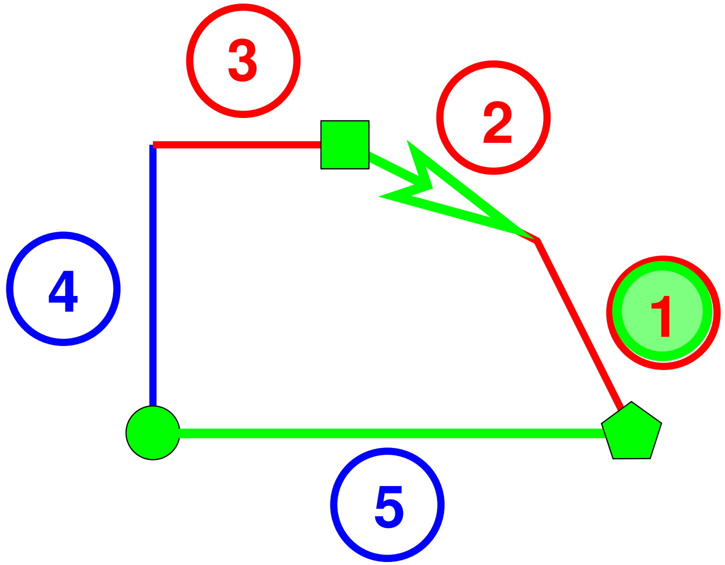
Drop label 3 from ■



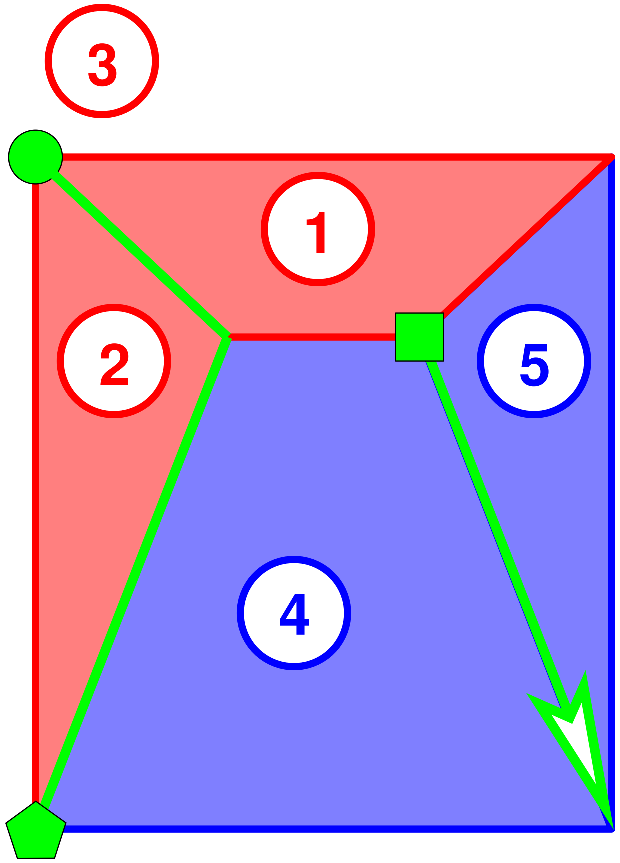
The Lemke–Howson algorithm



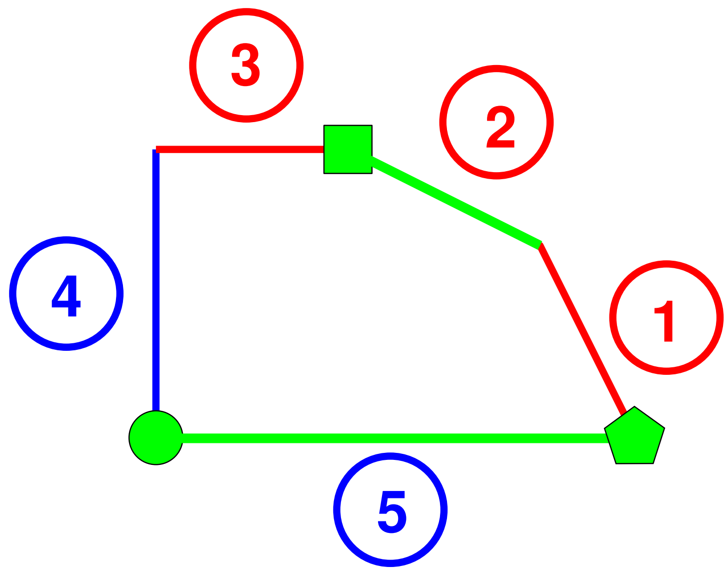
Drop label 3 from ■



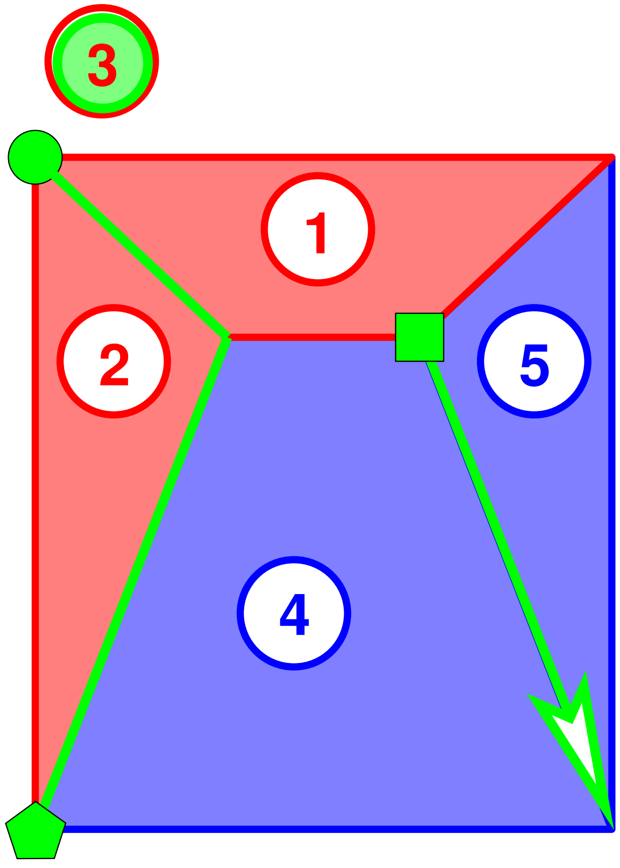
The Lemke–Howson algorithm

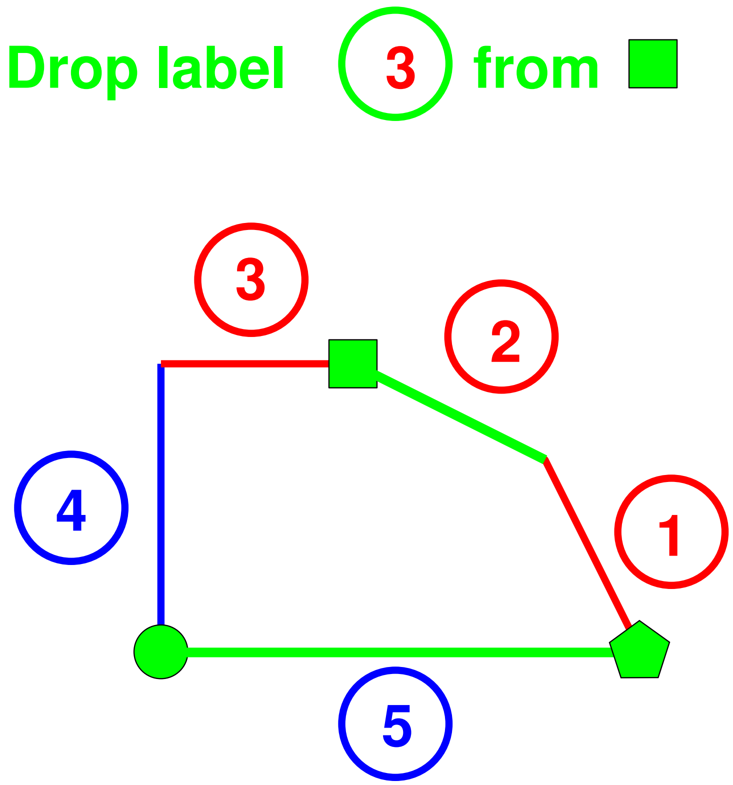


Drop label 3 from ■

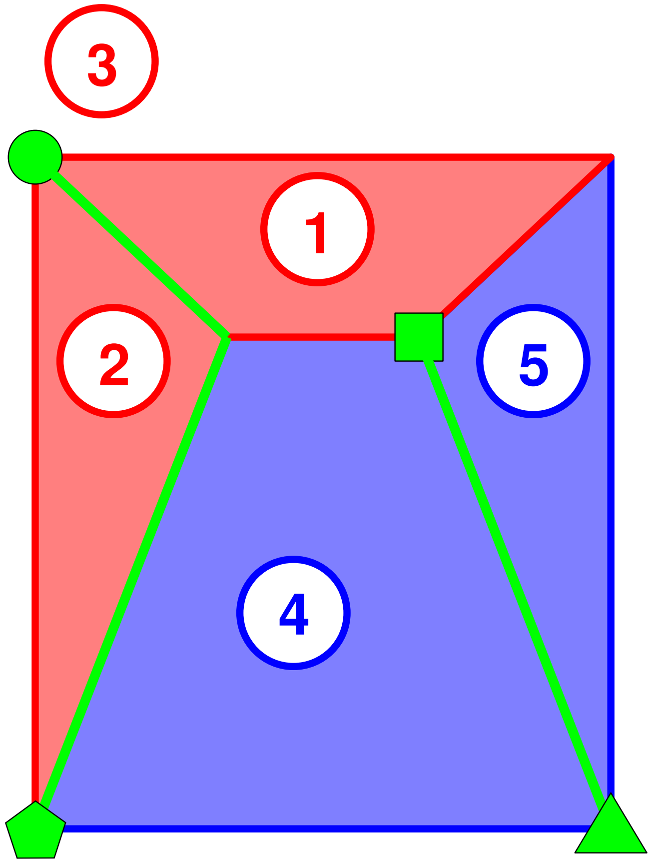


The Lemke–Howson algorithm

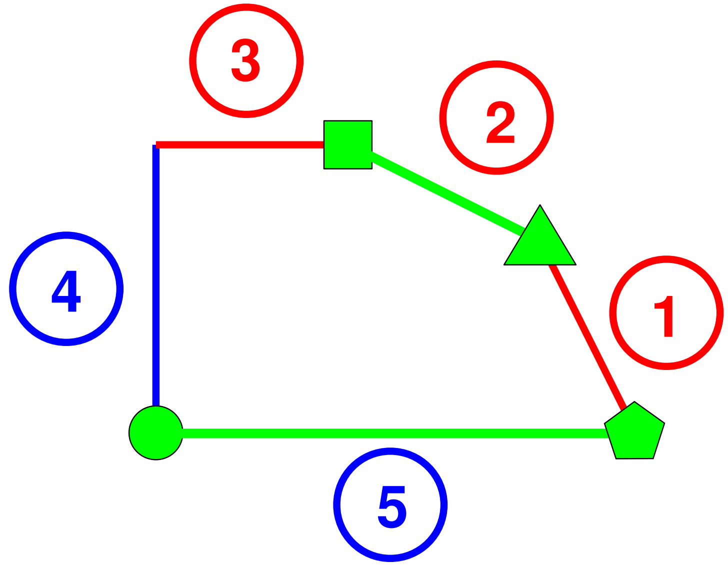




The Lemke–Howson algorithm



Drop label 3 from ■



Complexity of Lemke-Howson

- finds at least one Nash equilibrium,
  pivots like Simplex algorithm for linear programming

- Simplex may be exponential [Klee-Minty cubes]

- exponentially many steps of Lemke-Howson
  for **any** dropped label?

- **Yes**! This is our result.

Our result

There are $d \times d$ games with exactly one Nash equilibrium, for which the Lemke-Howson algorithm takes $\geq \phi^{3d/4}$ many steps for **any dropped label** (with Golden Ratio $\phi = (\sqrt{5} + 1) / 2 = 1.618...$)

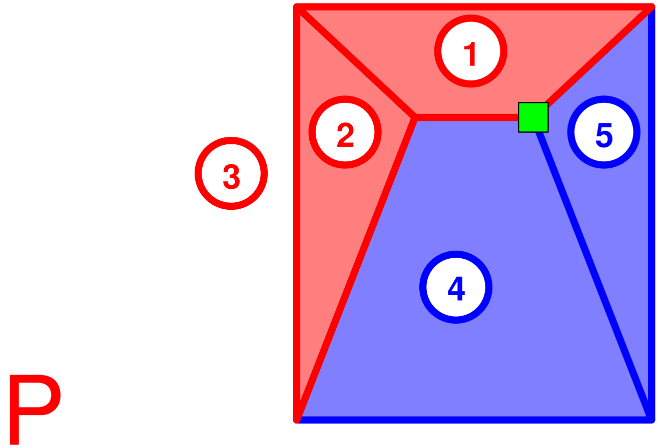
We will show this extending

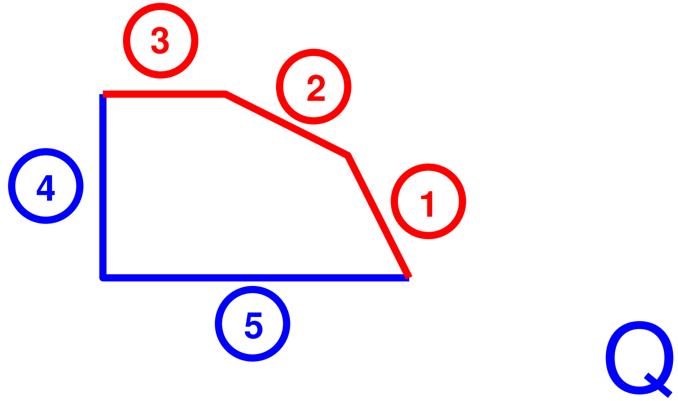
[Morris 1994] - exponentially long Lemke paths (finds symmetric equilibria of symmetric games)

[von Stengel 1999] - games with many equilibria

using **dual cyclic polytopes**

Vertices as bit patterns

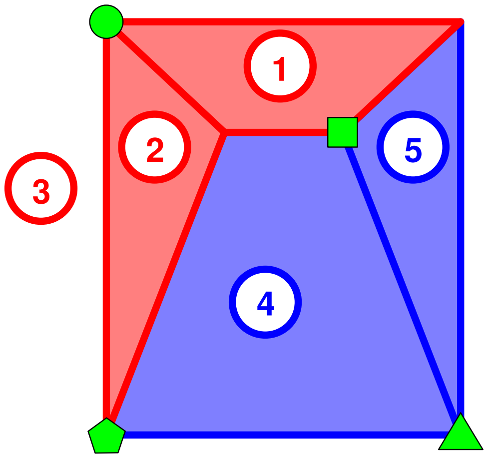




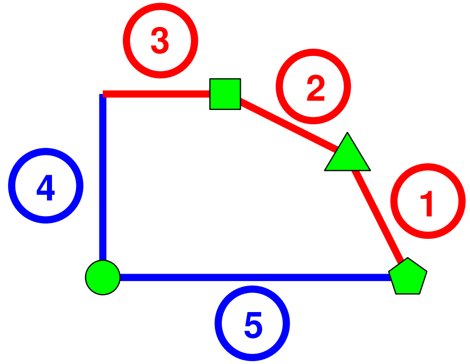
1	2	3	4	5
1	1	1	0	0
1	1	0	1	0
1	0	1	0	1
1	0	0	1	1
0	0	1	1	1
0	1	1	1	0

1	2	3	4	5
0	0	0	1	1
0	0	1	1	0
0	1	1	0	0
1	1	0	0	0
1	0	0	0	1

Vertices as bit patterns



P



Q

	P 1	P 2	P 3	P 4	P 5	Q 1	Q 2	Q 3	Q 4	Q 5
●	1	1	1	0	0	0	0	0	1	1
	1	1	0	1	0	0	0	1	1	0
	1	0	1	0	1	0	1	1	0	0
■	1	0	0	1	1	1	1	0	0	0
▲	0	0	1	1	1	1	0	0	0	1
⬟	0	1	1	1	0					

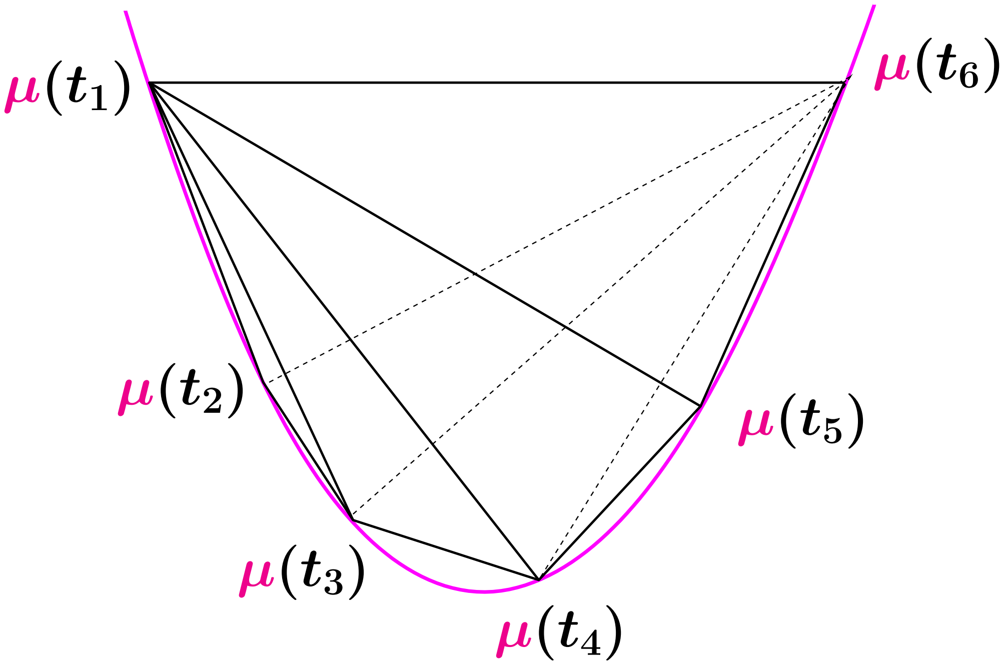
Cyclic polytopes

moment curve in $\mathbb{R}^d$

$$\mu : \mathbb{R} \longrightarrow \mathbb{R}^d \qquad t \mapsto \mu(t) = (t, t^2, \ldots, t^d)^\top.$$

cyclic polytope in dim d with N vertices: $\quad t_1 < t_2 < \cdots < t_N$

$$C_d(N) := \text{conv}\{\mu(t_1), \ldots, \mu(t_N)\}$$

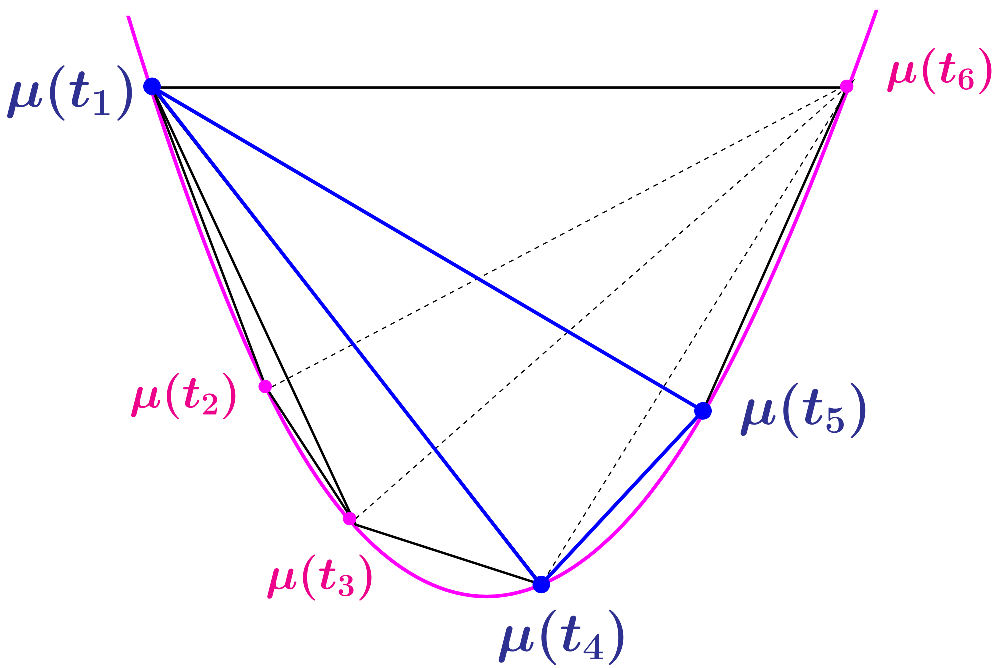


Facets of $C_d(N)$

Any d of the vertices $\mu(t_1), \ldots, \mu(t_N)$ define hyperplane F in $\mathbf{R}^d$.

F facet $\iff$ all **other** vertices are on one side of F

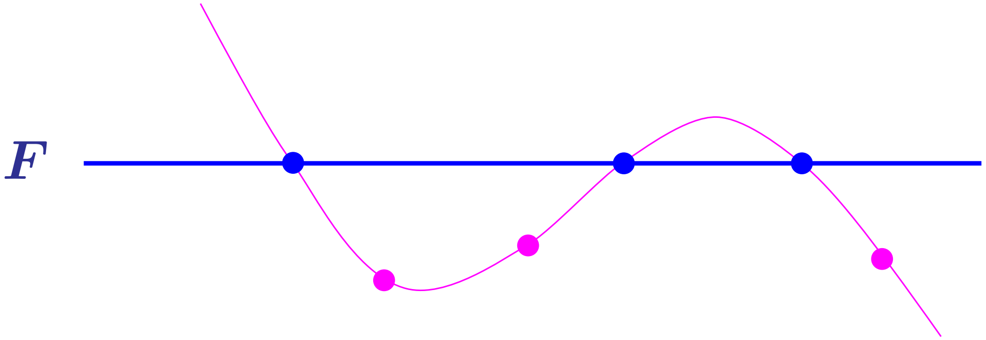
Example: $C_3(6)$, vertices 100110



Gale's Evenness condition

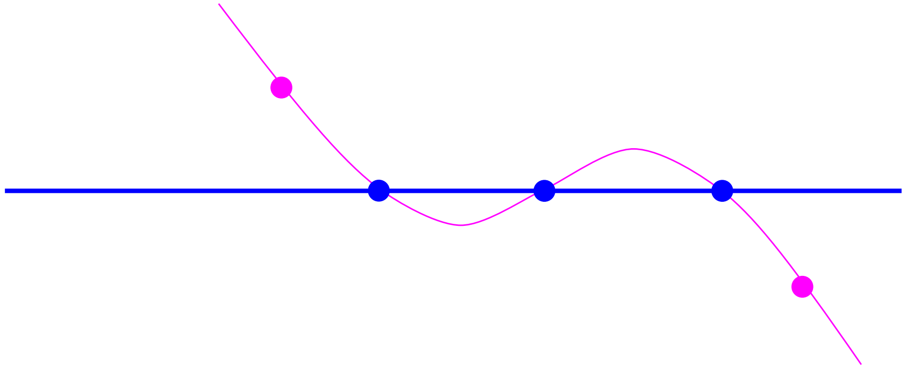
bitstring $s = s_1 s_2 \dots s_N$, $s_i \in \{0, 1\}$ e.g. 100110

defines facet F = conv$\{\mu(t_i) \mid s_i = 1\}$ of $C_d(N)$



$\Longleftrightarrow$ s has only even-length substrings 0110, 011110, 01111110,

forbidden: substrings 010, 01110, $\dots$ of odd length.



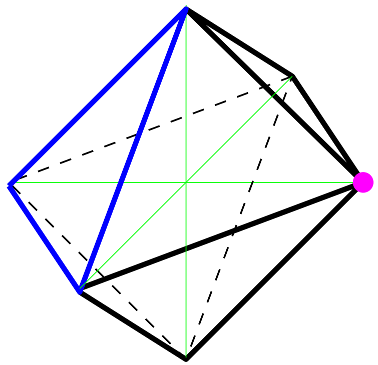
Polar polytopes

P = conv$\{c_1, \ldots, c_N\}$, $\quad 0 \in \text{int}(P) \quad$ vertices c_i

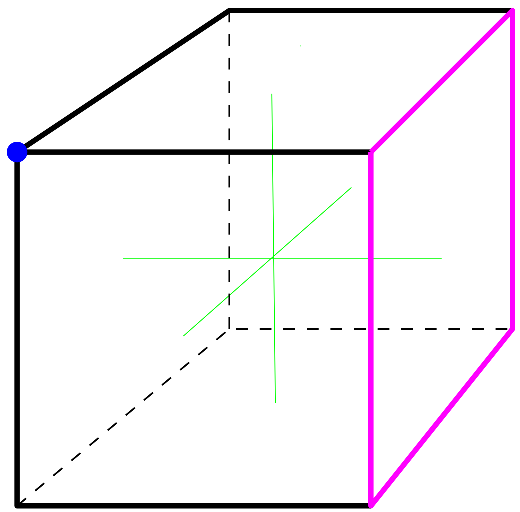
polar polytope

$$P^{\Delta} = \{\, z \mid c_1{}^{\top} z \le 1, \ldots, \; c_N{}^{\top} z \le 1\} \qquad \text{facets } \{z \in P^{\Delta} \mid c_i{}^{\top} z = 1\}$$

P



P^{Δ}



Dual cyclic polytopes

- vertices = strings of **N** bits with **d** bits "**1**",
- **no** odd substrings 010, 01110, 0111110, . . .
  [Gale **even**ness]

Example: d=4, N=6 d=2, N=6 (4×2 game)

d=4, N=6	d=2, N=6
111100	000011
111001	000110
110110	001100
110011	011000
101101	110000
100111	100001
011110	
011011	
001111	

Vertices of $C_d(2d)^\Delta$ and complementarity

$C_4(8)^\Delta$

vertex no.	defining facets	labels (example)
1	00001111	
2	00011011	
3	00011110	
4	00110011	
5	00110110	
6	00111100	
7	01100011	
8	01100110	② ③ ⑥ ⑦
9	01101100	
10	01111000	
11	10000111	
12	10001101	
13	10011001	① ④ ⑤ ⑧
14	10110001	
15	11000011	
16	11000110	
17	11001100	
18	11011000	
19	11100001	
20	11110000	

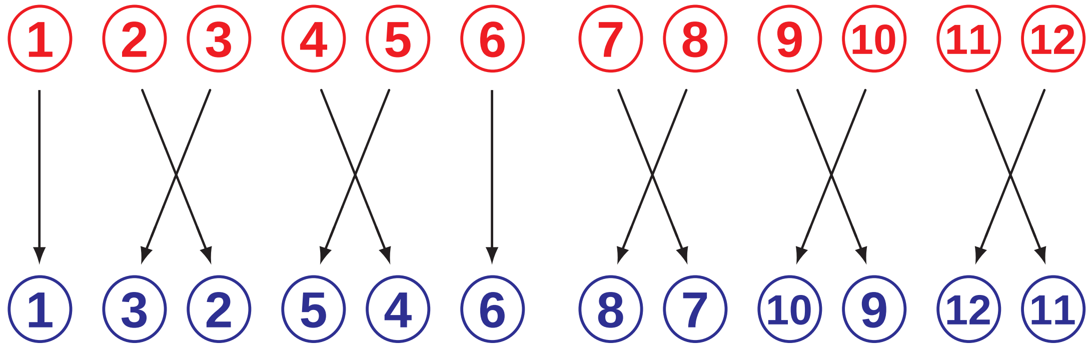
Permuted labels

P = dual cyclic polytope in dimension **d** with **2d** facets

with facets labeled

Q = P

with facets labeled



$\Longrightarrow$ only **one** non-artificial equilibrium:

0 0 0 0 0 0 1 1 1 1 1 1

1 1 1 1 1 1 0 0 0 0 0 0

Lemke–Howson will take long to find it!

Lemke-Howson on dual cyclic polytopes

P								Q							
1	2	3	4	5	6	7	8	1	3	2	4	6	5	8	7
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1

Lemke-Howson on dual cyclic polytopes

P								Q							
①	②	③	④	⑤	⑥	⑦	⑧	①	③	②	④	⑥	⑤	⑧	⑦
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
0	1	1	1	1	0	0	0								

Lemke-Howson on dual cyclic polytopes

P								Q							
①	②	③	④	⑤	⑥	⑦	⑧	①	③	②	④	⑥	⑤	⑧	⑦
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
0	1	1	1	1	0	0	0	0	0	0	1	1	0	1	1

Lemke-Howson on dual cyclic polytopes

P								Q							
1	2	3	4	5	6	7	8	1	3	2	4	6	5	8	7
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
0	1	1	1	1	0	0	0	0	0	0	1	1	0	1	1
0	1	1	0	1	1	0	0								

Lemke-Howson on dual cyclic polytopes

P ①	②	③	④	⑤	⑥	⑦	⑧	Q ①	③	②	④	⑥	⑤	⑧	⑦
1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
0	1	1	1	1	0	0	0	0	0	0	1	1	0	1	1
0	1	1	0	1	1	0	0	0	0	1	1	0	0	1	1
0	0	1	1	1	1	0	0	0	1	1	0	0	0	1	1
0	0	0	1	1	1	1	0	0	1	1	0	0	1	1	0
0	0	1	1	0	1	1	0	0	0	1	1	0	1	1	0
0	1	1	0	0	1	1	0	0	0	0	1	1	1	1	0
0	1	1	0	0	0	1	1	0	0	1	1	1	1	0	0
0	0	1	1	0	0	1	1	0	1	1	0	1	1	0	0
0	0	0	1	1	0	1	1	0	1	1	1	1	0	0	0
0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0

A(4) = path for d=4, label 1

1	2	3	4	5	6	7	8
1	1	1	1				
	1	1	1	1			
	1	1		1	1		
		1	1	1	1		
			1	1	1	1	
		1	1		1	1	
	1	1			1	1	
	1	1				1	1
		1	1			1	1
			1	1		1	1
				1	1	1	1

1	3	2	4	6	5	8	7
				1	1	1	1
			1	1		1	1
		1	1			1	1
	1	1				1	1
	1	1			1	1	
		1	1		1	1	
			1	1	1	1	
		1	1	1	1		
	1	1		1	1		
	1	1	1	1			
1	1	1	1				

B(6) = path for d=6, label 12

	1	2	3	4	5	6	7	8	9	10	11	12	1	3	2	5	4	6	8	7	10	9	12	11
	1	1	1	1	1	1													1	1	1	1	1	1
1		1	1	1	1	1	1						1					1	1		1	1		1
2		1	1	1	1		1	1					1				1	1			1	1		1
3		1	1		1	1	1	1					1			1	1				1	1		1
4		1	1			1	1	1	1				1			1	1			1	1			1
5		1	1		1	1		1	1				1				1	1		1	1			1
6		1	1	1	1			1	1				1					1	1	1	1			1
7		1	1	1	1				1	1			1				1	1	1	1				1
8		1	1		1	1			1	1			1			1	1		1	1				1
9		1	1			1	1		1	1			1			1	1	1	1					1
10		1	1				1	1	1	1			1		1	1	1	1						1
11			1	1			1	1	1	1			1		1	1		1	1					1
12			1	1		1	1		1	1			1		1	1			1	1				1
13			1	1	1	1			1	1			1	1	1				1	1				1
14				1	1	1	1		1	1			1	1	1			1	1					1
15				1	1		1	1	1	1			1	1	1		1	1						1
16					1	1	1	1	1	1			1	1	1	1	1							1
17						1	1	1	1	1	1		1	1	1	1	1							
18							1	1	1	1	1	1	1	1	1	1	1	1						

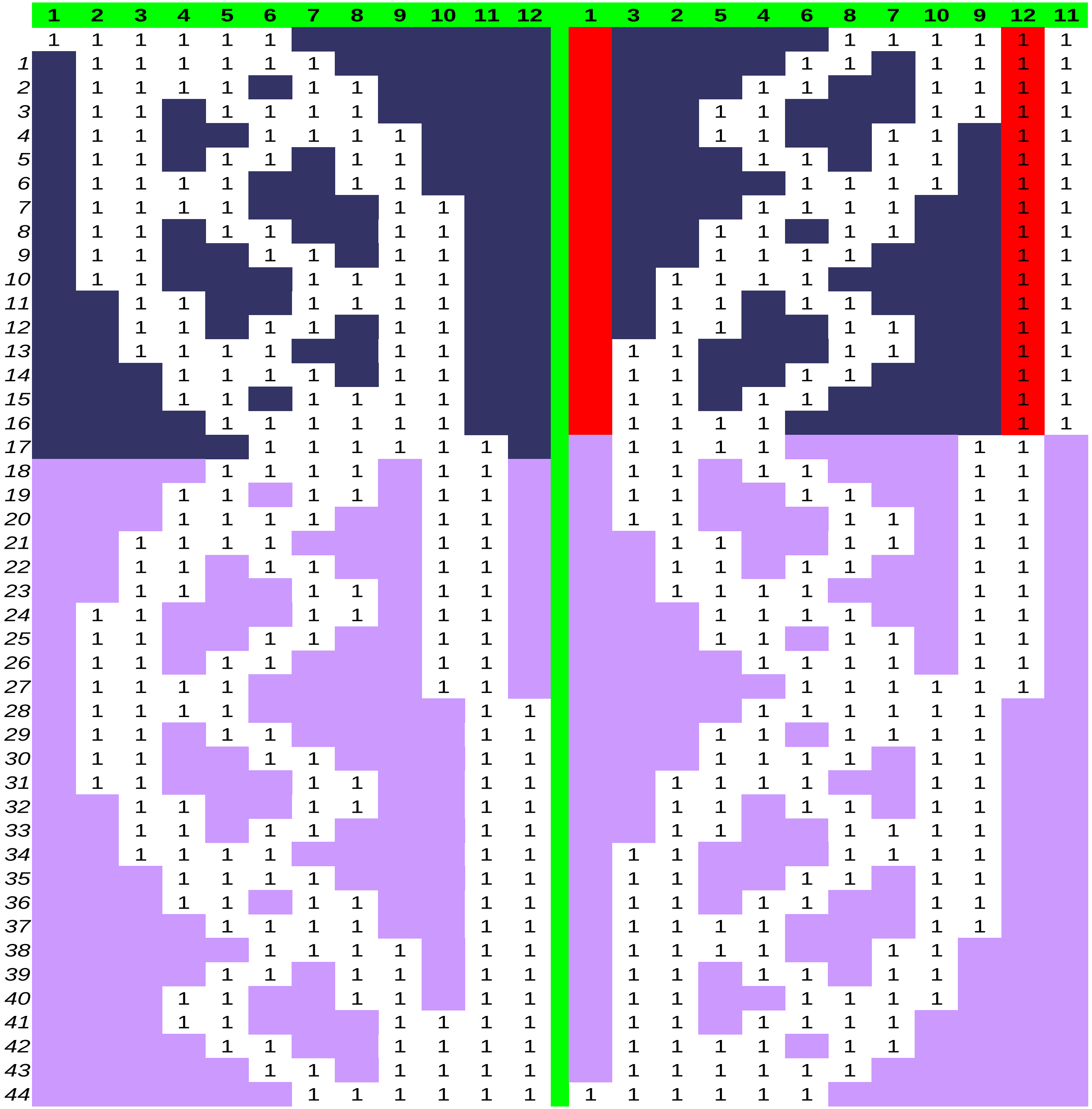
A(4) is prefix of B(6)

	1	2	3	4	5	6	7	8	9	10	11	12		1	3	2	5	4	6	8	7	10	9	12	11
	1	1	1	1	1	1														1	1	1	1	1	1
1		1	1	1	1	1	1							1					1	1		1	1		1
2		1	1	1	1		1	1						1				1	1			1	1		1
3		1	1		1	1	1	1						1			1	1				1	1		1
4		1	1			1	1	1	1					1			1	1			1	1			1
5		1	1		1	1		1	1					1				1	1		1	1			1
6		1	1	1	1			1	1					1					1	1	1	1			1
7		1	1	1	1				1	1				1				1	1	1	1				1
8		1	1		1	1			1	1				1			1	1		1	1				1
9		1	1			1	1		1	1				1			1	1	1	1					1
10		1	1				1	1	1	1				1		1	1	1	1						1
11			1	1			1	1	1	1				1		1	1		1	1					1
12			1	1		1	1		1	1				1		1	1			1	1				1
13			1	1	1	1			1	1				1	1	1				1	1				1
14				1	1	1	1		1	1				1	1	1			1	1					1
15				1	1		1	1	1	1				1	1	1		1	1						1
16					1	1	1	1	1	1				1	1	1	1	1							1
17						1	1	1	1	1	1			1	1	1	1	1							
18							1	1	1	1	1	1		1	1	1	1	1	1						

A(6) = path for d=6, label 1

	1	2	3	4	5	6	7	8	9	10	11	12	1	3	2	5	4	6	8	7	10	9	12	11
	1	1	1	1	1	1													1	1	1	1	1	1
1		1	1	1	1	1	1											1	1		1	1	1	1
2		1	1	1	1		1	1									1	1			1	1	1	1
3		1	1		1	1	1	1								1	1				1	1	1	1
4		1	1			1	1	1	1							1	1			1	1		1	1
5		1	1		1	1		1	1								1	1		1	1		1	1
6		1	1	1	1			1	1									1	1	1	1		1	1
7		1	1	1	1				1	1							1	1	1	1			1	1
8		1	1		1	1			1	1						1	1		1	1			1	1
9		1	1			1	1		1	1						1	1	1	1				1	1
10		1	1				1	1	1	1					1	1	1	1					1	1
11			1	1			1	1	1	1					1	1		1	1				1	1
12			1	1		1	1		1	1					1	1			1	1			1	1
13			1	1	1	1			1	1				1	1				1	1			1	1
14				1	1	1	1		1	1				1	1			1	1				1	1
15				1	1		1	1	1	1				1	1		1	1					1	1
16					1	1	1	1	1	1				1	1	1	1						1	1
17						1	1	1	1	1	1			1	1	1	1					1	1	
18					1	1	1	1		1	1			1	1		1	1				1	1	
19				1	1		1	1		1	1			1	1			1	1			1	1	
20				1	1	1	1			1	1			1	1				1	1		1	1	
21			1	1	1	1				1	1				1	1			1	1		1	1	
22			1	1		1	1			1	1				1	1		1	1			1	1	
23			1	1			1	1		1	1				1	1	1	1				1	1	
24		1	1				1	1		1	1					1	1	1	1			1	1	
25		1	1			1	1			1	1					1	1		1	1		1	1	
26		1	1		1	1				1	1						1	1	1	1		1	1	
27		1	1	1	1					1	1							1	1	1	1	1	1	
28		1	1	1	1						1	1					1	1	1	1	1	1		
29		1	1		1	1					1	1				1	1		1	1	1	1		
30		1	1			1	1				1	1				1	1	1	1		1	1		
31		1	1				1	1			1	1			1	1	1	1			1	1		
32			1	1			1	1			1	1			1	1		1	1		1	1		
33			1	1		1	1				1	1			1	1			1	1	1	1		
34			1	1	1	1					1	1		1	1				1	1	1	1		
35				1	1	1	1				1	1		1	1			1	1		1	1		
36				1	1		1	1			1	1		1	1		1	1			1	1		
37					1	1	1	1			1	1		1	1	1	1				1	1		
38						1	1	1	1		1	1		1	1	1	1			1	1			
39					1	1		1	1		1	1		1	1		1	1		1	1			
40				1	1			1	1		1	1		1	1			1	1	1	1			
41				1	1				1	1	1	1		1	1		1	1	1	1				
42					1	1			1	1	1	1		1	1	1	1		1	1				
43						1	1		1	1	1	1		1	1	1	1	1	1					
44							1	1	1	1	1	1	1	1	1	1	1	1						

B(6) is prefix of A(6)



Suffix of A(6) = C(6) = A(4)+B(6)

	1	2	3	4	5	6	7	8	9	10	11	12	1	3	2	5	4	6	8	7	10	9	12	11
	1	1	1	1	1	1													1	1	1	1	1	1
1		1	1	1	1	1	1											1	1		1	1	1	1
2		1	1	1	1		1	1									1	1			1	1	1	1
3		1	1		1	1	1	1								1	1				1	1	1	1
4		1	1			1	1	1	1							1	1			1	1		1	1
5		1	1		1	1		1	1								1	1		1	1		1	1
6		1	1	1	1			1	1									1	1	1	1		1	1
7		1	1	1	1				1	1							1	1	1	1			1	1
8		1	1		1	1			1	1						1	1		1	1			1	1
9		1	1			1	1		1	1						1	1	1	1				1	1
10		1	1				1	1	1	1					1	1	1	1					1	1
11			1	1			1	1	1	1					1	1		1	1				1	1
12			1	1		1	1		1	1					1	1			1	1			1	1
13			1	1	1	1			1	1				1	1				1	1			1	1
14				1	1	1	1		1	1				1	1			1	1				1	1
15				1	1		1	1	1	1				1	1		1	1					1	1
16					1	1	1	1	1	1				1	1	1	1						1	1
17						1	1	1	1	1	1			1	1	1	1					1	1	
18					1	1	1	1		1	1			1	1		1	1				1	1	
19				1	1		1	1		1	1			1	1			1	1			1	1	
20				1	1	1	1			1	1			1	1				1	1		1	1	
21			1	1	1	1				1	1				1	1			1	1		1	1	
22			1	1		1	1			1	1				1	1		1	1			1	1	
23			1	1			1	1		1	1				1	1	1	1				1	1	
24		1	1				1	1		1	1					1	1	1	1			1	1	
25		1	1			1	1			1	1					1	1		1	1		1	1	
26		1	1		1	1				1	1						1	1	1	1		1	1	
27		1	1	1	1					1	1							1	1	1	1	1	1	
28		1	1	1	1						1	1					1	1	1	1	1	1		
29		1	1		1	1					1	1				1	1		1	1	1	1		
30		1	1			1	1				1	1				1	1	1	1		1	1		
31		1	1				1	1			1	1			1	1	1	1			1	1		
32			1	1			1	1			1	1			1	1		1	1		1	1		
33			1	1		1	1				1	1			1	1			1	1	1	1		
34			1	1	1	1					1	1		1	1				1	1	1	1		
35				1	1	1	1				1	1		1	1			1	1		1	1		
36				1	1		1	1			1	1		1	1		1	1			1	1		
37					1	1	1	1			1	1		1	1	1	1				1	1		
38						1	1	1	1		1	1		1	1	1	1			1	1			
39					1	1		1	1		1	1		1	1		1	1		1	1			
40				1	1			1	1		1	1		1	1			1	1	1	1			
41				1	1				1	1	1	1		1	1		1	1	1	1				
42					1	1			1	1	1	1		1	1	1	1		1	1				
43						1	1		1	1	1	1		1	1	1	1	1	1					
44							1	1	1	1	1	1	1	1	1	1	1	1						

Recurrences for longest paths

A(d) = LH path dropping label 1 in dim d

B(d) = LH path dropping label 2d in dim d

C(d) = suffix of A(d)

lengths of

B(2) C(2) **A**(2) B(4) C(4) **A**(4) B(6) C(6) **A**(6) . . .

are the **Fibonacci** numbers

2 3 5 8 13 21 34 55 89 . . .

Exponential path lengths

longest paths: drop label 1 or $2d$, paths $A(d)$, $B(d)$

path length $\Omega(\phi^{3d/2})$

with Golden Ratio $\phi = (\sqrt{5} + 1) / 2 = 1.618...$

shortest paths: drop label $3d/2$, path $B(d/2)+B(d/2+2)$

path length $\Omega(\phi^{3d/4}) = \Omega(\mathbf{1.434...}^{d})$

Summary and extensions

- NE of a bimatrix game = combinatorial polytope problem

- label dual cyclic polytopes,
  equilibrium at end of **exponentially long** paths

- **but**: fully mixed equilibrium easily guessed
  by support enumeration algorithms

- can extend to $d \times 2d$ games with hard-to-guess
  support (exponentially many guesses on average)
  and exponentially long paths

The 1984 song „The longest time“ by Billy Joel was given the following „computer science“ version by Daniel Barrett, who wrote it as a graduate student at Johns Hopkins University,

„on May 1, 1988, during a difficult Algorithms II final exam“, and subsequently recorded it.

Woh oh-oh-oh find the longest path
Woh oh-oh find the longest path.

If you say P is NP tonight
there would still be
papers left to write
I have a weakness
I'm addicted to completeness
and I keep searching
for the longest path.

The algorithm I would like to see
is of polynomial degree
but it's elusive
nobody has found conclusive
evidence that we can find
the longest path.

I have been
hard working for so long
I swear it's right
and he marks it wrong
somehow I feel
sorry when it's done
GPA 2.1
is more than I hope for

Garey, Johnson,
Karp and other men (and women, too)
try to make it order N log N
am I a mad fool
if I spend my life in grad school
forever following the longest path

Woh oh-oh-oh find the longest path
Woh oh-oh find the longest path
Woh oh-oh find the longest path.